Curve diagrams, two dimensional quantum processes and quantum computing

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Atiyah's Axioms of (2+1)-TQFT

(TQFT w/o excitation and anomaly)

A functor (V, Z): cat of surfaces → Vect

Oriented closed surface Y → vector space V(Y)
Oriented 3-mfd X with $\partial X=Y$ → vector $Z(X)\in V(\partial X)$

- $V(\emptyset) \cong \mathbb{C}$
- $V(Y_1 \sqcup Y_2) \cong V(Y_1) \otimes V(Y_2)$
- $V(-Y) \cong V^*(Y)$
- $Z(Y \times I)=\mathrm{Id}_{V(Y)}$
- $Z(X_1 \cup_Y X_2)=Z(X_1)\cdot Z(X_2)$

Chern-Simons (CS) TQFTs

Using path integral (Witten) or quantum groups (Reshetikhin-Turaev), for each level k

$$Z_k(X^3)=\int_{\mathcal{A}} e^{2\pi i k\, cs(A)} DA$$

What is $V_k(Y)$?

a typical vector looks like a 3-mfd X s.t. $\partial X=Y$

(M. Atiyah, G. Segal, V. Turaev, K. Walker,…)

Reshetikhin-Turaev (R-T) or physically Witten-CS TQFTs do not satisfy the last axiom due to framing anomaly.
Gluing of 3-manifolds along boundaries does NOT correspond to the composition of linear maps, only up to a scalar.
Hence reps of mapping class groups are in general NOT linear, only projective.

But the Turaev-Viro (T-V) type TQFTs e.g. the diagram TQFTs are such examples.

Quantum Chern-Simons Theory

Topology		Algebra
4-dim	W^4	signature $\in$ Z
3-dim	X^3	invariant $\in$ ₵
2-dim	Y^2	vector space $\in$ Category
1-dim	S^1	category $\in$ 2-category
0-dim	pt	3-category or Lurie ?

Example of Diagrams TQFTs

Z_2-homology TQFT

Closed surface Y, $V(Y)=C[H_1(Y;Z_2)]$

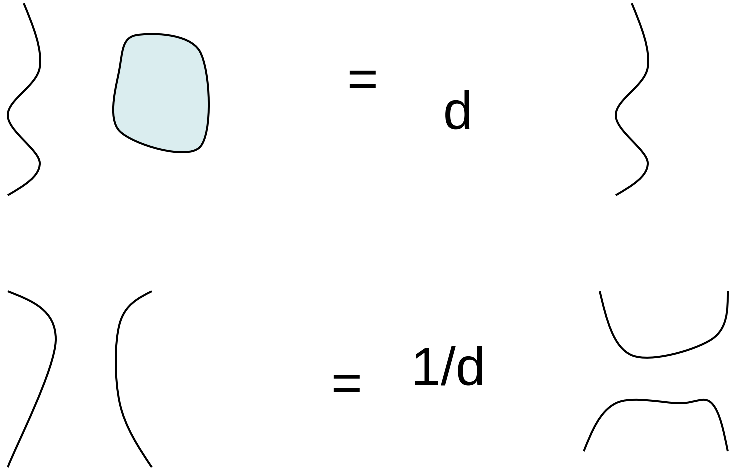
Closed 3-manifold X, $V(X)=2^{b\_1(X)-1}$

Z_2-homology TQFT

Closed surface Y, V(Y)=S(Y)/~

S(Y)=linear span of simple closed curves

=formal multicurves



isotopy and d=1

Formal=finite linear combination

Multicurve=simple closed curve, i.e., a collection of disjoint simple closed loops

(2+1)-Picture TQFTs

Given an oriented closed surface Y and d$\in$ C\{0},

S(Y)=vector space generated by pictures in Y, e.g. multicurves, oriented multicurves, trivalent graphs, oriented trivalent graphs with colors

Let V(Y) be S(Y) modulo

1. d-isotopy (isotopy+trivial loop=d)

