Topological Phases of Matter: Modeling and Application to Quantum Computing

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Outline

- Modeling of fractional quantum Hall liquids
- Theory of topological phases of matter
- Topological quantum computation

Classical Hall effect

On a new action of the magnet on electric currents

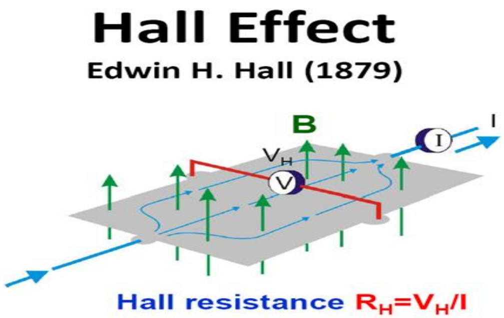
Am. J. Math. Vol. 2, No. 3, 287—292

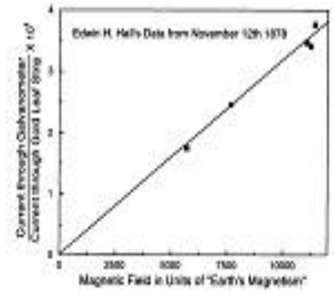
E. H. Hall, 1879

"It must be carefully remembered, that the mechanical force which urges a conductor carrying a current across the lines of magnetic force, acts, not on the electric current, but on the conductor which carries it…"

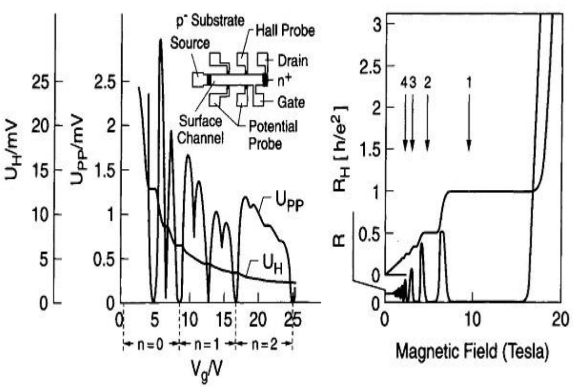
Maxwell, Electricity and Magnetism Vol. II, p.144

Birth of Integer Quantum Hall Effect





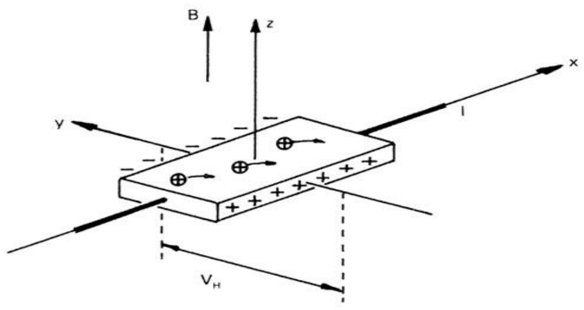
New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance,
K. v. Klitzing, G. Dorda and M. Pepper
Phys. Rev. Lett. 45, 494 (1980).

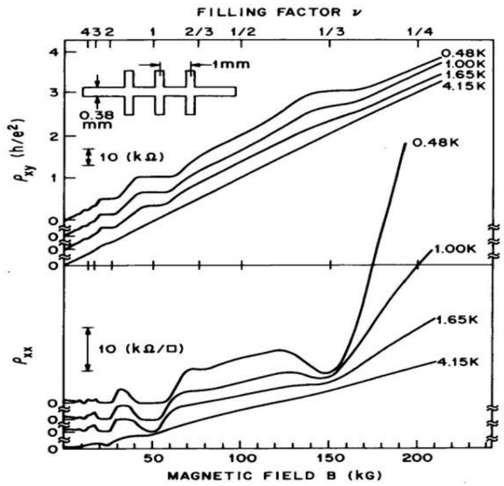


5.2.1980 BIRTHDAY OF QHE
(at 2 a.m.)

These experimental data, available to the public 3 years before the discovery of the quantum Hall effect, contain already all information of this new quantum effect so that everyone had the chance to make a discovery that led to the Nobel Prize in Physics 1985. The unexpected finding in the night of 4./5.2.1980 was the fact, that the plateau values in the Hall resistance x-y are not influenced by the amount of localized electrons and can be expressed with high precision by the equation $R_H = \frac{h}{\nu e^2}$

Fractional Quantum Hall Effect





D. Tsui enclosed the distance between B=0 and the position of the last IQHE between two fingers of one hand and measured the position of the new feature in this unit. He determined it to be three and exclaimed, "quarks!" H. Stormer

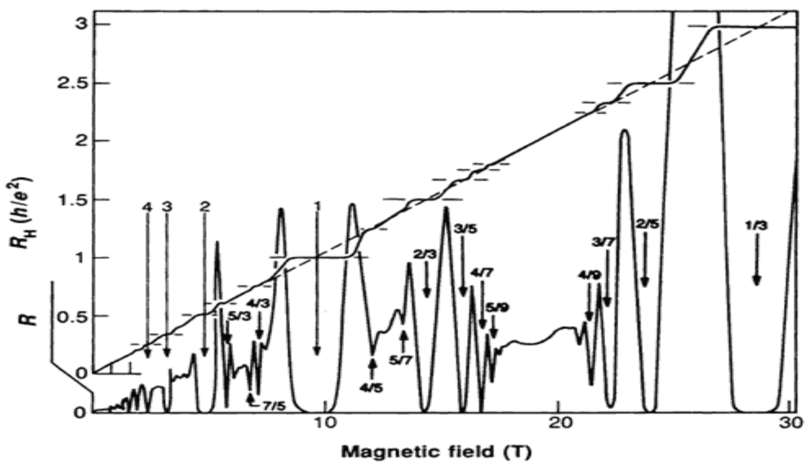
In 1998, Laughlin, Stormer, and Tsui are awarded the Nobel Prize

" for their discovery of a new form of quantum fluid with fractionally charged excitations."

