Journal of Knot Theory and Its Ramifications
Vol. 15, No. 4 (2006) 413–427

EXTRASPECIAL 2-GROUPS AND IMAGES OF BRAID GROUP REPRESENTATIONS

JENNIFER M. FRANKO*, ERIC C. ROWELL† and ZHENGHAN WANG‡

Department of Mathematics, Indiana University,
Bloomington, IN 47405, USA
**jefranko@indiana.edu*
†errowell@indiana.edu
‡zhewang@indiana.edu

Accepted 20 May 2005

ABSTRACT

We investigate a family of (reducible) representations of the braid groups $\mathcal{B}_n$ corresponding to a specific solution to the Yang–Baxter equation. The images of $\mathcal{B}_n$ under these representations are finite groups, and we identify them precisely as extensions of extra-special 2-groups. The decompositions of the representations into their irreducible constituents are determined, which allows us to relate them to the well-known Jones representations of $\mathcal{B}_n$ factoring over Temperley–Lieb algebras and the corresponding link invariants.

Keywords: Braid group; extraspecial 2-group; Arf invariant; Temperley–Lieb algebra.

Mathematics Subject Classification 2000: Primary 20F36; Secondary 20D15, 57M25

1. Introduction

Representations of Artin's braid groups $\mathcal{B}_n$ are of great importance to mathematicians [1], and physicists recently [15]. Certain representations of the braid groups have been proposed as the fractional statistics of anyons [15], and used in the topological models for quantum computing [4]. Therefore it is interesting to identify the images of such braid group representations. In this paper we analyze a particular representation of the braid groups afforded by a unitary solution of the braid relation, i.e. a flipped R-matrix $R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ that satisfies the *Yang–Baxter equation*

$$(R \otimes I_2)(I_2 \otimes R)(R \otimes I_2) = (I_2 \otimes R)(R \otimes I_2)(I_2 \otimes R), \tag{YBE}$$

where I_2 is the 2×2 identity matrix. All solutions to the YBE of the form $R : V \otimes V \to V \otimes V$ with V 2-dimensional have been listed in [9]. Dye [2] found all unitary solutions of this form to the braid relations based on this list. The importance of this particular braid operator R was pointed out in the work of Kauffman and Lomonaco [12], and the connection of R with quantum computing was explored there which is another reason for our interest.

As is well-known, any (invertible) matrix satisfying the YBE gives rise to representations of $\mathcal{B}_n$ for any n. The representation $(\pi_n, (\mathbb{C}^2)^{\otimes n})$ corresponding to the matrix R above is unitary and defined as follows:

$$\pi_n(\sigma_i) = I_2^{\otimes i-1} \otimes R \otimes I_2^{\otimes n-i-1},$$

where σ_i is the ith braid generator. We shall see that the images of the braid groups $\mathcal{B}_n$ under this representation are finite groups, and the image matrix of each braid generator σ_i has only two distinct eigenvalues. The image group of an irreducible constituent of π_n is generated by the conjugacy class of a braid generator with two distinct eigenvalues whose ratio is not -1, i.e. has the so-called 2-eigenvalue property defined in [3]. Such representations are completely classified [3], so in principle the image groups of the irreducible constituents of π_n can be identified by using the complete list in [3, Theorem 1.6]. But we will see that π_n is reducible, hence first we would need to find the irreducible constituents of π_n; then distinguish a few different cases in the complete list for the images of the irreducible constituents. Instead we choose to solve the problem in an elementary and self-contained way. We decompose these representations π_n (for all n) into their irreducible constituents and describe the images of $\mathcal{B}_n$ under π_n as abstract groups. We find that the images of the *pure* braid groups are (nearly) extra-special 2-groups $\mathbf{E}_{n-1}^{-1}$. The images of the full braid groups $\mathcal{B}_n$ are extensions of the (nearly) extra-special 2-groups $\mathbf{E}_{n-1}^{-1}$ by the symmetric groups S_n, and the restrictions of the representations π_n to the subgroup of pure braids are isotypic copies of the odd representations of $\mathbf{E}_{n-1}^{-1}$.

As already discussed in [12] we can define link invariants using the representations π_n. By observing that π_n is related to the Jones representation of the braid groups at a 4th root of unity, we improve slightly some earlier results of Jones about the images of the Jones representation of the braid groups at the 4th root of unity [10]. As a consequence we point out that the resulting link invariants are essentially the Jones polynomial at a 4th root of unity, hence really the Arf invariant of a link (see references in [11]). The slight improvement of Jones's result comes from two subtle points about the Jones representations. Firstly, in the Jones representation of the braid group, there is some freedom in choosing phases so it is convenient to state the results projectively, i.e. modulo scalars, while not losing any significance mathematically. We choose to work out the images in full generality (as opposed to projectively) as this is desirable in physics for the applications to the fractional statistics of quantum Hall fluid [13]. This changes the images of the pure braid groups from the elementary abelian groups $\mathbb{Z}_2^{n-1}$ to the (nearly) extra-special 2-groups $\mathbf{E}_{n-1}^{1}$. Secondly, when the number of strands of the braid groups is even, there are two irreducible sectors of the Jones representation [10]. Jones found the projective images for each sector, but we determine the images of the two sectors together. This brings up a subtlety about the centers of the (nearly) extraspecial 2-groups in those cases, which disappears when the two irreducible sectors are treated separately, and projectively.

These results lead to several questions for future research currently being worked out by the authors. What are the closed images of the braid groups under the representations afforded by the other R-matrices listed in [2] and what are the associated link invariants? What are the other extraspecial p-groups that appear as homomorphic images of the pure braid groups? Results in this direction have been obtained in [7, 6] where Heisenberg groups are used in place of extraspecial p-groups.

2. Preliminaries

2.1. *Definitions and computations*

Definition 2.1. Artin's braid group $\mathcal{B}_n$ on n strands has presentation in generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying relations:

(B1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$.
(B2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-2$.