2. a local relation (a relative formal picture supported on the disk)

Some Local Relations

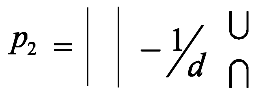
Fix 2n points on the boundary of the disk, and D_i=n disjoint arcs connecting the 2n points, i=1,2,…, Catalan number

A local relation is a formal equation

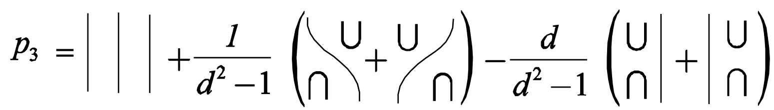
$$\sum_i \lambda_i \cdot D_i = 0.$$

Quotient of S(Y) by a local relation: any vector in S(Y), if restricted to some topological disk=the local relation, is set to 0.

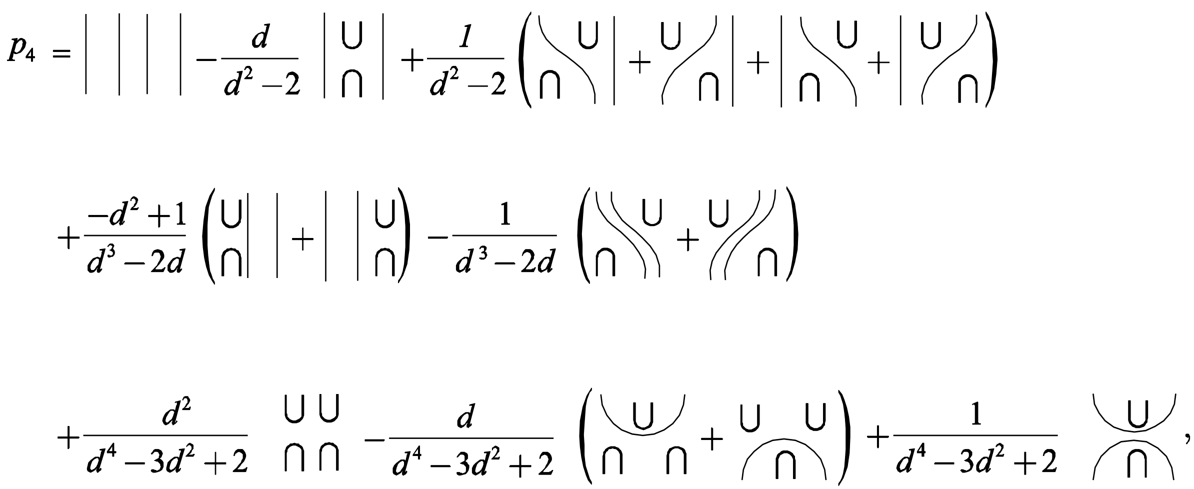
Jones-Wenzl projectors p_l



, p_2 generates a proper radical for $d = 1, -1$;



p_3 genrerates aproprer radical for $d = \pm\sqrt{2}$, and $d = 0$;



p_4 genrerates a proper radical for $d = \pm\left(\frac{1+\sqrt{5}}{2}\right), \pm\left(\frac{1-\sqrt{5}}{2}\right)$.

Diagram (2+1)-TQFTs

Fix a level $k=r-2 \geq 1$, $p_{k+1}=0$ as the local relation for a primitive $2r^{th}$ root of unity A, and $d=-A^2-A^{-2}$

$V^{diag}(Y)=S(Y)/\sim$ is the modular functor space.

Thm: 1. Diagram TQFTs are T-V type TQFTS defined intrinsically, i.e., without using triangulations.

2. Diagram TQFTs are quantum double or Drinfeld centers of the Jones-Kauffman TQFTs (Walker, Turaev)

3. Jones-Wenzl projectors are unique essentially.

Jones-Kauffman (J-K) TQFTs

- They are NOT the same as the R-T SU(2) TQFTs, though closely related.