The FQHE is fascinating for a long list of reasons, but it is important, in my view, primarily for one: It established experimentally that both particles carrying an exact fraction of the electron charge e and powerful gauge forces between these particles, two central postulates of the standard model of elementary particles, can arise spontaneously as emergent phenomena. R. Laughlin

D. C. Tsui, H. L. Stormer, and A. C. Gossard
Phys. Rev. Lett. 48, 1559 (1982)

How Many Fractions Have Been Observed? ~80



$$\nu = \frac{N_e}{N_\phi}$$

filling factor or fraction
N_e = # of electrons
N_ϕ =# of flux quanta

How to model the quantum state(s) at a filling fraction?

What are the electrons doing at a plateau?

1/3	1/5	1/7	1/9	2/11	2/13	2/15	2/17	3/19	5/21	6/23	6/25
2/3	2/5	2/7	2/9	3/11	3/13	4/15	3/17	4/19	10/21		
4/3	3/5	3/7	4/9	4/11	4/13	7/15	4/17	5/19			
5/3	4/5	4/7	5/9	5/11	5/13	8/15	5/17	9/19			
7/3	6/5	5/7	7/9	6/11	6/13	11/15	6/17	10/19			
8/3	7/5	9/7	11/9	7/11	7/13	22/15	8/17				
	8/5	10/7	13/9	8/11	10/13	23/15	9/17				
	11/5	12/7	25/9	16/11	20/13						
	12/5	16/7		17/11							
		19/7									

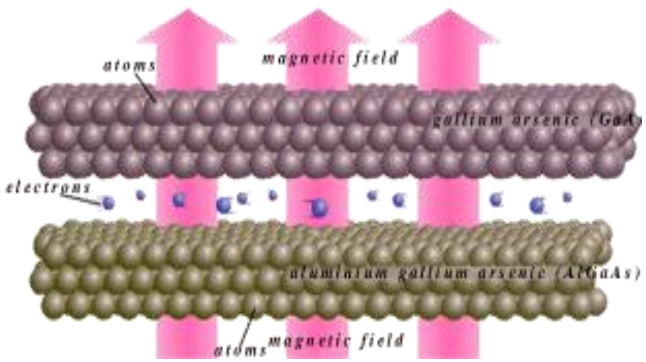
m/5, m=14,16, 19

5/2
7/2
19/8

Pan et al (2008)

Fractional Quantum Hall Liquids

N electrons in a plane bound to the interface between two semiconductors immersed in a perpendicular magnetic field



Classes of ground state wave functions that have similar properties or no phase transitions as N→∞ (N ~ $10^{11}\ cm^{-2}$)

Interaction is dynamical entanglement and quantum order is materialized entanglement

Fundamental Hamiltonian:

$$H = \Sigma_1^{N} \{\frac{1}{2m} [\nabla_j - q\, A(z_j)]^2 + V_{bg}(z_j)\} + \Sigma_{j<k} V(z_j - z_k)$$

Ideal Hamiltonian:

$$H = \Sigma_1^{N} \{\frac{1}{2m} [\nabla_j - q\, A(z_j)]^2 \} + \text{?}, \quad \text{e.g. } \Sigma_{j<k}\, \delta(z_j - z_k)$$

z_j position of j-th electron

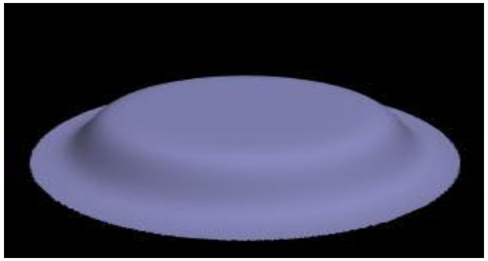
Laughlin wave function for ν=1/3

Laughlin 1983

Good trial wavefunction for N electrons at z_i in ground state

Gaussian

$$\Psi_{1/3} = \prod_{i<j}(z_i - z_j)^3 \, e^{-\Sigma_i |z_i|^2/4}$$



Physical Theorem:

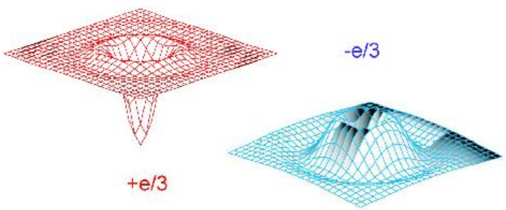
1. Laughlin state is incompressible: density and gap in limit (Laughlin 83)
2. Elementary excitations have charge e/3 (Laughlin 83)
3. Elementary excitations are abelian anyons (Arovas-Schrieffer-Wilczek 84)

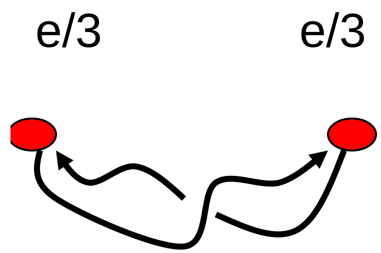
Experimental Confirmation:

1. and 2. √ , but 3. ?, thus Laughlin wave function is a good model

Excitations=Anyons

Quasi-holes/particles in ν=1/3 are abelian anyons





$$\Psi_{1/3} = \prod_k(\eta_0 - z_j)^3 \prod_{i<j}(z_i - z_j)^3 \ e^{-\Sigma_i |z_i|^2/4}$$

$$= \prod_k(\eta_1 - z_j) \prod_k(\eta_2 - z_j) \ \prod_k(\eta_3 - z_j) \ \prod_{i<j}(z_i - z_j)^3 \ e^{-\Sigma_i |z_i|^2/4}$$

n anyons at well-separated η_i, i=1,2,.., n,
there is a **unique** ground state

$$\psi \rightarrow e^{\pi \, i/3} \, \psi$$

Moore-Read or Pfaffian State

G. Moore, N. Read 1991

Pfaffian wave function (MR w/ ≈ charge sector)

$$\Psi_{1/2} = \mathrm{Pf}(1/(z_i - z_j)) \prod_{i<j} (z_i - z_j)^2 \, e^{-\sum_i |z_i|^2/4}$$

Pfaffian of a 2n×2n anti-symmetric matrix M=(a_{ij}) is

ω^n =n! Pf (M) dx^1∧dx^2∧…∧dx^{2n} if $\omega = \Sigma_{i<j}\, a_{ij}$ dx^i∧ dx^j

Physical Theorem:

1. Pfaffian state is gapped
2. Elementary excitations are non-abelian anyons, called Ising anyon σ

…… Read 09

Non-abelian Anyons in Pfaffian State

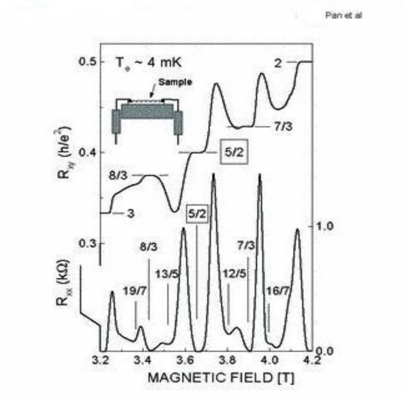
- 1-qh: Pf $\left(\frac{(\eta - z_i)(\eta - z_j)}{z_i - z_j}\right)$
- 2-qh: Pf$\left(\frac{(\eta_1 - z_i)(\eta_2 - z_j) + (\eta_1 - z_j)(\eta_2 - z_i)}{z_i - z_j}\right)$
- 4-qh: $P_{[12,34]}$=Pf$\left(\frac{(\eta_1 - z_i)(\eta_2 - z_i)(\eta_3 - z_j)(\eta_4 - z_j) + (i \leftrightarrow j)}{z_i - z_j}\right)$

  $P_{[13,24]}$, $P_{[14,23]}$.

There is one linear relation among the three. (Nayak-Wilczek 96)

Anyons w/ degeneracy in the plane are non-abelian anyons.

Enigma of ν=5/2 FQHE



R. Willett et al discovered ν=5/2 in1987

- Moore-Read State, Wen 1991
- Greiter-Wilczek-Wen 1991
- Nayak-Wilczek 1996
- Morf 1998
- …

MR (maybe some variation) is a good trial state for 5/2

- Bonderson, Gurarie, Nayak 2011,

Willett et al, PRL 59 1987

A landmark (physical) proof for the MR state

"Now we eagerly await the next great step: experimental confirmation." ---Wilczek

Experimental confirmation of 5/2:

gap and charge e/4 √ , but non-abelian anyons ???

Wave functions of bosonic FQH liquids

- Chirality:

  $\Psi(z_1,\ldots,z_N)$ is a polynomial (Ignore Gaussian)
- Statistics:

  symmetric=anti-symmetric divided by $\Pi_{i<j}(z_i-z_j)$
- Translation invariant:

  $\Psi(z_1+c,\ldots,z_N+c) = \Psi(z_1,\ldots,z_N)$ for any c
- Filling fraction:

  $\nu=\lim \frac{N}{N_\phi}$, where N_ϕ is max degree of any z_i

Pattern of zeros

joint work with X.-G. Wen

W.F. $\{\Psi(z_1,\ldots,z_N)\}$ "vanish" at certain powers $\{S_a\}$ when a particles are brought together, $a=1,2,\ldots$:

$$\Psi(z_1,\ldots,z_N)=\sum c_I Z^I,\ I=(i_1,\ldots,i_N),$$

$S_a=\min\{\sum^a i_j\}$---minimal total degrees of a variables.

Morally, $\{S_a\} \approx$ ideal wave function $\approx$ ideal Hamiltonian

These powers $\{S_a\}$ should be consistent to represent the same local physics of the quantum phase, and encode many topological properties of the FQH state.

Conformal field theory examples

Laughlin: $S_a = qa(a-1)/2$, $\nu = 1/q$

Pfaffian: $S_a = a(a-1)/2 - [a/2]$, $\nu = 1$ bosonic

In a CFT, if V_e is chosen as the electron operator and a conformal block as a W.F.