For tensor products of matrices we use the convention "left into right", that is, if $X = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $X \otimes A = \begin{pmatrix} aX & bX \\ cX & dX \end{pmatrix}$. Various matrices and quantities will be needed throughout, so we define them here:

(1) I_m is the $m \times m$ identity matrix.
(2) $R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$.
(3) $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
(4) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
(5) $P_s = \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix}$.
(6) $P_{\sigma_x} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.
(7) $P_n = (P_s \otimes P_{\sigma_x})^{\otimes \lfloor n/2 \rfloor} \otimes I_2^{\otimes(n-2\lfloor n/2 \rfloor)}$, where $\lfloor a \rfloor$ is the integer part of a.
(8) $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
(9) $g_i = \pi_n((\sigma_i)^2) = I_2^{\otimes(i-1)} \otimes R^2 \otimes I_2^{\otimes(n-i-1)}$ (observe we ignore the dependence of g_i on n; the value of n will always be clear from the context).
(10) $\zeta = \frac{1}{\sqrt{2}}(1 + \sqrt{-1})$.
(11) $d = \begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{pmatrix}$.

(12) $D = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \bar{\zeta} & 0 & 0 \\ 0 & 0 & \bar{\zeta} & 0 \\ 0 & & 0 & \zeta \end{pmatrix}$.

(13) $M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

We will also need a few simple computations involving these matrices so we record them in the following:

Lemma 2.2. *The matrices defined above satisfy*:

(a) $R^2 = s \otimes \sigma_x$.
(b) $R = \frac{1}{\sqrt{2}}(R^2 + I_4)$, $R^{-1} = \frac{1}{\sqrt{2}}(R^{-2} + I_4)$.
(c) $(R^2 \otimes I_2)(I_2 \otimes R^2) = -(I_2 \otimes R^2)(R^2 \otimes I_2)$.
(d) $g_i g_{i+1} = -g_{i+1} g_i$.
(e) $(R^{-1} \otimes I_2)(I_2 \otimes R^2)(R \otimes I_2) = (I_2 \otimes R^2)(R^2 \otimes I_2)$.
(f) $\pi_n(\sigma_i^{-1}) g_{i\pm1} \pi_n(\sigma_i) = g_{i\pm1} g_i$.
(g) $g_i g_j = g_j g_i$ *and* $\pi_n(\sigma_i) g_j = g_j \pi_n(\sigma_i)$ *if* $|i - j| \geq 2$.
(h) $R^4 = -I_4$, $(g_i)^2 = -I_{2^n}$.
(i) $(P_s)^{-1} s P_s = \sqrt{-1}\sigma_z$, *and* $(P_s)^{-1} \sigma_x P_s = \sigma_x$.
(j) $(P_{\sigma_x})^{-1} \sigma_x P_{\sigma_x} = \sigma_z$, $(P_{\sigma_x})^{-1} s P_{\sigma_x} = s$.
(k) $(P_n)^{-1} g_{2i+1} P_n = \sqrt{-1}(I_2^{\otimes 2i} \otimes \sigma_z^{\otimes 2} \otimes I_2^{\otimes n-2i-2})$, *and* $(P_n)^{-1} g_{2i} P_n = g_{2i}$.

Proof. The first assertions (a) and (b) are straightforward computations. Having checked that s and σ_x anti-commute (c) follows, and (d) is immediate from (c). Using (b) and the observation $R^{-2} = -R^2$, we express the left-hand side of the equality in (e) in terms of R^2 and then use (c) to derive the right-hand side. Assertion (f) is immediate from (e). Assertion (g) is a consequence of the "far commutation" relations satisfied by the braid group, and (h) follows from (b) and the definition of g_i. The matrix P_s (respectively, P_{σ_x}) is a change of bases matrix that diagonalizes s (respectively, σ_x) and commutes with σ_x (respectively, s). This is the statement (j), and (k) follows directly from this fact and the definition of P_n. □

2.2. *Restriction to* $\mathcal{P}_n$

The homomorphism from $\mathcal{B}_n$ to the symmetric group on n letters S_n given by $\sigma_i \to (i, i+1)$ has kernel $\mathcal{P}_n$ the so-called pure braid group. $\mathcal{P}_n$ is generated by all conjugates of the squares of the generators of $\mathcal{B}_n$: $(\sigma_i)^2$. Actually a more economical presentation of $\mathcal{P}_n$ can be found (see e.g. [1]), but we shall not need it here. To exploit this relationship between $\mathcal{B}_n$ and $\mathcal{P}_n$ we shall restrict π_n to the subgroup $\mathcal{P}_n$. For convenience of notation we introduce the following notation.

Definition 2.3. $H_n := \pi_n(\mathcal{P}_n)$ and $G_n := \pi_n(\mathcal{B}_n)$

We can describe H_n very succinctly:

Lemma 2.4. *H_n is generated by $g_1, \ldots, g_{n-1}$.*

Proof. Observe that H_n is generated by all conjugates of g_i, so that H_n is the smallest normal subgroup of G_n containing the subgroup $\langle g_1, \ldots, g_{n-1} \rangle$ generated by the g_i. But by Lemma 2.2(f),(g) $\langle g_1, \ldots, g_{n-1} \rangle$ is normal in G_n so $\langle g_1, \ldots, g_{n-1} \rangle = H_n$. □

Remark 2.5. Combining this with Lemma 2.2(a), we have a very powerful tool for studying the representation π_n of $\mathcal{B}_n$. After decomposing the representation π_n restricted to $\mathcal{P}_n$ into its irreducible components and computing the corresponding images of the $(\sigma_i)^2$, we can immediately determine the decomposition of the images of the σ_i under π_n as $\pi_n(\sigma_i) = \frac{1}{\sqrt{2}}(g_i + I_{2^n})$.