  Perhaps should not be regarded as the math realization of Witten-CS SU(2)-TQFTs.

- They are not anomaly-free, so are not picture TQFTs. Surfaces and 3-manifolds need to be endowed with extra structures such as Lagrangian subspaces and 2-framings, respectively.

Skein Spaces

Let A be a primitive $4r^{th}$ root of unity, X be an oriented closed 3-manifold, S(X) be the vector space of formal framed links.

Set $K_A(X)=S(X)/\sim$, where ~

1. Regular d-isotopy of framed links, **$d=-A^2-A^{-2}$**
2. Kauffman bracket
3. Jones-Wenzl projectors $p_{r-1}=0$

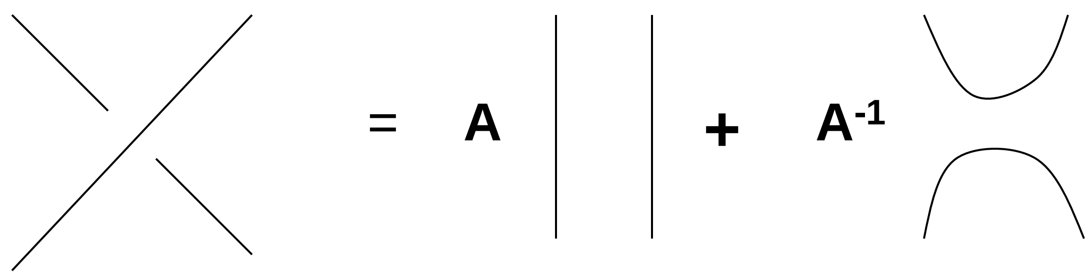
All three relations are used inside a 3-ball.

Kauffman Bracket

Theorem: $K_A(X) \cong C$, but not canonically.

Overstrand counterclockwise rotated to the understrand, smooth two sweptout regions---A, other two---A^{-1}, independent of orientation

Kauffman Bracket



J-K Modular functor $V_A(Y)$

Let Y be an oriented closed surface, and X be an oriented 3-manifold such that $\partial X=Y$.

Set $V_A(Y;X)=K_A(X)$

Thm: dim $V_A(Y;X)$ is independent of X, but no canonical identification of $V_A(Y;X)$'s.

Extended Surfaces

- An extension of an oriented closed surface Y is a choice of a Lagrangian subspace λ of $H_1(Y;R)$

- Given an extended surface $(Y;\lambda)$, choose an oriented 3-mfd X such that

$$\ker (H_1(Y;R) \rightarrow H_1(X;R)) = \lambda,$$

Thm: $V_A(Y;X)$ can be canonically identified.
Denoted it as $V_A(Y;\lambda)$

Projective Reps of MCGs

Oriented closed surface Y,

f: Y$\rightarrow$Y orientation preserving diffeo.,

its mapping cylinder M_f gives rise to

$$V_A(f): V_A(Y) \rightarrow V_A(Y)$$

since $\partial M_f = -Y \sqcup Y$, hence

$Z(M_f) \in V_A(-Y \sqcup Y)$

$$\cong V^*_A(Y) \otimes V_A(Y) \cong \mathrm{Hom}(V_A(Y), V_A(Y))$$

Jones-Kauffman and Diagram TQFTs

Thm: Let Y be an oriented closed surface, and $A^{4r}=1$ primitive root of unity, $d=-A^2-A^{-2}$

then:

1. $K_A(Y\times I)=V^{diag}(Y)$

2. $K_A(Y\times I)=\mathrm{End}(V_A(Y))$

Diagram TQFTs

- Given a spherical tensor category $\mathcal{C}$, there is a procedure to construct a T-V type TQFT using triangulations. But the known literature seems inadequate.
- The intrinsic diagram approach generalizes using colored trivalent graphs.
- Theorem: T-V type TQFTs are well-defined for unimodal ribbon fusion categories (Turaev)
- Theorem: Drinfeld center Z($\mathcal{C}$) or quantum double of a spherical category $\mathcal{C}$ is modular (M. Mueger)
- Conjecture: R-T TQFT from Drinfeld center Z($\mathcal{C}$) is the same as T-V from $\mathcal{C}$.