If $V_a = (V_e)^a$ has scaling dimension h_a, then

$$S_a = h_a - a\, h_1$$

Classification of FQH states

Find necessary and sufficient conditions for patterns of zeros

a) to be realized by polynomials

b) to represent a topological phase

Thm (Wen, W.) If translation inv. symm. polys. $\{\Psi(z_i)\}$ satisfy UFC and nCF for n, then

1) Set $m=S_{n+1}-S_n$, mn even, and $\nu=n/m$

2) $S_{a+b}-S_a-S_b \geq 0$

3) $S_{a+b+c}-S_{a+b}-S_{b+c}-S_{c+a} +S_a +S_b +S_c \geq 0$

4) S_{2a} even

5) $2S_n=0$ mod n

6) $S_{a+kn}=S_a+kS_n+kma+k(k-1)mn/2$

Further works with Y. Lu and Z. Wang show pattern of zeros is not complete, though many topological properties can be derived from pattern of zeros.

More complete data use vertex operator algebra.

Puzzle: Need to impose $S_{a+b+c}-S_{a+b}-S_{b+c}-S_{c+a} +S_a +S_b +S_c$ to be EVEN!

Theory of Topological Phases of Matter

- Low energy effective theory is a topological quantum field theory (TQFT)

- Quantum information characterization

1) error correction codes

Kitaev, Freedman, Bravyi-Hastings-Michalakis

2) long range entanglement

Kitaev-Preskill, Levin-Wen

Sensitivity to Topology

Given a theory H, i.e. a Hamiltonian schema, and a surface Y, put H on Y. Let V(Y) be its ground state manifold,

(ground state (GS) manifold=Hilbert space of GSs. Degenerate if Dim GS manifold > 1)

e.g. “Laughlin w.f.”s on T^2 consist of classical θ-functions, which form a 3-dimensional Hilbert space.

So Laughlin theory has a 3-fold degeneracy on torus (3^g on genus g surface.)

Ground state degeneracy depends on topology.

TQFT as Effective Theory

A theory H is topological if the functor

Surface Y $\rightarrow$ V(Y) (GS manifold) is a TQFT.

Rm: H is the Hamiltonian for all degrees of freedom. Restricted to the topological degrees of freedom, the effective Hamiltonian is constant (or 0).

Physical Thm: Topological properties of abelian bosonic FQH liquids are modeled by Witten-Chern-Simons theories with abelian gauge groups T^n.

Conjecture: NA statistics sectors of FQH liquids at $\nu=2+\frac{k}{k+2}$ are modeled by $SU(2)_k$-WCS theories. k=1,2,3,4, $\nu= \frac{7}{3}, \frac{5}{2}, \frac{13}{5}, \frac{8}{3}$. (Read-Rezayi). 5/2 √

Atiyah's Axioms of (2+1)-TQFT
(TQFT w/o excitations and anomaly)

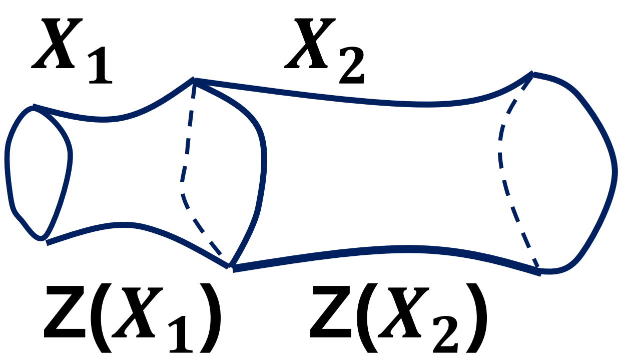
A functor (V,Z): category of surfaces → Vec

(Hilbert spaces for unitary TQFTs)

Oriented closed surface Y → vector space V(Y)

Oriented 3-mfd X with ∂X=Y → vector Z(X)$\in$V(∂X)

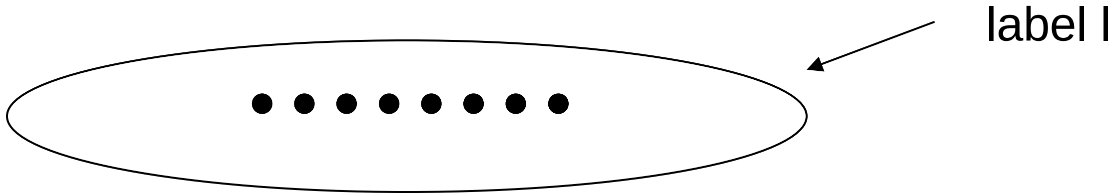
- $\mathbf{V(\varnothing) \cong \mathbb{C}}$
- $\mathbf{V(Y_1 \cup Y_2) \cong V(Y_1) \otimes V(Y_2)}$
- $\mathbf{V(-Y) \cong V^*(Y)}$
- $\mathbf{Z(Y \times I) = Id_{V(Y)}}$
- $\mathbf{Z(X_1 \cup_Y X_2) = Z(X_1) \bullet Z(X_2)}$



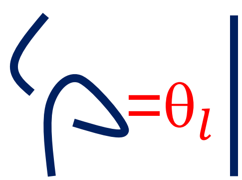
Modeling Anyons

Put a theory H on a closed surface Y with anyons $a_1, a_2, \ldots, a_n$ at $\eta_1,\ldots,\eta_n$ (punctures), the (relative) ground states of the system "outside" $\eta_1,\ldots,\eta_n$ is a Hilbert space $V(Y; a_1, a_2, \ldots, a_n)$.

For anyons in a surface w/ boundaries (e.g. a disk), the boundaries need conditions.



