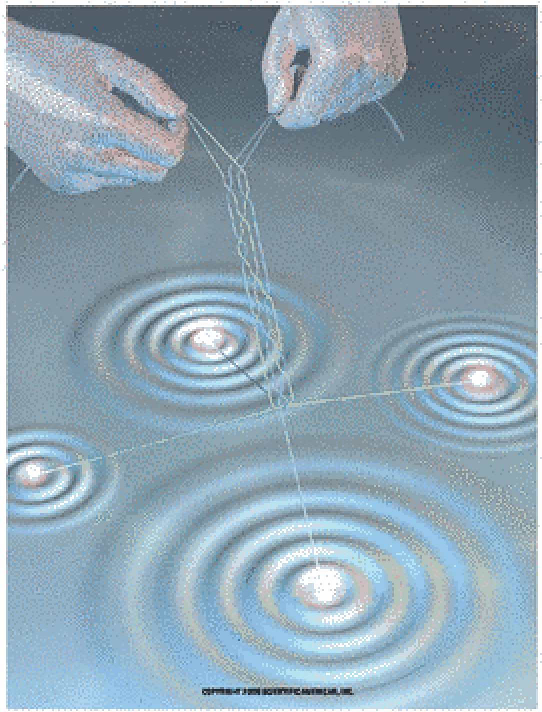
Topological Quantum Computation



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Quantum Information Science:

---Storage, processing and communicating information using quantum systems.

Four important results in QIS:

1. Shor's poly-time factoring algorithm (1994)

2. Error-correcting code, and fault-tolerant quantum computing (Shor, Stean, 1996)

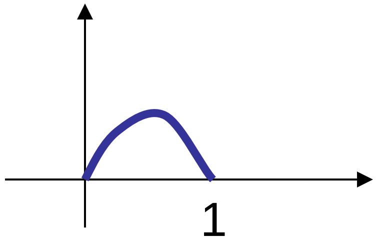
3. Security of private key exchange (BB84 protocol)

4. A Counterexample to Additivity of Minimum Output Entropy (Hastings, 2009)

- Classical information source is modeled by a random variable X

The bit---a random variable $X \in \{0,1\}$ with equal probability. Physically it is a switch

$I_X(p) = -\sum_{i=1}^{n} p_i \log_2 p_i$,



- A state of a quantum system is an information source

The qubit---a quantum system whose states given by non-zero vectors in C^2 up to non-zero scalars. Physically it is a 2-level quantum system.

Paradox: A qubit contains both more and less than 1 bit of information.

The average amount information of a qubit is $\frac{1}{2ln2}$.

A computing problem is given by a family of Boolean maps $\{0,1\}^n \longrightarrow \{0,1\}^{m(n)}$

Name: Factoring

Instance: an integer N>0

Question: Find the largest prime factor of N

Encode N as a bit string of length=n~ $\log_2 N$, the factoring problem is a family of Boolean functions $f_n: \{0,1\}^n \longrightarrow \{0,1\}^{m(n)}$:

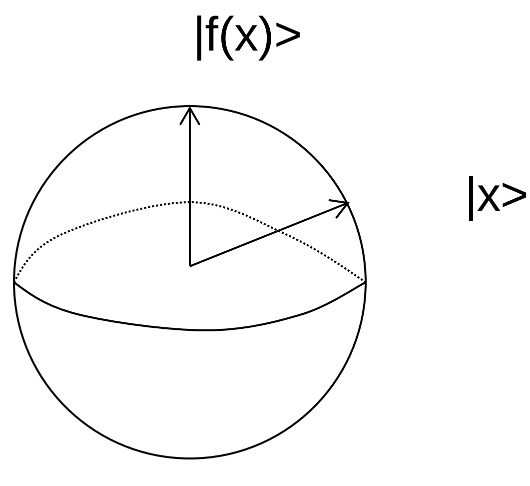
e.g. n=4, $f_4(1111)=101$

How a quantum computer works

Given a Boolean map f: $\{0,1\}^n \longrightarrow \{0,1\}^n$,

for any $x \in \{0,1\}^n$, represent x as a basis

$|x\rangle \in (C^2)^{\otimes n}$, then find a unitary matrix U so that $U(|x\rangle) = |f(x)\rangle$.



Basis of $(C^2)^{\otimes n}$ is in1-1correspondence with n-bit strings or $0,1,\ldots,2^n-1$

Problems:

- **x, f(x) does not have same # of bits**
- **f(x) is not reversible**
- **The final state is a linear combination**
- …
- **Not every U_x is physically possible**

Universal Gate Set

Fix a collection of unitary matrices (called gates) and use only compositions of local unitaries from gates, e.g. standard gate set

Hadamard

$$\sigma_z^{1/4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i/4} \end{pmatrix} \qquad H = 2^{-1/2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |11\rangle$

$|11\rangle \rightarrow |10\rangle$

$$C^2 \otimes C^2 \rightarrow C^2 \otimes C^2$$

The class BQP (bounded error quantum polynomial-time)

Fix a physical universal gate set

A computing problem $f_n: \{0,1\}^n \rightarrow \{0,1\}^{m(n)}$ is in BQP if

1) there exists a classical algorithm of time poly (n) (i.e. a Turing machine) that computes a function $x \rightarrow D_x$, where $x \in \{0,1\}^n$, and D_x encodes a poly(n)-qubit circuit U_x.