Asymptotic Faithfulness

Thm: Any infinite direct sum of Jones Kauffman TQFT representations faithfully represents the mapping class groups of oriented closed surfaces modulo center

i.e. for every non-central h in the MCG of an oriented closed surface, there is an integer $r_0(h)$ such that for any $r>r_0(h)$, and any primitive $4r^{th}$ root of unity A, the operator $V_A(h)$ is not the identity projectively. (Freedman, Walker, W.)

Ideas of Proof

- Y oriented closed surface, h: Y$\rightarrow$Y orientation preserving diffeo., and V_A(h): V_A(Y)$\rightarrow$$V_A$(Y) the J-K rep. Suppose there exists an unoriented ssc a in Y such that h(a) is not isotopic to a as a set, then V_A(h) is identity for at most finitely many r's.
- Let a, b be two non-trivial, non-isotopic sccs on an oriented closed surface Y. Then there exists a pants decomposition of Y such that a is a decomposing curve and b a non-trivial graph geodesic (ie no turn-backs wrt pants curves).

Topological Phase(=State) of Matter

- A quantum system whose low energy effective theory is described by a TQFT

- Some features:

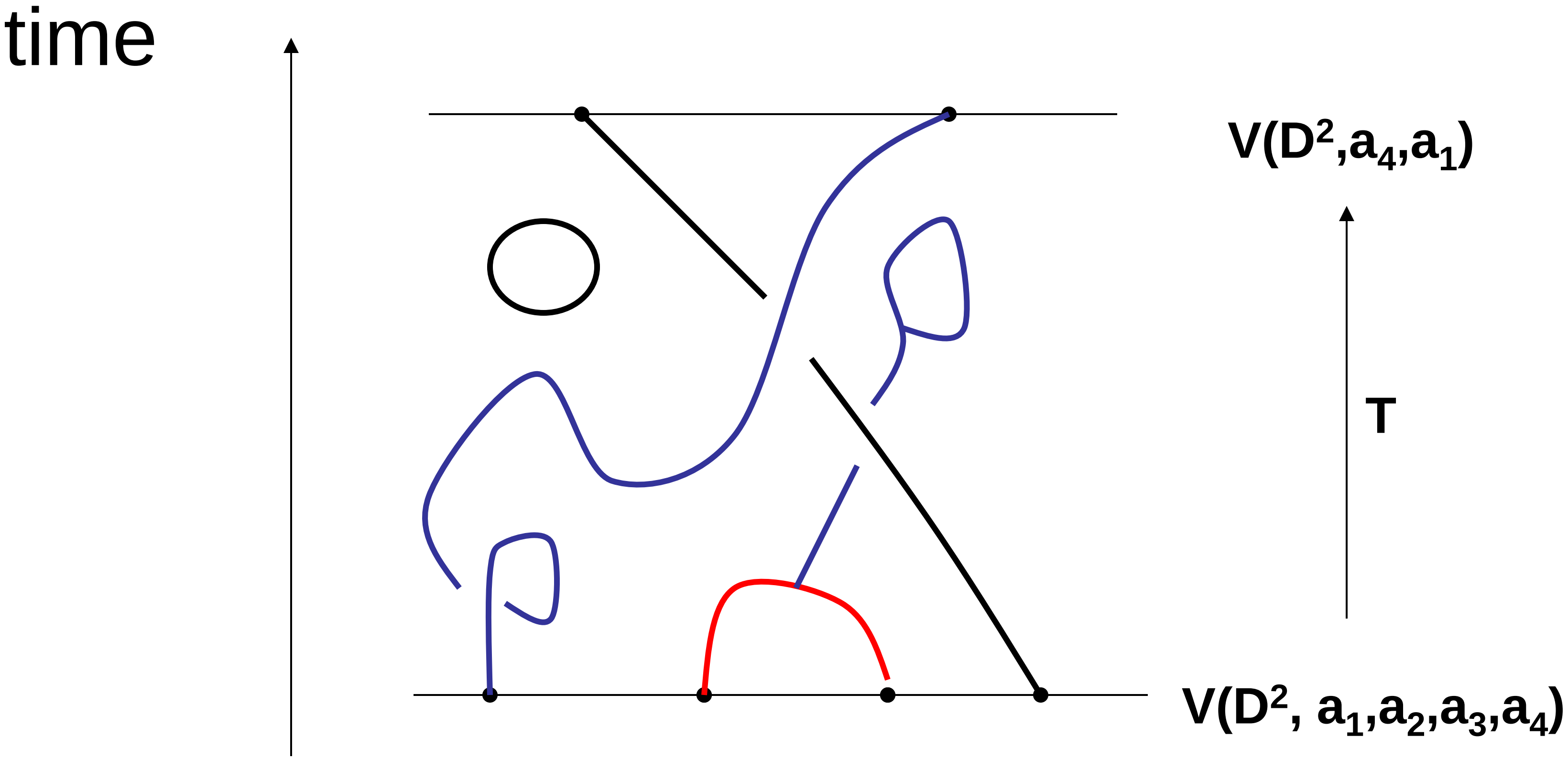
1) Ground states degeneracy
2) No continuous evolution
3) Energy gap