Stable boundary conditions correspond to anyon types (labels, super-selection sectors, topological charges). Moreover, each puncture (anyon) needs a tangent direction, so anyon is modeled by a small arrow, not a point. Topological twist:



$\theta_l \neq 1$ in general

Non-abelian Anyons

Given n anyons of type x in a disk D, their ground state degeneracy

dim(V(D,x,…,x))=$D_n \sim d^n$

The asymptotic growth rate d is called the quantum dimension.

An anyon d=1 is called an abelian anyon, e.g. Laughlin anyon, d=1

An anyon with d >1 is an non-abelian anyon, e.g. the Ising anyon σ, d=$\sqrt{2}$.

For n even, $D_n = \frac{1}{2}\, 2^{\frac{n}{2}}$ with fixed boundary conditions,

n odd, $D_n = 2^{\frac{n-1}{2}}$. (Nayak-Wilczek 96)

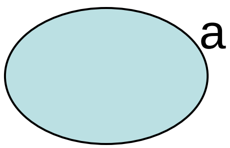
Degeneracy for non-abelian anyons in a disk grows exponentially with # of anyons, while for an abelian anyon, no degeneracy---it is always 1.

(Extended) TQFT Axioms

Moore-Seiberg, Walker, Turaev,…

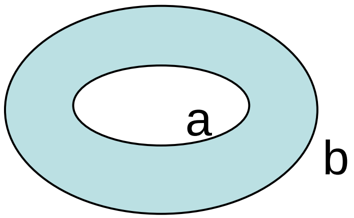
Let L={a,b,c,…d} be the labels (particle types), a $\rightarrow$ a*, and a**=a,

0 (or 1) =trivial type



Disk Axiom:

V(D^2; a)=0 if a$\neq$ 0, $\mathbb{C}$ if a=0

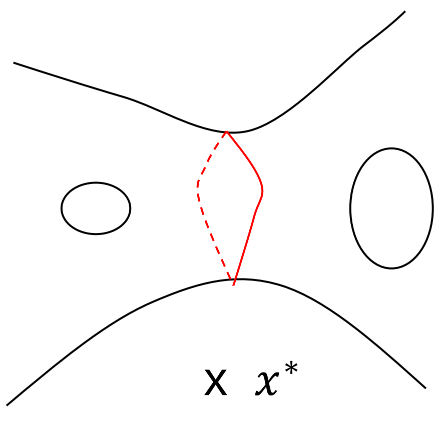


Annulus Axiom:

V(A; a,b)=0 if a$\neq$ b*, $\mathbb{C}$ if a=b*

Gluing Axiom:

V(Y; l) $\cong \oplus_{x\varepsilon L}$ V(Y_{cut}; l,x,x^*)

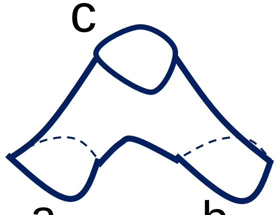


Algebraic Structure of Anyons

L={a,b,c,…d} a label set and $P_{ab,c}$ a pair of pants labeled by a,b,c.

$N_{ab,c}$=dim V($P_{ab,c}$), then $N_{ab,c}$ is the fusion rule of the theory.

a⊗b=⊕$N_{ab,c}$c



Every orientable surface Y can be cut into disks D, annuli A, and pairs of pants.
If V(D), V(A), V($P_{ab,c}$) are known, then V(Y) is determined by the gluing axiom.

Conversely a TQFT can be constructed from V(Y) of disk, annulus and pair of pants. Need **consistent conditions: a modular tensor category**

Unitary modular categories (UMC) are algebraic data of unitary TQFTs and algebraic theories of anyons: anyon=simple object, fusion=tensor product, statistics of anyons are representations of the mapping class groups.

Rank < 5 Unitary Modular Categories

joint work w/ E. Rowell and R. Stong

	A 1 Trivial			
	A 2 Semion		NA 2 Fib BU	
	A 2 (U(1),3)	NA 8 Ising	NA 2 (SO(3),5) BU	
A 5 Toric code	A 4 (U(1),4)	NA 4 Fib x Semion BU	NA 2 (SO(3),7) BU	NA 3 DFib BU

The ith-row is the classification of all rank=i unitary modular tensor categories. Middle symbol: fusion rule. Upper left corner: A=abelian theoy, NA=non-abelian. Upper right corner number=the number of distinct theories. Lower left corner BU=there is a universal braiding anyon.

Code Subspace Property

Conjecture: H: $V_\Gamma \rightarrow V_\Gamma$ is a topological theory on a lattice Γ (graph in a surface Y), where $V_\Gamma = \otimes_{e \varepsilon \Gamma} C^m$ for some m, then GS(H) $\subset V_\Gamma$ is an error correction code.

If true, then local operators do not act on the ground states

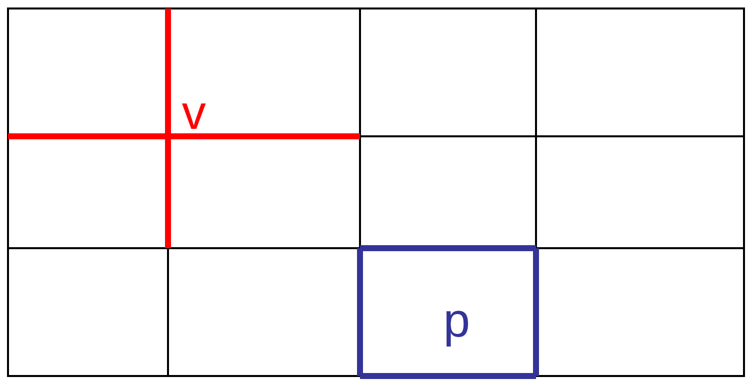
For some k, all k-local operators O_k: $V_\Gamma \rightarrow V_\Gamma$ the following composition is $\lambda \bullet$Id for some scalar λ (possibly 0),

$$\mathrm{GS(H)} \subset V_\Gamma \longrightarrow V_\Gamma \rightarrow \mathrm{GS(H)}$$

GS manifolds are fault-tolerant quantum memory.