2) when the state $U_x|0\cdots 0\rangle$ is measured in the standard basis $\{|i_1\cdots i_{p(n)}\rangle\}$, the probability to observe the value $f_n(x)$ for any $x \in \{0,1\}^n$ is at least ¾.

Remarks:

1) Any function that can be computed by a QC can be computed by a TM.

2) Any function can be efficiently computed by a TM can be computed efficiently by a QC, i.e. $BPP \subseteq BQP$

Factoring is in BQP (Shor's algorithm), but not known in FP (although Primality is in P).

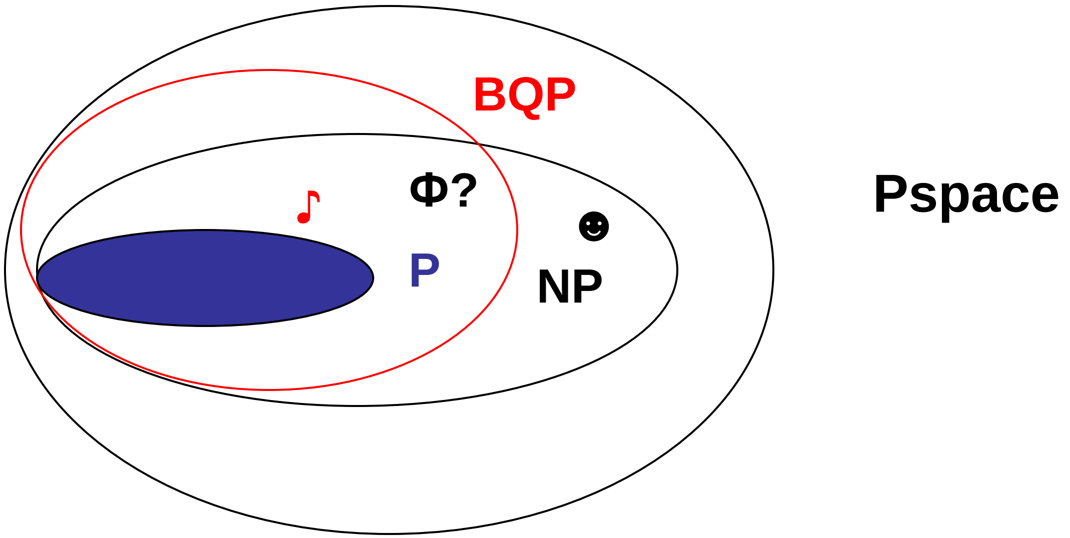
Given an n bit integer $N \sim 2^n$

Classically $\sim e^{c\, n^{1/3}\, \mathrm{poly}\,(\log n)}$

Quantum mechanically $\sim n^2$ poly (log n)

For $N=2^{500}$, classically ~ billion years

Quantum computer ~ a few days



Can we build a large scale universal QC?

The obstacle is mistakes and errors (decoherence)

Error correction by simple redundancy
0 → 000, 1 → 111
Not available due to the No-cloning theorem:

The cloning map $|\psi\rangle \otimes |0\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$ is not linear.

Fault-tolerant quantum computation shows if hardware can be built up to the accuracy threshold ~10^{-4}, then a scalable QC can be built.

Possible Solution---TOPOLOGY

History

- 1997

  M. Freedman, (2+1)-Topological quantum field theory (TQFT) computing model

  A. Kitaev, fault-tolerant QC by anyons

- 2000, Freedman, Kitaev, Larsen, Wang

  Two ideas lead to the same model, and equivalent to the standard QCM

- TQFTs found in real systems would be inherently fault-tolerant quantum computers

Topological Quantum Computing

- TQC is an implementation of fault-tolerant quantum computation **at hardware level** (vs traditional quantum computation **at software level**)

- Non-abelian **topological phases of matter (=topological quantum field theories in Nature)** are the hardware.

(2+1)-TQFTs in Nature

- **FQHE**

  1980 Integral Quantum Hall Effect (QHE)---von Klitzing

  (1985 Nobel)

  1982 Fractional QHE---Stormer, Tsui, Gossard at v=1/3

  (1998 Nobel for Stormer, Tsui and Laughlin)

  1987 Non-abelian FQHE**???**---R. Willet et al at v=5/2

  (All are more or less Witten-Chern-Simons TQFTs)

- **Topological superconductors p+ip (Ising TQFT)**
- Engineered topological materials (ISH)