Anyons=Simple Objects of MTCs

Elementary excitations (called quasi-particles or particles) in a topological quantum system are anyons.

In general the vector space V(Y) describes the ground states of a quantum system on Y, and the rep of the mapping class groups describes the evolutions.

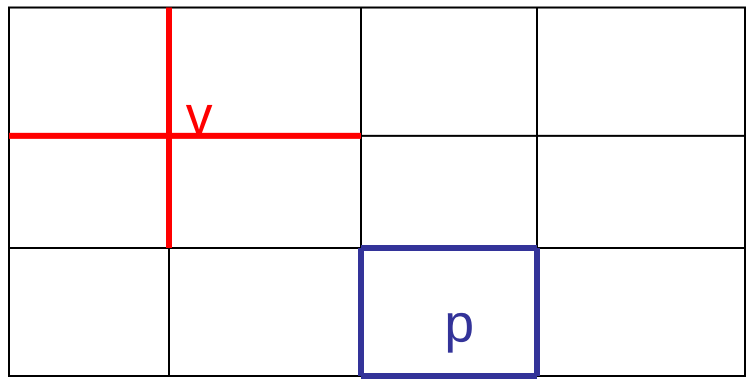
Invariant for Anyon Trajectories



Each line is labeled by an anyon. Topological invariant=amplitude of the quantum process.

Doubled Topological Phases

Kitaev’s toric code:



$=T^2$

$V=_e \mathbb{C}^2$, $A_v=_{e\in\, st(v)}\sigma^z{}_{others}Id_e$,

$B_p=_{e\in\,\partial\, p}\sigma^x{}_{others}Id_e$,

$H=\sum_v (I-A_v) +\sum_p(I-B_p)$

Toric code exactly solvable

- A_v, B_p all commute with each other
- Ground states are $\simeq \mathbb{C}^4$, ie 4-fold degenerate
- Gapped in the thermodynamic limit: $\lambda_1 - \lambda_0 \geq c > 0$
- Excitations are mutual anyons