Once we understand H_n as an abstract group and decompose its defining representation (as it is presented to us as a matrix group), we will need to consider the group G_n/H_n. We can immediately see that G_n/H_n is a homomorphic image of S_n as π_n induces a surjective homomorphism $\overline{\pi}_n : \mathcal{B}_n/\mathcal{P}_n \to G_n/H_n$ and $\mathcal{B}_n/\mathcal{P}_n \cong S_n$. We would like to know if $\overline{\pi}_n$ is an isomorphism, so we must determine if $\mathrm{Ker}(\overline{\pi}_n)$ is trivial. Observing that the kernel of $\overline{\pi}_n$ is (isomorphic to) a normal subgroup of S_n we need only check that the kernel is not S_n, A_n or the normal subgroup of S_4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. For $n \geq 4$ it is sufficient to check that the element $(12)(34)$ is not in the kernel, while for $n = 3$ we should check that (123) is not in the kernel. Since H_n is a normal subgroup of G_n, we also have a homomorphism $\vartheta : G_n \to \mathrm{Aut}(H_n)$ where $\mathrm{Aut}(H_n)$ is the automorphism group of H_n and $\vartheta(s)$ is conjugation by $s \in G_n$. Restricting to H_n we see that $\vartheta(H_n) = \mathrm{Inn}(H_n) \subset \mathrm{Aut}(H_n)$ the normal subgroup of inner automorphisms of H_n and so we have the induced homomorphism $\overline{\vartheta} : G_n/H_n \to \mathrm{Aut}(H_n)/\mathrm{Inn}(H_n)$. Since $\mathrm{Ker}(\overline{\pi}_n) \subset \mathrm{Ker}(\overline{\vartheta} \circ \overline{\pi}_n)$, if we can show the composition has trivial kernel then $\overline{\pi}_n$ must be an isomorphism. By Lemma 2.2(d), the generators g_i of H_n commute or anti-commute, so the elements of $\mathrm{Inn}(H_n)$ act by sign changes. So if we can show that the automorphisms corresponding to $(12)(34)$ (for $n \geq 4$) and (123) are not simply sign changes, we will have shown that $\overline{\pi}_n$ is an isomorphism. The corresponding elements of $\mathcal{B}_n$ are $(\sigma_1\sigma_3)$ and $(\sigma_2\sigma_1)$ and we use Lemma 2.2(f) to compute that under $\overline{\vartheta}\overline{\pi}_n$ the element $(\sigma_1\sigma_3)$ maps g_2 to $g_2g_1g_3$, and $(\sigma_2\sigma_1)$ maps g_2 to $g_2g_1g_2 = g_1$. We check directly that $g_1g_3 \neq \pm 1$ using Lemma 2.2(a), (i) and (j), so $\overline{\pi}_n$ is an isomorphism for $n \geq 3$. In the case $n = 2$ we see that G_2 is the group generated by the matrix R which is isomorphic to $\mathbb{Z}_8$, so combining, we have:

Theorem 2.6. *We have an exact sequence*:

$$1 \to H_n \xrightarrow{\subseteq} G_n \xrightarrow{\vartheta} S_n \to 1$$

for all $n \geq 2$. *In other words*, G_n *is an extension of* H_n *by* S_n.

3. Extraspecial 2-Groups and Related Groups

Definition 3.1. The group $\mathbf{E}_m^\nu$ is the abstract group generated by

$$x_1, \ldots, x_m$$

with relations:

$$x_i^2 = \nu, \qquad 1 \leq i \leq m, \tag{1}$$

$$x_i x_j = x_j x_i, \qquad |i-j| \geq 2, \tag{2}$$

$$x_{i+1} x_i = -x_i x_{i+1}, \qquad 1 \leq i \leq m, \tag{3}$$

where -1 is an order two central element, and $\nu = \pm 1$.

These groups appear classically and have important connections with Clifford algebras. The case $\nu = -1$ appears in Exercise 3.9 in the text by Fulton and Harris [5], and other cases appeared in [8]. The necessary facts about these groups are found in various places, but are elementary so we reprove them here for the reader's convenience.

3.1. *Properties of* $\mathbf{E}_m^\nu$

Any element in $\mathbf{E}_m^\nu$ can be expressed in the normal form: $\pm x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ where $\alpha_i \in \mathbb{Z}_2$. The following lemma will show that it is unique.

Lemma 3.2. *Denote by* $Z(\mathbf{E}_m^\nu)$ *the center of* $\mathbf{E}_m^\nu$. *We have*:

(a) $Z(\mathbf{E}_m^\nu) = \begin{cases} \{\pm 1\}, & m \text{ even}, \\ \{\pm 1, \pm x_1 x_3 \cdots x_m\}, & m \text{ odd}. \end{cases}$

(b) $\mathbf{E}_m^\nu / \{\pm 1\} \cong (\mathbb{Z}_2)^m$.

(c) *Any* $x \in \mathbf{E}_m^\nu \backslash Z(\mathbf{E}_m^\nu)$ *is conjugate to* $-x$.

(d) *Any nontrivial normal subgroup of* $\mathbf{E}_m^\nu$ *intersects* $Z(\mathbf{E}_m^\nu)$ *nontrivially.*

(e) *For* $m = 2k-1$ *odd*, $Z(\mathbf{E}_{2k-1}^\nu) \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2, & \text{if } \nu = 1 \text{ or } k \text{ even}, \\ \mathbb{Z}_4 & \text{if } \nu = -1 \text{ and } k \text{ odd}. \end{cases}$

(f) *The normal form* $\pm x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ *is unique.*

Proof. Using the above-mentioned normal form, we may assume, without loss of generality, that $z = x_1^{\alpha_1} \cdots x_{2k}^{\alpha_m} \in Z(\mathbf{E}_m^\nu)$ since if z is central, so is $-z$. By the commutation/anti-commutation relations in $\mathbf{E}_m^\nu$, we have $x_i z = (-1)^{\alpha_{i-1}+\alpha_{i+1}} z x_i = z x_i$ for all i where we take $\alpha_0 = \alpha_{m+1} = 0$. Thus we get the system of equations over $\mathbb{Z}_2$:

$$\alpha_2 = 0,$$

$$\alpha_{m-1} = 0,$$

$$\alpha_i + \alpha_{i+2} = 0 \pmod 2, \quad 1 \leq i \leq m-2.$$

If m is even then the system has only the trivial solution $\boldsymbol{\alpha} = \mathbf{0}$, but if m is odd there are two solutions $\mathbf{0}$ and $(1, 0, 1, \ldots, 0, 1)$, that is, all the $\alpha_{2i} = 0$ and $\alpha_{2i+1} = 1$.