Classical Hall effect

On a new action of the magnet on electric currents

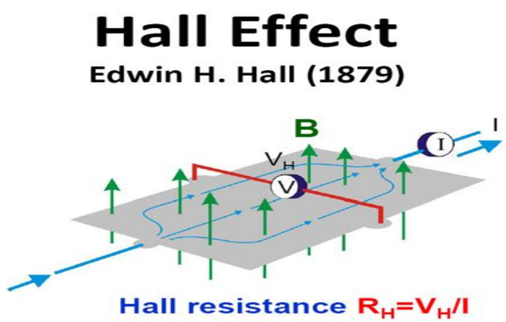
Am. J. Math. Vol. 2, No. 3, 287—292

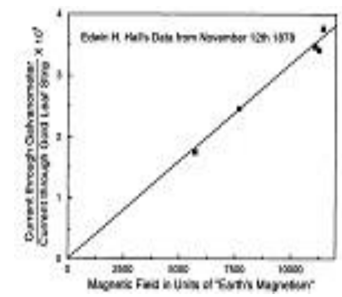
E. H. Hall, 1879

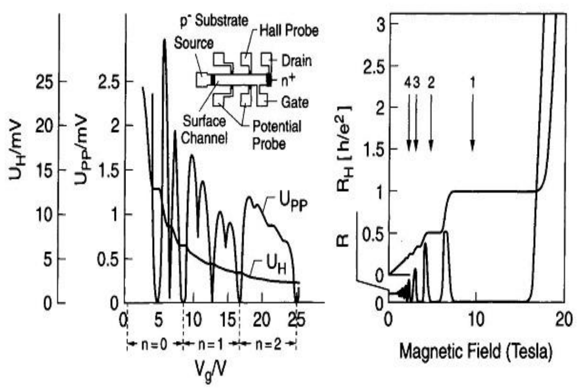
"It must be carefully remembered, that the mechanical force which urges a conductor carrying a current across the lines of magnetic force, acts, not on the electric current, but on the conductor which carries it…"

Maxwell, Electricity and Magnetism Vol. II, p.144

Birth of Integer Quantum Hall Effect



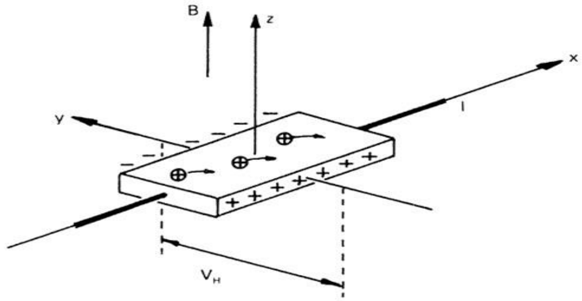


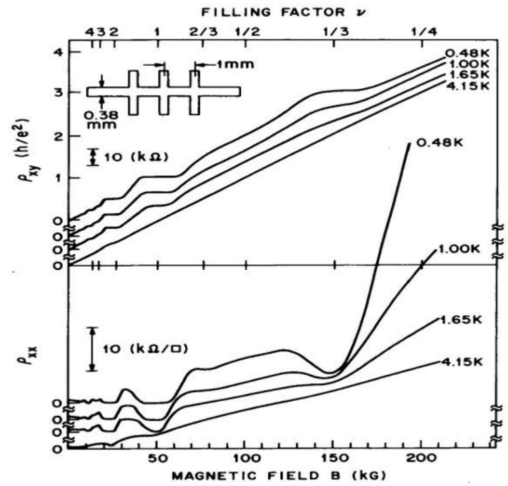


New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance,
K. v. Klitzing, G. Dorda and M. Pepper
Phys. Rev. Lett. 45, 494 (1980).

These experimental data, available to the public 3 years before the discovery of the quantum Hall effect, contain already all information of this new quantum effect so that everyone had the chance to make a discovery that led to the Nobel Prize in Physics 1985. The unexpected finding in the night of 4./5.2.1980 was the fact, that the plateau values in the Hall resistance x-y are not influenced by the amount of localized electrons and can be expressed with high precision by the equation $R_H = \frac{h}{\nu e^2}$

Fractional Quantum Hall Effect





D. Tsui enclosed the distance between B=0 and the position of the last IQHE between two fingers of one hand and measured the position of the new feature in this unit. He determined it to be three and exclaimed, "quarks!" H. Stormer

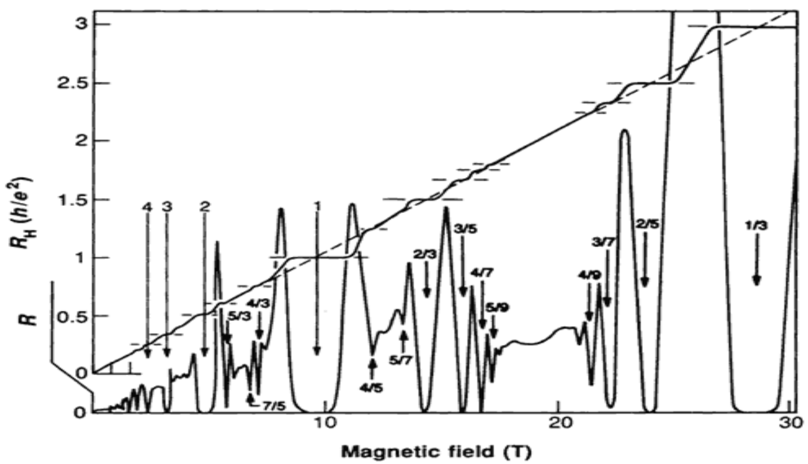
The FQHE is fascinating for a long list of reasons, but it is important, in my view, primarily for one: It established experimentally that both particles carrying an exact fraction of the electron charge e and powerful gauge forces between these particles, two central postulates of the standard model of elementary particles, can arise spontaneously as emergent phenomena. R. Laughlin

In 1998, Laughlin, Stormer, and Tsui are awarded the Nobel Prize

" for their discovery of a new form of quantum fluid with fractionally charged excitations."