Fault-tolerant

The embedding of the ground states

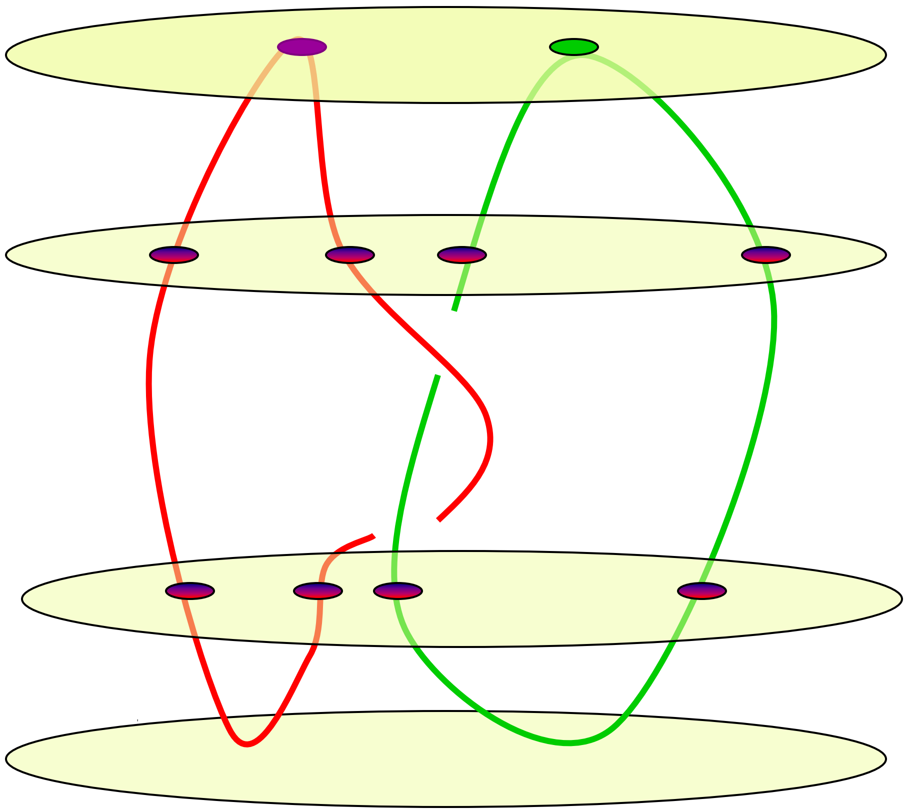
$\mathbb{C}^2 \oplus \mathbb{C}^2$ (2 qubits $\cong \mathbb{C}^4$) $\rightarrow$ $V =_e \mathbb{C}^2$

is an error correction code. Information encoded in the ground states is protected.

Conjecture: True for all picture TQFTs

Topological Computation

Computation	**Physics**
output	**measure=fusion**
apply operators	**braid anyons**
initialize	**create anyons**



(2+1)-TQFT in Nature=Topological State of Matter

Topological quantum computing

- Topological phases of matter “are” quantum computers, and form the foundation for building a scalable universal quantum computer.