Kitaev's Toric Code

$$\mathbf{H=\sum_v (I-A_v) + \sum_p (I-B_p)}$$



$=T^2$

$$\mathbf{V}=\otimes_{edges}\ \mathbf{C}^2$$

$$\mathbf{A_v}=\otimes_{e\varepsilon v}\ \sigma^z\ \otimes_{others}\ \mathbf{Id_e},$$

$$\mathbf{B_p}=\otimes_{e\varepsilon p}\ \sigma^x\ \otimes_{others}\ \mathbf{Id_e},$$

GS Manifolds as Quantum Memory

- **Thm:** If a TQFT is from a Drinfeld center (or quantum double), then GS manifolds of the Levin-Wen model/Kitaev model are error correction codes.

- Chiral theories (those with anomaly)? Open including all WCS theories so FQH states

  a) a holographic solution by Walker-W.

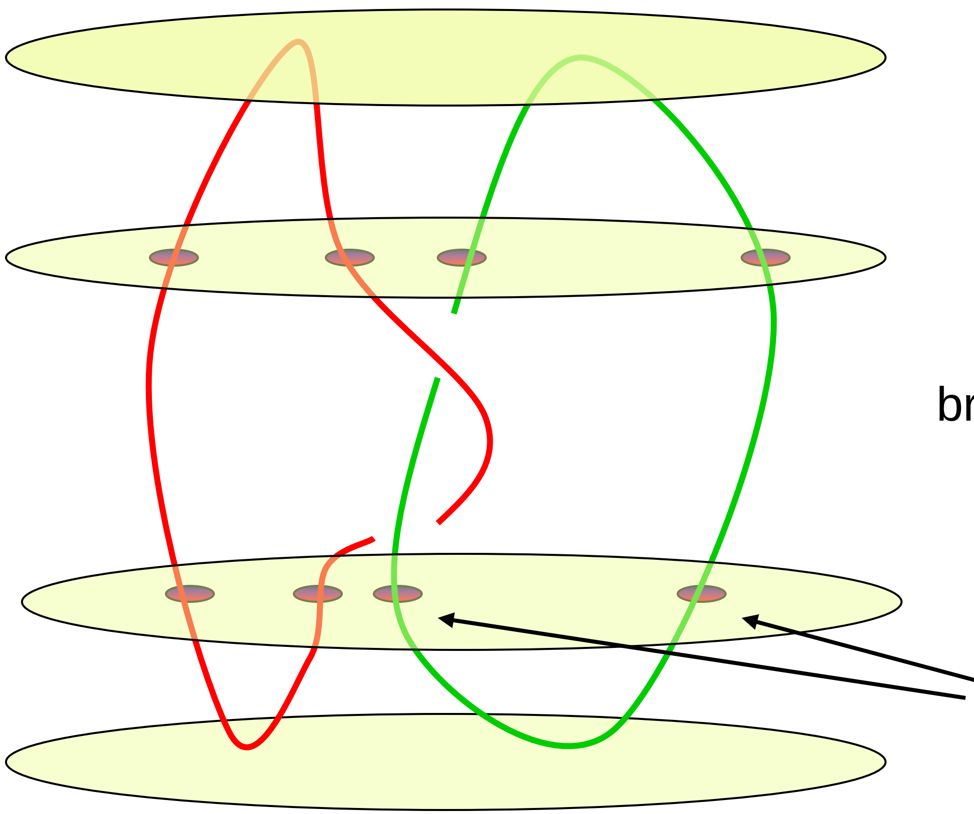
  b) local degrees of freedom might be infinite.

Topological phases of matter exist in both real systems (FQHE) and theories, what are they good for?

Topological Quantum Computation

Freedman 97, Kitaev 97, FKW 00, FLW 00

Computation	**Physics**
readout	fusion
applying gates	braiding particles
initialize	create anyons



Mathematical Theorems

Theorem 1 (FKW): Any unitary TQFT can be efficiently simulated by the quantum circuit model.

There are efficient additive approximation algorithms of quantum invariants by the quantum circuit model.

Theorem 2 (FLW): Anyonic quantum computers based on SU(2)-Chern-Simons theory at level k are braiding universal except k=1,2,4.

The approximation of Jones poly of links at the $(k+2)^{th}$ root of unity ($k\neq 1,2,4$) is a BQ(F)P-complete problem.

Estimation of braid closure is DQC1-complete for k=3 (Shor-Jordan 07)

Exact or FPRAS approximation of Jones poly of links at the (k+2)th root of unity ($k\neq 1,2,4$) is #P-hard. (Vertigan 05, Kuperberg 09)

Density Theorem

In 1981, Jones proved that $\rho_{SU(2),k,l}(B_n)$ is infinite

if k$\neq$ 1,2,4 n$\geq$3 or k=8, n$\geq$4 (k=r-2).

and asked:

What are the closed images of $\rho_{SU(2),k,l}(B_n)$?