D. C. Tsui, H. L. Stormer, and A. C. Gossard
Phys. Rev. Lett. 48, 1559 (1982)

FQHE States? ~80



$\nu = \frac{N_e}{N_\phi}$ filling factor or fraction

N_e = # of electrons

N_ϕ =# of flux quanta

How to model the quantum state(s) at a filling fraction?

What are the electrons doing at a plateau?

1/3	1/5	1/7	1/9	2/11	2/13	2/15	2/17	3/19	5/21	6/23	6/25
2/3	2/5	2/7	2/9	3/11	3/13	4/15	3/17	4/19	10/21		
4/3	3/5	3/7	4/9	4/11	4/13	7/15	4/17	5/19			
5/3	4/5	4/7	5/9	5/11	5/13	8/15	5/17	9/19			
7/3	6/5	5/7	7/9	6/11	6/13	11/15	6/17	10/19			
8/3	7/5	9/7	11/9	7/11	7/13	22/15	8/17				
	8/5	10/7	13/9	8/11	10/13	23/15	9/17				
	11/5	12/7	25/9	16/11	20/13						
	12/5	16/7		17/11							
		19/7									

m/5, m=14,16, 19

5/2

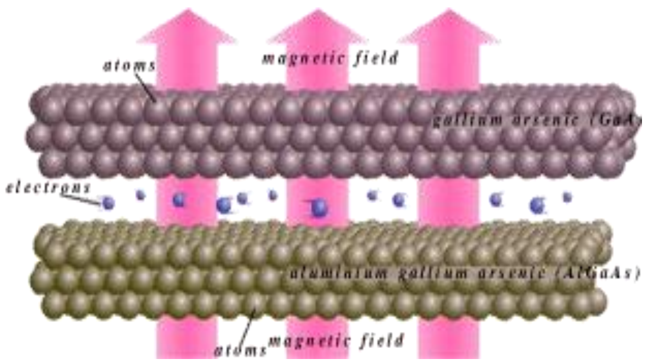
7/2

19/8

Pan et al (2008)

Fractional Quantum Hall Liquids

N electrons in a plane bound to the interface between two semiconductors immersed in a perpendicular magnetic field



Classes of ground state wave functions that have similar properties or no phase transitions as N→∞ (N ~ $10^{11}\ cm^{-2}$)

Interaction is dynamical entanglement and quantum order is materialized entanglement

Fundamental Hamiltonian:

$$H = \Sigma_1^{N} \{ \frac{1}{2m} [\nabla_j - q\, A(z_j)]^{2} + V_{bg}(z_j) \} + \Sigma_{j<k} V(z_j - z_k)$$

Ideal Hamiltonian:

$$H = \Sigma_1^{N} \{ \frac{1}{2m} [\nabla_j - q\, A(z_j)]^{2} \} + \textbf{?}, \text{ e.g. } \Sigma_{j<k}\, \delta(z_j - z_k)$$

z_j position of j-th electron

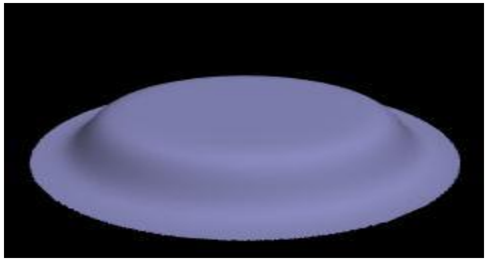
Laughlin wave function for ν=1/3

Laughlin 1983

Good trial wavefunction for N electrons at z_i in ground state

Gaussian

$$\Psi_{1/3} = \prod_{i<j}(z_i - z_j)^3 \; e^{-\sum_i |z_i|^2/4}$$



Physical Theorem:

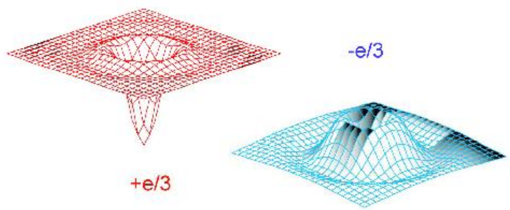
1. Laughlin state is incompressible: density and gap in limit (Laughlin 83)
2. Elementary excitations have charge e/3 (Laughlin 83)
3. Elementary excitations are abelian anyons (Arovas-Schrieffer-Wilczek 84)

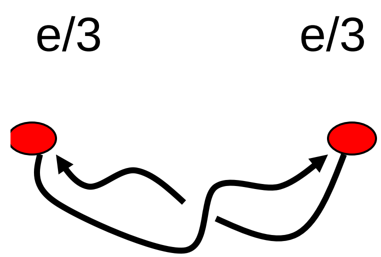
Experimental Confirmation:

1. and 2. √ , but 3. ?, thus Laughlin wave function is a good model

Quasi-particles=Anyons

Quasi-holes/particles in ν=1/3 are abelian anyons





$$\Psi_{1/3} = \prod_k (\eta_0 - z_j)^3 \prod_{i<j} (z_i - z_j)^3 \; e^{-\Sigma_i |z_i|^2/4}$$

$$= \prod_k (\eta_1 - z_j) \prod_k (\eta_2 - z_j) \; \prod_k (\eta_3 - z_j) \; \prod_{i<j} (z_i - z_j)^3 \; e^{-\Sigma_i |z_i|^2/4}$$

n anyons at well-separated η_i, i=1,2,.., n,
there is a **unique** ground state

$$\psi \rightarrow e^{\pi\, i/3}\, \psi$$

Moore-Read or Pfaffian State

G. Moore, N. Read 1991

Pfaffian wave function (MR w/ ≈ charge sector)

$$\Psi_{1/2} = \mathrm{Pf}(1/(z_i - z_j)) \prod_{i<j} (z_i - z_j)^2 \, e^{-\sum_i |z_i|^2/4}$$

Pfaffian of a 2n×2n anti-symmetric matrix M=(a_{ij}) is

ω^n =n! Pf (M) dx^1∧dx^2∧…∧dx^{2n} if $\omega = \sum_{i<j} a_{ij}\, dx^i \wedge dx^j$

Physical Theorem:

1. Pfaffian state is gapped
2. Elementary excitations are non-abelian anyons, called Ising anyon σ

…… Read 09

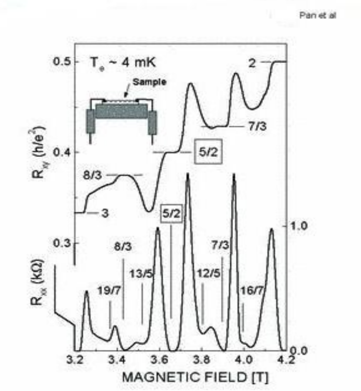
Enigma of ν=5/2 FQHE

R. Willett et al discovered ν=5/2 in1987

- Moore-Read State, Wen 1991
- Greiter-Wilczek-Wen 1991
- Nayak-Wilczek 1996
- Morf 1998
- …

MR (maybe some variation) is a good trial state for 5/2

- Bonderson, Gurarie, Nayak 2011,



Willett et al, PRL 59 1987

A landmark (physical) proof for the MR state

"Now we eagerly await the next great step: experimental confirmation."   ---Wilczek

Experimental confirmation of 5/2:

gap and charge e/4 √ , but non-abelian anyons ???

Topological Phases

Given a quantum theory H on a surface Y with Hilbert space $L_Y \cong \oplus V_i(Y)$,

where $V_i(Y)$ has energy λ_i, and $V_0(Y)$ is the groundstate manifold

Assume energy gap $(\lambda_1 - \lambda_0 \geq 0)$,

$Y \rightarrow V_0(Y)$ is well-defined

TQFT as Effective Theory

A theory H is topological if the functor surface Y → V(Y) (GS manifold) is a TQFT.

Rm: H is the Hamiltonian for all degrees of freedom. Restricted to the topological degrees of freedom, the effective Hamiltonian is constant (or 0).

Physical Thm: Topological properties of abelian bosonic FQH liquids are modeled by Witten-Chern-Simons theories with abelian gauge groups T^n.

Conjecture: NA statistics sectors of FQH liquids at $\nu=2+\frac{k}{k+2}$ are modeled by $SU(2)_k$-WCS theories. k=1,2,3,4, $\nu=\frac{7}{3}, \frac{5}{2}, \frac{13}{5}, \frac{8}{3}$. (Read-Rezayi). 5/2 √

Atiyah's Axioms of (2+1)-TQFT

(TQFT w/o excitations and anomaly)

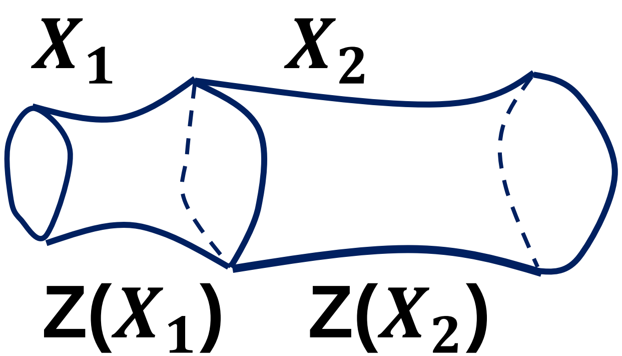
A functor (V,Z): category of surfaces → Vec

(Hilbert spaces for unitary TQFTs)

Oriented closed surface Y → vector space V(Y)

Oriented 3-mfd X with $\partial X=Y$ → vector $Z(X)\in V(\partial X)$

- $V(\emptyset) \cong \mathbf{C}$
- $V(Y_1 \cup Y_2) \cong V(Y_1)\otimes V(Y_2)$
- $V(-Y) \cong V^*(Y)$
- $Z(Y\times I)=\mathrm{Id}_{V(Y)}$
- $Z(X_1\cup_Y X_2)=Z(X_1) \bullet Z(X_2)$