Thus we have (a). It is clear from the relations in $\mathbf{E}_m^\nu$ that $\mathbf{E}_m^\nu/\{\pm 1\}$ is presented by m commuting generators of order 2, i.e. $(\mathbb{Z}_2)^m$. To prove (c) observe that any non-central element $x \in \mathbf{E}_m^\nu$ must anti-commute with some x_i. So (d) follows from (c) as any nontrivial normal subgroup N must either be central or contain $\{x, -x\}$ for some non-central element x so that $-1 \in N$ as well. For (e) we compute the order of the central element $x_1 x_3 \cdots x_{2k-1}$ and find that it is 2 or 4, which gives us the two cases. Assertion (f) follows from a simple counting argument as $|\mathbf{E}_m^\nu| = 2^{m+1}$ by (b). □

Definition 3.3. A group G of order 2^{m+1} is an *extraspecial 2-group* if (see [8]):

(1) The center $Z(G)$ and the commutator subgroup G' coincide and are isomorphic to $\mathbb{Z}_2$.
(2) $G/Z(G) \cong (\mathbb{Z}_2)^m$.

It is immediate from the anti-commutation relations that the commutator subgroup of $\mathbf{E}_m^\nu$ is $\{\pm 1\}$, and for $m = 2k$ the other conditions were verified in Lemma 3.2, so we have:

Proposition 3.4. $\mathbf{E}_{2k}^\nu$ *is an extraspecial* 2*-group.*

Remark 3.5. Since the group $\mathbf{E}_{2k+1}^\nu$ contains $\mathbf{E}_{2k}^\nu$, we call the groups $\mathbf{E}_m^\nu$ *nearly* extraspecial 2-groups for any m (so they include extraspecial 2-groups). This should not be confused with *almost* extraspecial 2-groups found in the literature which are central products of extraspecial 2-groups with $\mathbb{Z}_4$. The cases where the center of $\mathbf{E}_m^\nu$ is isomorphic to $\mathbb{Z}_4$ are almost extraspecial, but when the center is $\mathbb{Z}_2 \times \mathbb{Z}_2$ they are not (see [8]).

3.2. *Representations of* $\mathbf{E}_m^\nu$

We wish to construct the irreducible representations of $\mathbf{E}_m^\nu$. There are 4 cases corresponding to the parity of m and the choice of ν. For the reader's convenience we recall the following standard facts from the character theory of finite groups (see any standard text, e.g. [5]):

Proposition 3.6. *Let G be a finite group, and* $\mathrm{Irr}(G) = \{\chi_i\}_{i \in \mathcal{I}}$ *the set of irreducible characters of G, corresponding to irreducible representations V_i.*

(a) $|\mathrm{Irr}(G)|$ *is equal to the number of conjugacy classes of elements of G.*
(b) $|G| = \sum_{\mathcal{I}} (\dim V_i)^2$.
(c) *For* $\chi_i,\ \chi_j \in \mathrm{Irr}(G)$ $\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \begin{cases} 0, & \text{if } V_i \not\cong V_j, \\ |G|, & \text{if } V_i \cong V_j. \end{cases}$
(d) *If g and h are not conjugate then* $\sum_{\mathcal{I}} \chi_i(h)\overline{\chi_i(g)} = 0$.

3.2.1. $\mathbf{E}_{2k}^{-1}$

To determine the number of irreducible representations we count conjugacy classes. The center $\{\pm 1\}$ gives us two singleton classes, and Lemma 3.2(c) shows that the non-singleton conjugacy classes are given by $\left[\pm x_1^{\alpha_1} \cdots x_{2k}^{\alpha_{2k}}\right]$ for any $\alpha \in (\mathbb{Z}_2)^{2k} \setminus \{\mathbf{0}\}$. So we have $2+(2^{2k}-1) = 2^{2k}+1$ inequivalent irreducible representations. Let $\mathrm{Irr}(\mathbf{E}_{2k}^{-1}) = \{V_1, \ldots, V_{2^{2k}+1}\}$ denote a set of inequivalent irreducible representations of $\mathbf{E}_{2k}^{-1}$. By Lemma 3.2 we can induce 1-dimensional representations of $\mathbf{E}_{2k}^{-1}$ from any representation of $(\mathbb{Z}_2)^{2k}$ by letting the center act trivially. Thus we have 2^{2k} 1-dimensional representations (say, $V_2, \ldots, V_{2^{2k}+1}$) leaving only one representation, V_1 to determine. Using the class equation

$$2^{2k+1} = |\mathbf{E}_{2k}^{-1}| = (\dim V_1)^2 + \sum_{2}^{2^{2k}+1} (\dim V_i)^2 = (\dim V_1)^2 + 2^{2k},$$

we find that $\dim V_1 = 2^k$. The 1-dimensional representations are equal to their characters so for $2 \le i \le 2^{2k}+1$ we have $\chi_i(1) = \chi_i(-1) = 1$, and $\chi_i([\pm x_j]) = \pm 1$ for all possible choices of sign. From Proposition 3.6(c),(d), we find that $\chi_1(1) = -\chi_1(-1) = 2^k$, and $\chi_1\left(\left[\pm x_1^{\alpha_1} \cdots x_{2k}^{\alpha_{2k}}\right]\right) = 0$. We can construct the representation (ρ_1, V_1) as follows (recall the definition of the matrices s and σ_z from Sec. 2.1):

$$\begin{aligned}
\rho_1(x_1) &= \sqrt{-1}\left(\sigma_z \otimes I_2^{\otimes k-1}\right),\\
\rho_1(x_2) &= s \otimes I_2^{\otimes k-1},\\
&\vdots\\
\rho_1(x_{2i}) &= I_2^{\otimes i-1} \otimes s \otimes I_2^{\otimes k-i},\\
\rho_1(x_{2i+1}) &= \sqrt{-1}\left(I_2^{\otimes i-1} \otimes \sigma_z \otimes \sigma_z \otimes I_2^{\otimes k-i-1}\right),\\
&\vdots\\
\rho_1(x_{2k}) &= I_2^{\otimes k-1} \otimes s.
\end{aligned}$$

As $(\sigma_z)^2 = I_2$, $s^2 = -I_2$ and $\sigma_z s \sigma_z = -s$ we see that this is indeed a representation of $\mathbf{E}_{2k}^{-1}$, and since $\mathrm{tr}(s) = \mathrm{tr}(\sigma_z) = 0$ it follows from the orthogonality of characters that this is the irreducible 2^k-dimensional representation of $\mathbf{E}_{2k}^{-1}$.