Theorem (FLW 02):

Always contain SU if k$\neq$ 1,2,4, n$\geq$3 or k=8, n$\geq$4.

Others are finite groups which can be identified (using the classification of simple groups for all SU(n) theories).

FQHE at ν=5/2 w/ ≈ charge sector

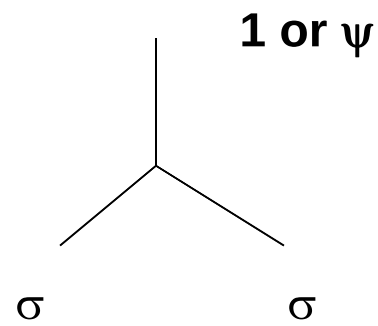
The effective theory is Ising TQFT

(Fradkin-Nayak-Tsvelik-Wilczek 98)

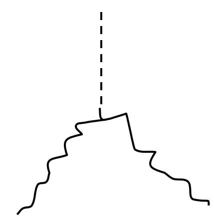
Ising=M(3,4) minimal model

= $\frac{SU(2)_2}{U(1)}$ coset

=TL at 4th root



Particle types are {1,σ,ψ}



Fusion rules:

1---ground state

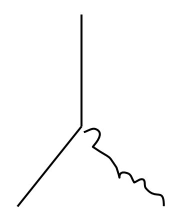
ψ---Majarona fermion

σ---Ising anyon

$$\sigma^2 \cong 1+\psi,$$

$$\psi^2 \cong 1,$$

$$\sigma\ \psi \cong \psi\ \sigma \cong \sigma$$

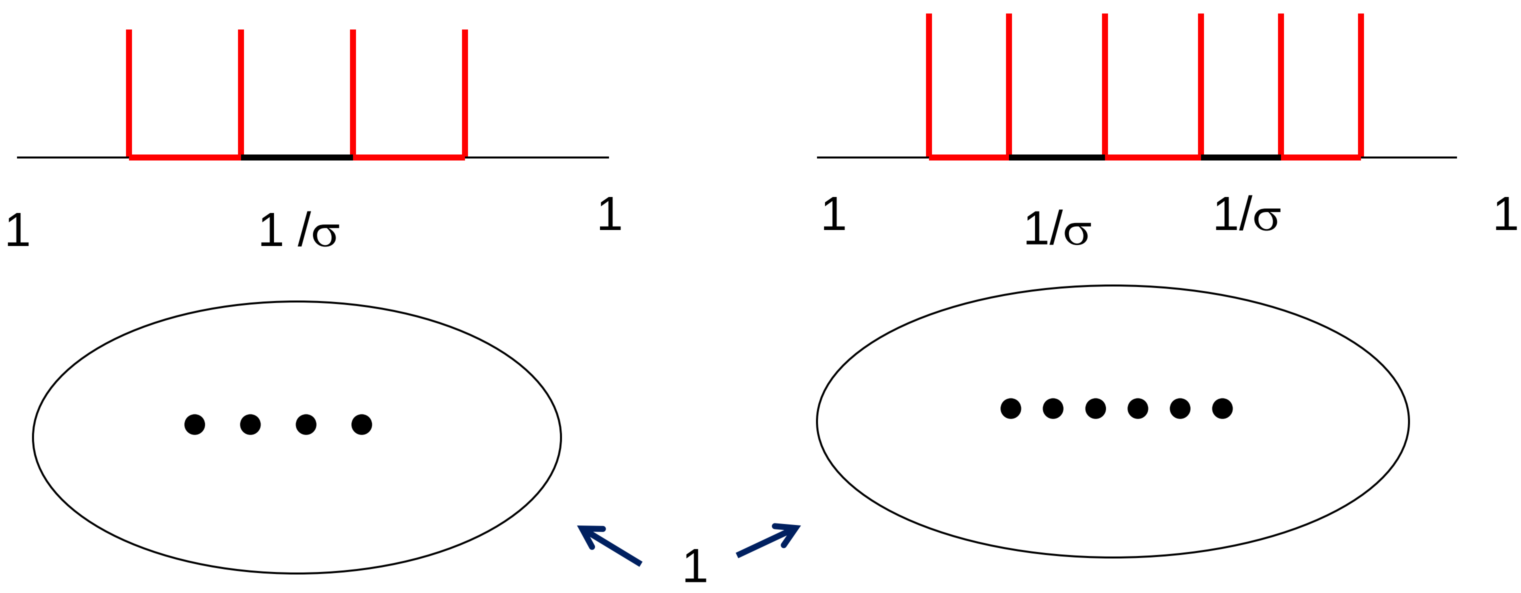


Ising Quantum Computer

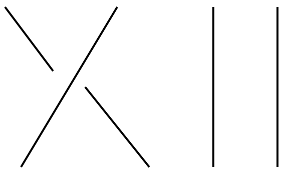
4 Ising σ's in a disk is C^2-qubit. 6 σ's C^4-2 qubits.