Topological Phases of Matter

- **Gapped** quantum phases of matter at **T=0** with topological order (physics)
- Phases of matter whose low energy physics modeled by TQFTs (math)
- Gapped quantum phases whose groundstates are quantum error-correction codes (information science)

Phases of matter of many interacting constituents such as electrons, characterized by entanglement rather than by thermal energy (classical states of matter)

Anyons=Elementary Excitations

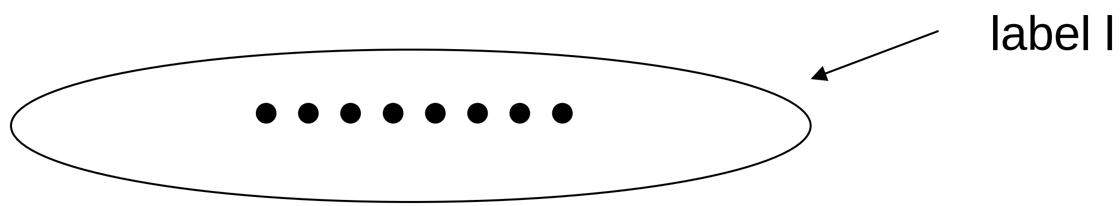
Elementary excitations in topological phases of matter are predicted to be **anyons** (with physical proof and experimental evidences)

Quasi-particles in 2D (space) whose statistics given by unitary matrices not +1 (bosons) or -1 (fermions)

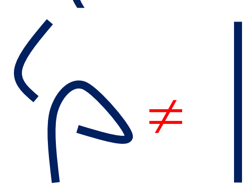
(Extended) TQFT Models Anyons

Put a theory H on a closed surface Y with anyons $a_1, a_2, \ldots, a_n$ at $\eta_1,\ldots,\eta_n$ (punctures), the (relative) ground states of the system "outside" $\eta_1,\ldots,\eta_n$ is a Hilbert space $V(Y; a_1, a_2, \ldots, a_n)$.

For anyons in a surface w/ boundaries (e.g. a disk), the boundaries need conditions.



Stable boundary conditions correspond to anyon types (labels, super-selection sectors, topological charges). Moreover, each puncture (anyon) needs a tangent direction, so anyon is modeled by a small arrow (combed point), not just a point.



Non-abelian Anyons

Given n anyons of type x in a disk D, their ground state degeneracy

dim(V(D,x,…,x))=$D_n \sim d^n$

The asymptotic growth rate d is called the quantum dimension.

An anyon d=1 is called an abelian anyon, e.g. Laughlin anyon, d=1

An anyon with d >1 is an non-abelian anyon, e.g. the Ising anyon σ, d=$\sqrt{2}$.

For n even, D_n=$\frac{1}{2}\, 2^{\frac{n}{2}}$ with fixed boundary conditions,

n odd, D_n=$2^{\frac{n-1}{2}}$. (Nayak-Wilczek 96)

Degeneracy for non-abelian anyons in a disk grows exponentially with # of anyons, while for an abelian anyon, no degeneracy---it is always 1.

Non-abelian Statistics

If the ground state is not unique, and has a basis $\psi_1, \psi_2, \ldots, \psi_k$

Then after braiding some particles:

$$\psi_1 \longrightarrow a_{11}\psi_1 + a_{12}\psi_2 + \ldots + a_{k1}\psi_k$$

$$\psi_2 \longrightarrow a_{12}\psi_1 + a_{22}\psi_2 + \ldots + a_{k2}\psi_k$$

....