3.2.2. $\mathbf{E}_{2k-1}^{-1}$

We now construct the irreducible representations of $\mathbf{E}_{2k-1}^{-1}$. Denote by z the central element $x_1 x_3 \cdots x_{2k-1}$ for convenience. Using Lemma 3.2 we find that there are $2^{2k-1}+2$ distinct conjugacy classes in $\mathbf{E}_{2k-1}^{-1}$ and therefore we may label the inequivalent classes of irreducible representations by $\mathrm{Irr}(\mathbf{E}_{2k-1}^{-1}) = \{W_1, \ldots, W_{(2^{2k-1}+2)}\}$. We get 2^{2k-1} distinct 1-dimensional representations from $(\mathbb{Z}_2)^{2k-1}$ by composing with the projection onto $\mathbf{E}_{2k}/\{\pm 1\}$; denote them by $W_3, \ldots, W_{(2^{2k-1}+2)}$. We compute their characters ψ_i for $3 \leq i \leq 2^{2k-1}+2$ as in the $\mathbf{E}_{2k}^{-1}$ case $\psi_i(1) = \psi_i(-1) = 1$ and $\psi_i(\pm x_j) = \pm 1$ which determines their values on all classes (observe that $\psi_i(z) = \psi_i(-z)$ for nontrivial central elements $\pm z$ for these 1-dimensional representations). From Proposition 3.6(b), we get $\dim W_1 + \dim W_2 = 2^k$ for the remaining two irreducible representations. Since $\dim W_i | 2^{2k}$ we see that in fact, $\dim W_1 = \dim W_2 = 2^{k-1}$. Using Proposition 3.6(c), we find that the characters ψ_1 and ψ_2 vanish on all equivalence classes except for the central classes: $[1]$, $[-1]$, $[z]$ and $[-z]$. Observing that the restrictions of W_1 and W_2 to the subgroup $\mathbf{E}_{2k-2}^{-1} \subset \mathbf{E}_{2k-1}^{-1}$ must both be the unique non-trivial irreducible 2^{k-1}-dimensional representation we find that $\psi_1(-1) = \psi_2(-1) = -2^{k-1}$. Proposition 3.6(c),(d) then implies first that $\psi_1(z) = \psi_2(-z) = -\psi_1(-z) = -\psi_2(z)$, and then using this and the orthogonality of ψ_1 and ψ_2 to see that $\|\psi_1(z)\| = 2^{k-1}$. Restricting to $Z\big(\mathbf{E}_{2k-1}^{-1}\big)$ and recalling that $\mathbb{Z}_2 \times \mathbb{Z}_2$ has only real characters while the non-trivial characters of $\mathbb{Z}_4$ have pure complex values on its generators we determine the value of $\psi_1(z)$ up to a choice of sign coming from switching W_1 and W_2. For the purpose of simplifying notation later we include a sign depending on the value of k (mod 4) and define:

$$\psi_1(x) = \begin{cases} \pm 2^{k-1}, & \text{for } x = \pm 1, \\ \pm(-1)^{(k/2)}(2^{k-1}), & \text{for } x = \pm z, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

and

$$\psi_2(x) = \begin{cases} \pm 2^{k-1}, & \text{for } x = \pm 1, \\ \mp(-1)^{(k/2)}(2^{k-1}), & \text{for } x = \pm z, \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

Next we give explicit matrix realizations of W_1 and W_2. Since $Z\big(\mathbf{E}_{2k-1}^{-1}\big)$ must act non-trivially (although not necessarily faithfully) on W_1 and W_2 we use the inclusion $\mathbf{E}_{2k-1}^{-1} \subset \mathbf{E}_{2k}^{-1}$ to observe:

$$\mathrm{Ind}_{\mathbf{E}_{2k-1}^{-1}}^{\mathbf{E}_{2k}^{-1}}(W_1) = \mathrm{Ind}_{\mathbf{E}_{2k-1}^{-1}}^{\mathbf{E}_{2k}^{-1}}(W_2) = V_1,$$

where V_1 is the 2^k-dimensional irreducible representation of $\mathbf{E}_{2k}^{-1}$ given in Sec. 3.2. Thus by Frobenius reciprocity (and a dimension count) we have that

$$\mathrm{Res}_{\mathbf{E}_{2k-1}^{-1}}^{\mathbf{E}_{2k}^{-1}}(V_1) = W_1 \oplus W_2. \tag{6}$$

From this we get explicit realizations (λ_1, W_1) and (λ_2, W_2). (*N.b.* the only difference of λ_1 and λ_2 on the generators is that the image of x_{2k-1} differs in sign.)

$$\begin{aligned}
\lambda_1(x_1) &= \lambda_2(x_1) = \sqrt{-1}\sigma_z \otimes I_2^{\otimes k-2},\\
\lambda_1(x_2) &= \lambda_2(x_2) = s \otimes I_2^{\otimes k-2},\\
&\vdots\\
\lambda_1(x_{2i}) &= \lambda_2(x_{2i}) = I_2^{\otimes i-1} \otimes s \otimes I_2^{\otimes k-i-1},\\
\lambda_1(x_{2i+1}) &= \lambda_2(x_{2i+1}) = \sqrt{-1}I_2^{\otimes i-1} \otimes \sigma_z \otimes \sigma_z \otimes I_2^{\otimes k-i-2},\\
&\vdots\\
\lambda_1(x_{2k-2}) &= \lambda_2(x_{2k-2}) = I_2^{\otimes k-2} \otimes s,\\
\lambda_1(x_{2k-1}) &= -\lambda_2(x_{2k-1}) = \sqrt{-1}I_2^{\otimes k-2} \otimes \sigma_z.
\end{aligned}$$