- **“Electrons”$\rightarrow$ TQFTs or UMTCs**
  $\rightarrow$ Quantum computers

Physical Conjectures

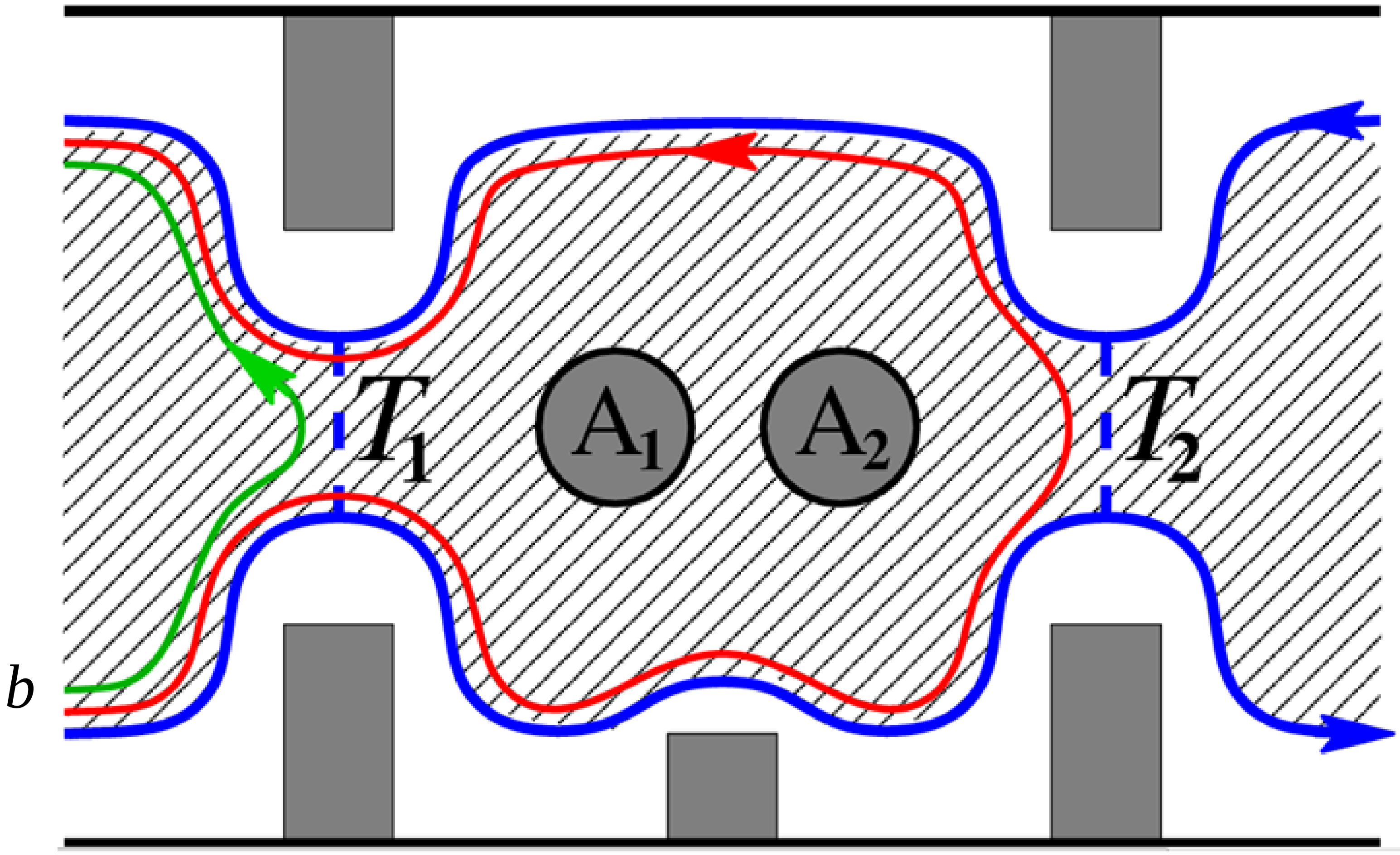
- Jones-Kauffman TQFTs are “realized” in fractional quantum Hall liquids.

  Experimental confirmation is making progress.

- Materials are designed to realize diagram TQFTs e.g. Kitaev’s toric code and Levin-Wen model (=Hamiltonian formulaion of Turaev-Viro type TQFTs.)