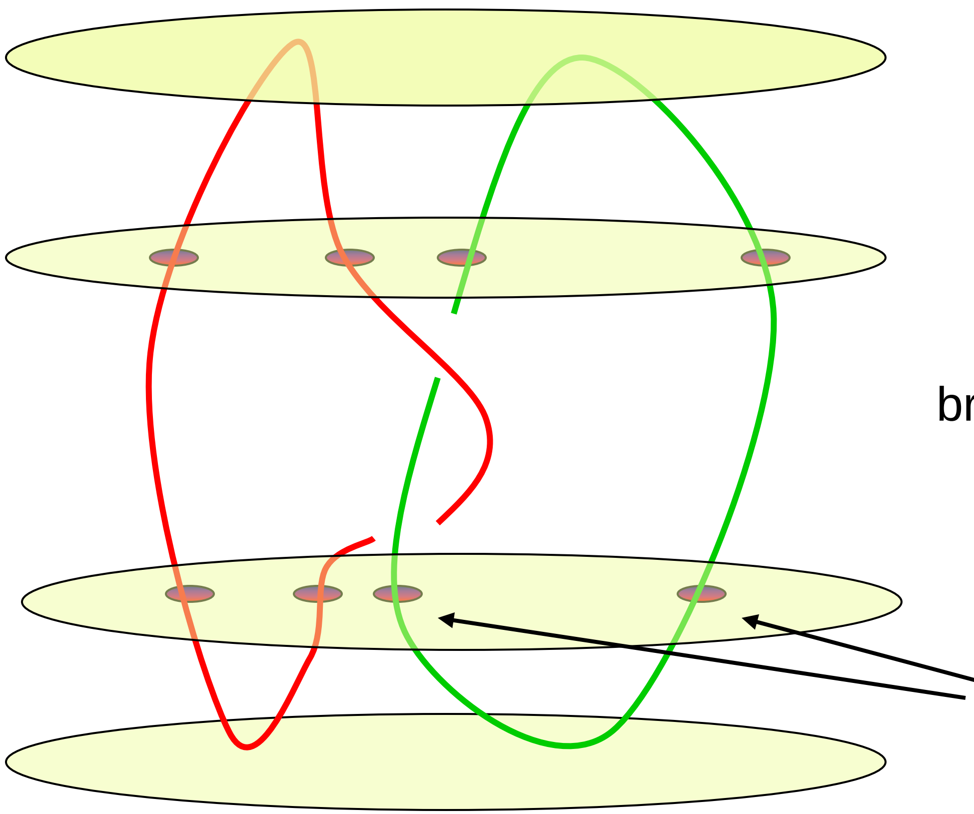
$$\lambda: B_n \longrightarrow U(k),$$

when k>1, non-abelian anyons.

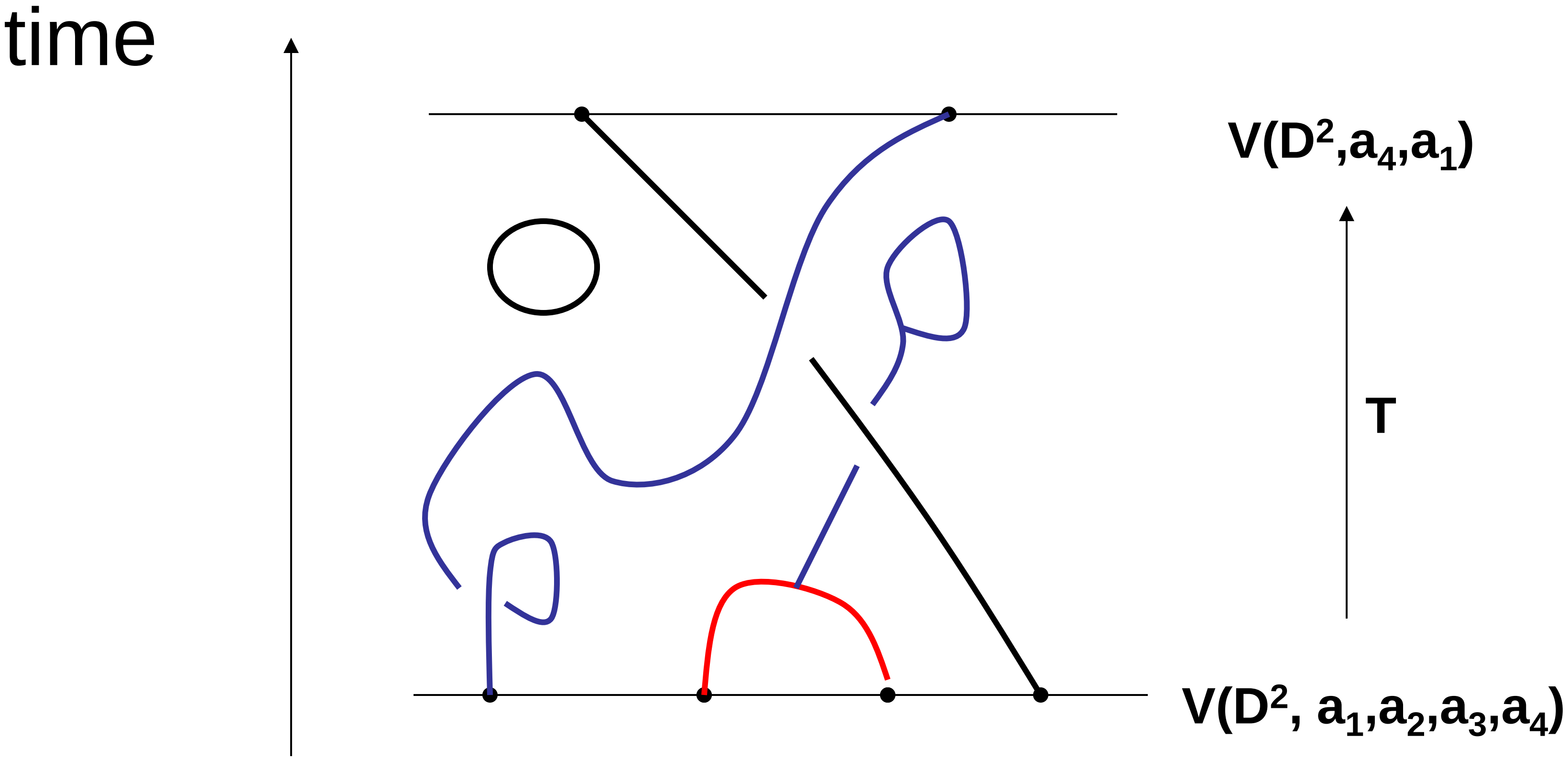
How Do We Compute: Circuits=Braids

Freedman 97, Kitaev 97, FKW 00, FLW 00

Computation		**Physics**
readout		fusion
applying gates		braiding particles
initialize		create anyons