One easily checks that these indeed define irreducible representations of $\mathbf{E}_{2k-1}^{-1}$ just as in the $m = 2k$ case. It is perhaps worth computing the traces of the images of the central element z under λ_1 and λ_2. We have: $\lambda_1(z) = -\lambda_2(z) = (\sqrt{-1})^k((\sigma_z)^2 \otimes \cdots \otimes (\sigma_z)^2) = (\sqrt{-1})^k I_{2^{k-1}}$ so that:

$$\mathrm{tr}(\lambda_1(z)) = -\mathrm{tr}(\lambda_2(z)) = \begin{cases} 2^{k-1}, & \text{if } k \equiv 0 \pmod 4,\\ -2^{k-1}, & \text{if } k \equiv 2 \pmod 4,\\ \sqrt{-1}(2^{k-1}), & \text{if } k \equiv 1 \pmod 4,\\ -\sqrt{-1}(2^{k-1}), & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

The traces of the images of ± 1 are also easily computed, and comparing these values with the above formulas (4) and (5), we check that the characters of λ_1 and λ_2 are ψ_1 and ψ_2 respectively.

3.2.3. $\mathbf{E}_m^1$

Suppose that (ρ, V) is any representation of $\mathbf{E}_m^{-1}$ defined on generators $\rho(x_i) = A_i$ for some set of matrices $\{A_i\}_{1\leq i\leq m}$. Denote by $x'_1, \ldots, x'_m$ the generators of $\mathbf{E}_m^1$ and define $\rho'(x'_i) = \sqrt{-1}A_i$. Then since $(A_i)^2 = -Id_V$ we have $(\rho'(x'_i))^2 = Id_V$ and (ρ', V) defines a representation of $\mathbf{E}_m^1$ (observe that the commutation/anti-commutation relations are homogeneous and hence also satisfied). Obviously this process is reversible, so that all representations of $\mathbf{E}_m^1$ are obtained in this way. If we define representations λ'_1 and λ'_2 of $\mathbf{E}_{2k-1}^1$ corresponding to the two 2^{k-1}-dimensional representations of $\mathbf{E}_{2k-1}^{-1}$ then we find that the characters ψ'_1 and ψ'_2 *always* have real values on the central elements $\pm z' = \pm x'_1 x'_3 \cdots x'_{2k-1}$ as they should — since according to Lemma 3.2 the center of $\mathbf{E}_{2k-1}^1$ is always isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4. Applications

In this section we describe the abstract structure of the groups G_n and H_n and decompose the representation $\pi_n : \mathcal{B}_n \to (\mathbb{C}^2)^{\otimes n}$ into its irreducible constituents. We then extend these ideas to the re-normalized representation of $\mathcal{B}_n$ that factors over the Temperley–Lieb algebra.

4.1. *H_n and G_n as abstract groups*

Theorem 4.1. $H_n \cong \mathbf{E}_{n-1}^{-1}$.

Proof. To verify that the map $\phi : \mathbf{E}_{n-1}^{-1} \to H_n$ defined by $x_i \to g_i$ extends to a (surjective) group homomorphism one just checks that the g_i satisfy the defining relations of $\mathbf{E}_{n-1}^{-1}$. Since $\mathrm{Ker}(\phi)$ is normal it must be trivial or intersect $Z(\mathbf{E}_m^{-1})$ by Lemma 3.2(d). We check that $\phi(-1) = \phi(x_1^2) = g_1^2 = -I_{2^n}$ so $-1 \notin \mathrm{Ker}(\phi)$ and we have proved the theorem for $n-1$ even. If $n-1 = 2k-1$ is odd, we must also check that $\pm z \notin \mathrm{Ker}(\phi)$ where z is the nontrivial central element defined in Lemma 3.2. For this we must use Lemma 2.2(k) which shows that there is a change of basis which diagonalizes the odd-indexed g_{2i+1} while fixing the even indexed g_{2i}. We compute the image of z in this basis:

$$(P_n)^{-1}\phi(\pm z)P_n = (P_n)^{-1}(\pm g_1 g_3 \cdots g_{2k-1})P_n = \pm(\sqrt{-1})^k(\sigma_z^{\otimes 2k}),$$

which is a diagonal matrix of trace 0, so not the identity. □

Combining with Theorem 2.6, we have:

Theorem 4.2. *The image of $\mathcal{B}_n$ under the representation π_n is an extension of $\mathbf{E}_{n-1}^{-1}$ by S_n.*

4.2. *Decomposition of π_n*

By Theorem 4.1, we have $\mathbf{E}_{n-1}^{-1} \cong H_n$ as an abstract group so the (defining) representation $(\pi_n, (\mathbb{C}^2)^{\otimes n})$ of H_n induces a representation $\phi_n := \pi_n \circ \phi$ of $\mathbf{E}_{n-1}^{-1}$.

4.2.1. *n odd*

Assume that $n = 2k+1$ is odd. Then we may decompose $(\mathbb{C}^2)^{\otimes 2k+1} \cong \bigoplus_i m_i V_i$ as representations of $\mathbf{E}_{2k}^{-1}$ for some multiplicities m_i. Let χ be the character of ϕ_{2k+1}. Since

$$\phi_{2k+1}(-1) = (I_2 \otimes \cdots \otimes g_i^2 \cdots \otimes I_2) = -I_{2^n},$$

we see that $\chi(-1) = -2^{2k+1}$ and $\chi(1) = 2^{2k+1}$. By Proposition 3.6 we can compute the multiplicities m_i of the irreducible components V_i:

$$m_i = \frac{1}{2^{2k+1}} \sum_{x \in \mathbf{E}_{2k}^{-1}} \chi_i(x)\overline{\chi(x)}.$$

The character χ_1 of the 2^k-dimensional representation V_1 vanishes on the non-central elements of $\mathbf{E}_{2k}^{-1}$ so we compute the multiplicity

$$m_1 = \frac{2^k \cdot 2^{2k+1} + 2^k \cdot 2^{2k+1}}{2^{2k+1}} = 2^{k+1},$$

so V_1 appears 2^{k+1} times. But $\dim V_1 = 2^k$ so $\dim(2^{k+1}V_1) = 2^{2k+1} = \dim(\mathbb{C}^2)^{\otimes 2k+1}$, so in fact π_{2k+1} decomposes diagonally as 2^{k+1} copies of the unique 2^k-dimensional representation (ρ_1, V_1) of $\mathbf{E}_{2k}^{-1}$.