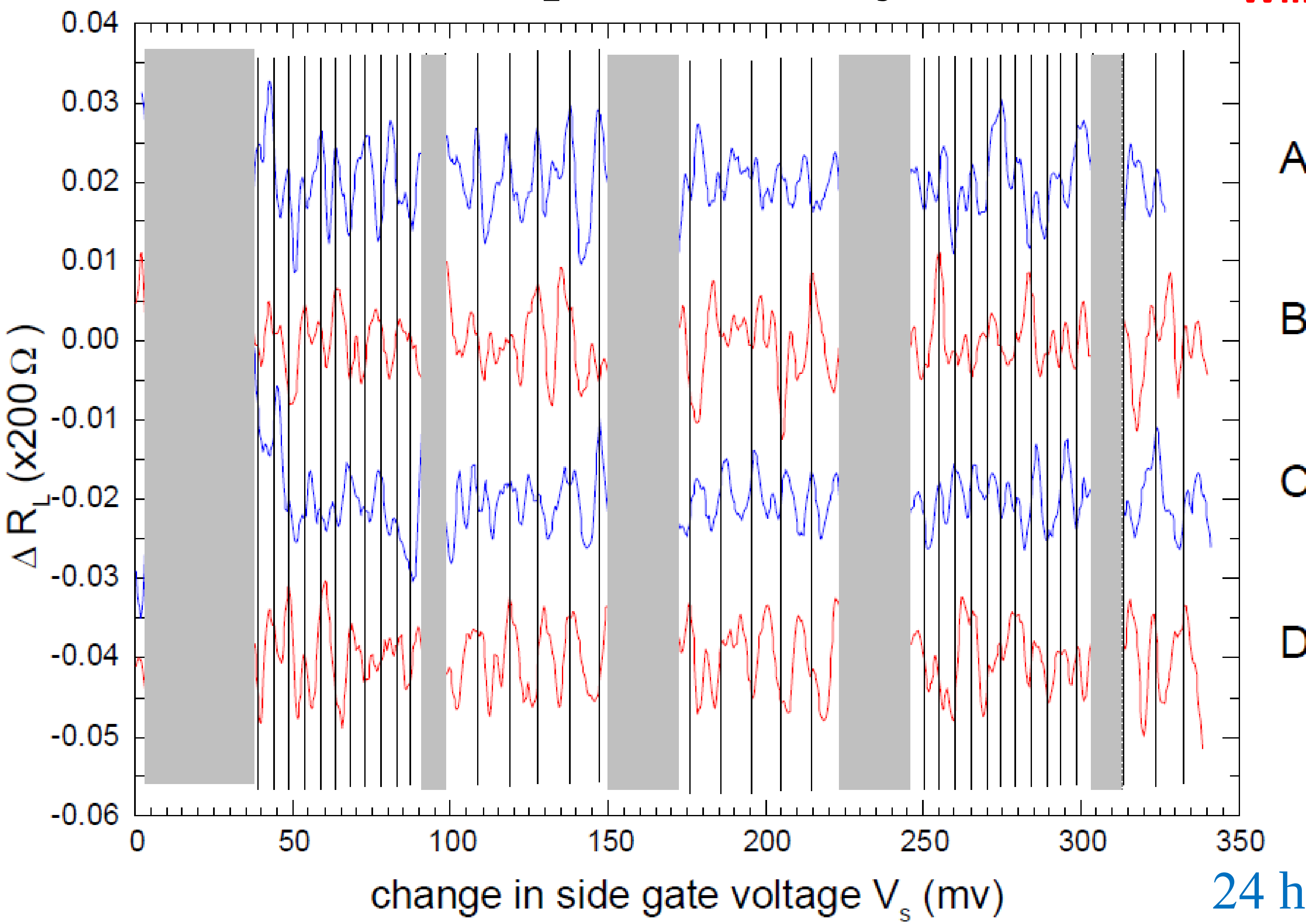
Topological Charge Measurement

e.g. FQH double point contact interferometer



Reproducibility

Bob Willett



24 hrs/run

$\tau_{error} \sim 1$ week!!

References

- **A Magnetic model with a possible CS phase (Freedman)**
- **On (2+1)-picture TQFTs (Freedman, Nayak, Walker, and W.)**
- **A class of P,T invariant topological phases of interacting electrons (Freedman, Nayak, Shentgel, Walker and W.)**
- **Quantum SU(2) faithfully detects MCGS modulo center (Freedman, Walker, W.)**
- **Topological quantum computation (W., CBMS book April, 2010?)**