What Do We Compute:
Approximation of Link Invariants

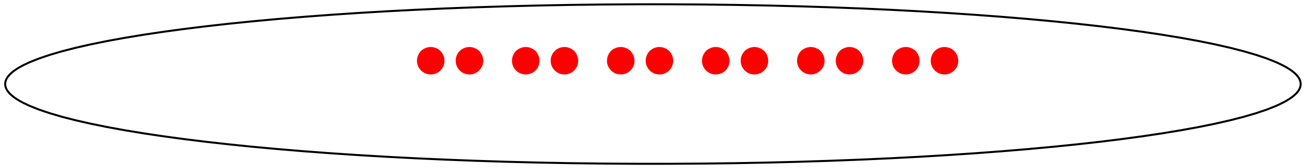


Each line is labeled by an anyon. Topological invariant=amplitude of the quantum process.

How To Implement Shor's Algorithm

For n qubits, consider the 4n Fibonacci anyons

$\rho: B_{4n} \longrightarrow U(F_{4n-2})$, F_{4n-2}---4n-2 Fib number



Given a quantum circuit on n qubits

$$U_L: (\mathbb{C}^2)^{\otimes n} \longrightarrow (\mathbb{C}^2)^{\otimes n}$$

Topological compiling: find a braid $b \in B_{4n}$ so that the following commutes for any U_L:

$$\begin{tikzcd} (\mathbb{C}^2)^{\otimes n} \arrow[r] \arrow[d, "U_L"'] & V_{4n} \arrow[d, "\rho(b)"] \\ (\mathbb{C}^2)^{\otimes n} \arrow[r] & V_{4n} \end{tikzcd}$$

V_{4n}-gs of 4n anyons

Mathematical Theorems

Theorem 1 (FKW): Any unitary TQFT can be efficiently simulated by the quantum circuit model.

There are efficient additive approximation algorithms of quantum invariants by the quantum circuit model.

Theorem 2 (FLW): Anyonic quantum computers based on SU(2)-Chern-Simons theory at level k are braiding universal except k=1,2,4.

The approximation of Jones poly of links at the $(k+2)^{th}$ root of unity ($k \neq 1,2,4$) is a BQ(F)P-complete problem.

Exact or FPRS approximation of Jones poly of links at the (k+2)th root of unity ($k \neq 1,2,4$) is #P-hard. (Vertigan 05, Kuperberg 09)

Math Questions

- **Classify TQFTs or modular categories**
- **Conjecture:** Fix the rank, there are only finitely many isomorphism classes of MCs

- **When an anyon leads to universal QC**
- **Conjecture**: only if the square of its quantum dimension is an integer

Physics Questions

- What is the **microscopic mechanism** of topological phases
- What are the **experimental signatures** of non-abelian statistics

Can we have a smoking-gun experiment for non-abelian anyons w/o building a small TQC? (which will be a large scale TQC!)

Information Questions

- What is the **architecture** of TQCs?

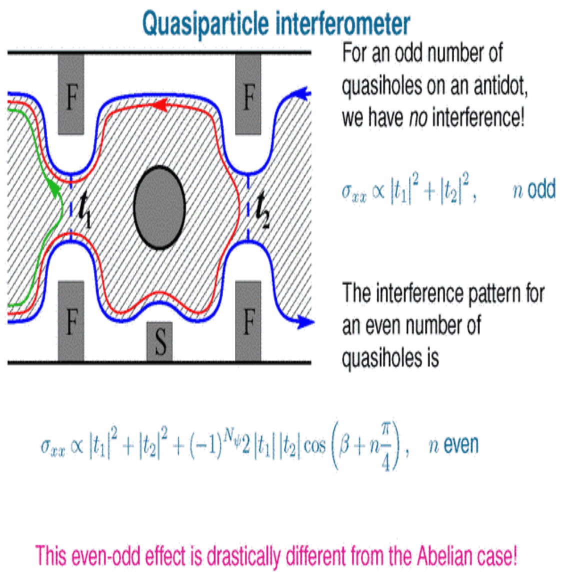
- How to **program** TQCs?

- Braiding gates form the machine language, are there **higher order languages**?

Future Directions

- Topological orders in (3+1)-dimension
- Topological orders for fermions or anyons
- Topological order with symmetries
- Topological order at finite temperature

Are we close to building a TQC?



Freedman, Nayak, Das Sarma, 2005

Halperin-Stern 06
Bonderson-Kitaev-Shtengel 06

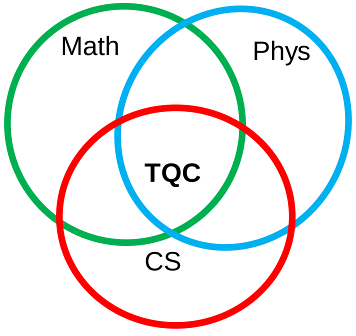
Willett reported data 09
Heiblum data on neutral mode
Spin polarization?

Hard:

Little correlation between anyons and local measurement

Extreme conditions

Are we stupid? We have to build a small topological quantum computer to confirm non-abelian anyons



Some References

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