4.2.2. *n even*

Suppose $n = 2k$ is even. We have already established (see (6) in Sec. 3.2) that the restriction of the irreducible 2^k-dimensional representation V_1 of $\mathbf{E}_{2k}^{-1}$ to $\mathbf{E}_{2k-1}^{-1}$ decomposes as the direct sum $W_1 \oplus W_2$ of the two inequivalent irreducible 2^{k-1} dimensional representations W_1 and W_1. So the 2^{2k}-dimensional representation ϕ_{2k} decomposes diagonally as the direct sum of 2^k copies of each of (λ_1, W_1) and (λ_2, W_2). One could also use the characters ψ_i to determine these multiplicities.

Remark 4.3. As $\pi_n(\mathcal{P}_n) = \phi_n(\mathbf{E}_{n-1}^{-1})$, all of the arguments above hold *mutatis mutandis* for decomposing π_n restricted to $\mathcal{P}_n$.

4.2.3. *Extension to* $\mathcal{B}_n$

With the explicit formulas for the representations ρ_1, λ_1 and λ_2 in hand, we easily compute the extensions $\hat{\rho}_1$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ to $\mathcal{B}_n$ using Lemma 2.2(a). Using the matrices d, M and D from Sec. 2.1, we give the explicit matrices for the 2^k-dimensional irreducible representation $\hat{\rho}_1$ with $n = 2k+1$ noting that the $\hat{\lambda}_1 \oplus \hat{\lambda}_2$ is just the restriction of $\hat{\rho}_1$.

$$\begin{aligned}
\hat{\rho}_1(\sigma_1) &= d \otimes I_2^{\otimes k-1}, \\
&\vdots \\
\hat{\rho}_1(\sigma_{2i}) &= I_2^{\otimes i-1} \otimes M \otimes I_2^{\otimes k-i}, \\
\hat{\rho}_1(\sigma_{2i+1}) &= I_2^{\otimes i-1} \otimes D \otimes I_2^{\otimes k-i-1}, \\
&\vdots \\
\hat{\rho}_1(\sigma_{2k}) &= I_2^{\otimes k-1} \otimes M.
\end{aligned}$$

The decomposition of π_n remains the same, so summarizing we have:

Theorem 4.4. *The representation π_n of $\mathcal{B}_n$ decomposes as*

$$(\mathbb{C}^2)^{\otimes n} \cong \begin{cases} (\mathbb{C}^2)^{\otimes (n+1)/2} \otimes V_1, & n \text{ odd}, \\ (\mathbb{C}^2)^{\otimes n/2} \otimes (W_1 \oplus W_2), & n \text{ even}. \end{cases}$$

5. Jones Representation and Jones Polynomial

The Jones representation of the braid groups $\mathcal{B}_n$ are defined using the Temperley–Lieb algebras $TL_n(q)$. Jones representation ρ_r in the following means the unitary representation of the braid groups at $q = e^{2\pi i/r}$ factoring through the semisimple Temperley–Lieb algebras, which are quotients of the Hecke algebras in [10]. The specific formulas that we use are the ones in [3].

Definition 5.1. Let $q = \sqrt{-1}$. The Temperley–Lieb algebra $TL_n(q)$ is defined as the (semisimple) quotient of the braid group algebra $\mathbb{C}[\mathcal{B}_n]$ by (the ideal generated by) the relations:

TL1: $(\sigma_i + 1)(\sigma_i - q) = 0$.
TL2: $\sigma_i\sigma_{i+1}\sigma_i + \sigma_i\sigma_{i+1} + \sigma_{i+1}\sigma_i + \sigma_i + \sigma_{i+1} + 1 = 0$.
TL3: $(\sigma_i - \sigma_{i+1})^2 = \sqrt{-1}$ (i.e. Jones–Wenzl projector $p_3 = 0$).

Observing that the Yang–Baxter operator R satisfies $(R - \zeta I_4)(R - \overline{\zeta} I_4) = 0$ we can define a new matrix $R' = -\overline{\zeta}R$ that satisfies $(R' - \sqrt{-1}I_4)(R' + I_4) = 0$. Since the equation (YBE) is homogeneous, (YBE) is satisfied by R' also. It is a (mildly tedious) computation to verify that the matrices $A_1 = (R' \otimes I_2)$ and $A_2 = (I_2 \otimes R')$ satisfy $A_1A_2A_1 + A_1A_2 + A_2A_1 + A_1 + A_2 + I_4 = 0$, and $(A_1 - A_2)^2 = \sqrt{-1}I_4$. Thus the representation π'_n of $\mathcal{B}_n$ afforded us by R' (or $\mathbb{C}\mathcal{B}_n$ if we prefer) factors over the Temperley–Lieb algebra $TL_n(\sqrt{-1})$. We can easily extend what we have learned about the representation π_n of $\mathcal{B}_n$ to this slight variation by observing the effect of renormalizing R. We record the result in the following (compare to [10]):

Corollary 5.2. *Denote by $H'_n = \pi'_n(\mathcal{P}_n)$ and $G'_n = \pi'_n(\mathcal{B}_n)$. Then we have $H'_n \cong \mathbf{E}^1_{n-1}$, and $G'_n/H'_n \cong S_n$.*

Proof. This follows easily from the observation that renormalizing R by $-\overline{\zeta}$ has the effect of multiplying the generators g_i of H_n by $-\sqrt{-1}$. Doing the same to the generators of the group $\mathbf{E}^{-1}_{n-1}$ just gives us a presentation of the group $\mathbf{E}^1_{n-1}$, and the same arguments as in the original representation π_n go through verbatim. □

To relate π'_n to the Jones representation ρ_4 of $\mathcal{B}_n$, we recall some facts about the Jones representation. The Temperley–Lieb algebras at a 4th root of unity are complex Clifford algebras and are isomorphic to the matrix algebra of $2^{n-1} \times 2^{n-1}$ matrices if n is odd, and the direct sum of two matrix algebras of $2^{\frac{n}{2}-1} \times 2^{\frac{n}{2}-1}$ matrices if n is even [11]. (Note here n is the number of strands in the geometric realization of $\mathcal{B}_n$, and differs by 1 from Jones' notation in [10].) So the Jones representation ρ_4 consists of a single irreducible sector if n is odd, and the direct sum of two irreducible sectors if n is even. Comparing with the comments in Sec. 3.2.3 we can also determine the decomposition of the representation π'_n as before. It follows that the restriction of the Jones representation ρ_4 to $\mathcal{P}_n$ for n even is the odd representation V_1 of the extra-special 2-group $\mathbf{E}^1_{n-1}$, and for n odd, $W_1 \oplus W_2$ as in Theorem 4.4. The images $\rho_4(\mathcal{B}_n)$ fit into the following exact sequence:

$$1 \to \mathbf{E}^1_{n-1} \to \rho_4(\mathcal{B}_n) \to S_n \to 1.$$

Projectively, we have

$$1 \to \mathbb{Z}_2^{n-1} \to \rho_4(\mathcal{B}_n) \to S_n \to 1.$$

The symmetric group S_n acts on the coordinates of $\mathbb{Z}_2^n$, hence $\mathbb{Z}_2^{n-1}$ when n is even. This action splits the exact sequence. But when n is odd, this sequence does not split as is shown in [10].

The Jones polynomial of a link at $\sqrt{-1}$ is given by the following formula [3]:

$$J_4(\hat{\sigma}) = (-1)^{n-1+\frac{e(\sigma)}{4}} \cdot (\sqrt{2})^{-\frac{1+(-1)^n}{2}} \cdot \mathrm{Trace}(\rho_4(\sigma)),$$

where $e(\sigma)$ is the sum of all exponents of the standard braid generators appearing in σ, and $\hat{\sigma}$ is the closure of σ. We can also define link invariants using the flipped R-matrix R. The conditions for enhancement (μ_i, α, β) is given in [14, Theorem 2.3.1.]. Working through the conditions, we found two link invariants: $T_R(\hat{\sigma}, \alpha) = \alpha^{n-e(\sigma)} \cdot (\sqrt{2})^{-n} \cdot \mathrm{Trace}(\pi_n(\sigma))$, where $\alpha = \pm 1$. Comparing with the Jones polynomial we get the relation:

$$T_R(\hat{\sigma}, \alpha) = (-1)^{n-1+e(\sigma)} \cdot \alpha^{n-e(\sigma)} \cdot \sqrt{2} \cdot J_4(\hat{\sigma}).$$

As we know that Jones polynomial $J_4(\hat{\sigma})$ is $(\sqrt{2})^{c(\hat{\sigma})-1} \cdot (-1)^{\mathrm{Arf}(\hat{\sigma})}$ if $\mathrm{Arf}(\hat{\sigma})$ is defined and 0 otherwise, where $c(\hat{\sigma})$ is the number of components of the link $\hat{\sigma}$ [11]. It follows that $T_R(\hat{\sigma}, \alpha)$ computes essentially the Arf invariant of a link.

Acknowledgments

The first author is supported in part by NSF grant EIA 0130388. The second author is supported in part by an NSF VIGRE grant. The third author is partially supported by NSF grants EIA 0130388 and FRG 0354772.

References

[1] J. Birman, *Braids, Links, and Mapping Class Groups*, Annals of Mathematics Studies, No. 82 (Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974).

[2] H. Dye, Unitary solutions to the Yang-Baxter equation in dimension four, *Quant. Infor. Proc.* **2**(1–2) (2002) 117–150.

[3] M. Freedman, M. Larsen and Z. Wang, The two-eigenvalue problem and density of Jones representation of braid groups, *Commun. Math. Phys.* **228** (2002) 177–199.

[4] M. Freedman, A. Kitaev, M. Larsen and Z. Wang, Topological quantum computation, *Bull. Amer. Math. Soc. (N.S.)* **40**(1) (2003) 31–38.

[5] W. Fulton and J. Harris, *Representation Theory, A First Course*, Graduate Texts in Mathematics, Vol. 129 (Springer-Verlag, New York, 1991).

[6] B. van Geemen and A. J. de Jong, On Hitchin's connection, *J. Amer. Math. Soc.* **11**(1) (1998) 189–228.
[7] D. Goldschmidt and V. F. R. Jones, Metaplectic link invariants, *Geom. Dedicata* **31**(2) (1989) 165–191.
[8] R. Griess, Automorphisms of extra special groups and nonvanishing degree two cohomology, *Pacific J. Math.* **48**(2) (1973) 403–422.
[9] J. Hietarinta, All solutions to the constant quantum Yang-Baxter equation in two dimensions, *Phys. Lett. A* **165** (1992) 2452–52.
[10] V. F. R. Jones, Braid groups, Hecke algebras and type II_1 factors, in *Geometric Methods in Operator Algebras (Kyoto, 1983)*, Pitman Research Notes in Mathematics Series, Vol. 123 (Longman Scientific and Technical, Harlow, 1986), pp. 242–273.
[11] ————, Hecke algebra representations of braid groups and link polynomials, *Ann. Math.* **126** (1987) 335–388.
[12] L. Kauffmann and S. Lomonaco Jr., Braiding operators are universal quantum gates, *New J. Phys.* **6** (2004) 134.1–134.40 (electronic).
[13] N. Read, Non-abelian braid statisitcs versus projective permutation statistics, *J. Math. Phys.* **44**(2) (2003) 558–563.
[14] V. Turaev, The Yang-Baxter equation and invaraints of links, *Invent. Math.* **92** (1988) 527–553.
[15] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scentific, 1990).