For 1-qubit gates, $\rho: B_4 \to U(2)$

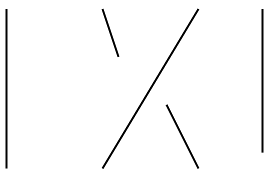
For 2-qubits gates, $\rho: B_6 \to U(4)$



Ising Braiding Gates

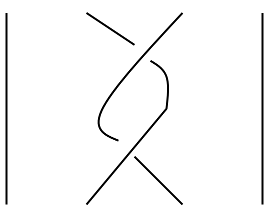


$$\longrightarrow \quad e^{-\pi i/8}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$



$$\longrightarrow \quad e^{-\pi i/8}\begin{pmatrix} (1-i)/2 & (1+i)/2 \\ (1+i)/2 & (1-i)/2 \end{pmatrix}$$

$\boldsymbol{\sigma_1 \sigma_2 \neq \sigma_2 \sigma_1}$



$$\longrightarrow \quad e^{-\pi i/4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

4 σ's

NOT Gate

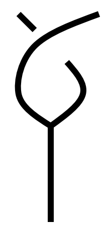
$\frac{\pi}{8}$ -gate cannot be realized

CNOT can be realized

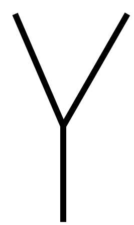
Fibonacci TQFT (FQHE at ν=12/5?)

G_2 level=1 CFT, c=14/5 mod 8

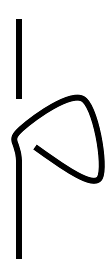
- Particle types: $\{1,\tau\}$, τ---Fib anyon
- Quantum dimensions: $\{1,\phi\}$, ϕ=golden ratio
- Fusion rules: $\tau^2=1\oplus\tau$
- Braiding:



$=e^{3\pi i/5}$



- Twist:

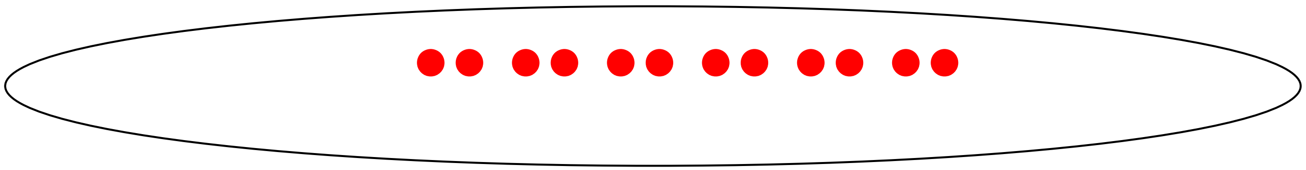


$=e^{4\pi i/5}$

Fibonacci Quantum Computer

for n qubits, consider the 4n Fibonacci anyons

$\rho: B_{4n} \longrightarrow U(F_{4n-2})$, F_{4n-2}---4n-2 Fib number



Given a quantum circuit on n qubits

$$U_L: (C^2)^{\otimes n} \longrightarrow (C^2)^{\otimes n}$$

Topological compiling: find a braid $b \in B_{4n}$ so that the following commutes for any U_L:

$$\begin{tikzcd}
(C^2)^{\otimes n} \arrow[r] \arrow[d, "U_L"'] & V_{4n} \arrow[d, "\rho(b)"] \\
(C^2)^{\otimes n} \arrow[r] & V_{4n}
\end{tikzcd}$$

V_{4n}-gs of 4n anyons

Universal Braiding Gates

- Ising anyon σ does not lead to universal braiding gates, but Fib anyon τ does

- Quantum dimension of Ising anyon σ has quantum dimension=$\sqrt{2}$, while Fib anyon τ has quantum dimension $\phi=(\sqrt{5}+1)/2$---golden ratio

- Given an anyon type x, when does it lead to universal braiding gate sets ?

  Related: Can a NA-anyon has its own local Hilbert space---an explicit locality of TQFT? No! Someway?

Yang-Baxterizable Anyon

joint work w/ E. Rowell

Conjecture: An anyon type x does not lead to universal braiding gates

if and only if its quantum dimension d_x is $\sqrt{q}$ for some integer q

if and only it is Yang-Baxterizable:

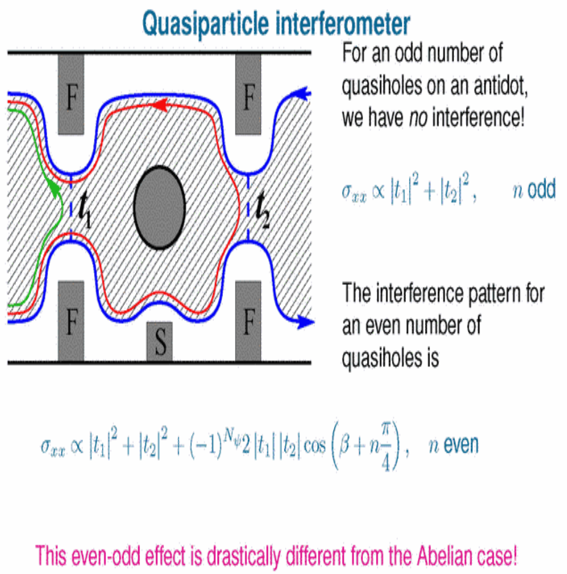
there is a unitary R-matrix R such that the rep $V_{n,x}$ of the n-strand braid group B_n from x is Yang-Baxterized by R for all n in the sense:

Let $V_{n,x} = \oplus \; V_{n,x,l}$, then

$$\oplus \; m_{n,x,l} \; V_{n,x,l} \cong V_{n,R}$$

for some $m_{n,x,l}$, and $V_{n,R}$-the rep of B_n from R-matrix R

Are we close to confirm non-abelian anyons?



Freedman, Nayak, Das Sarma, 2005

Halperin-Stern 06
Bonderson-Kitaev-Shtengel 06

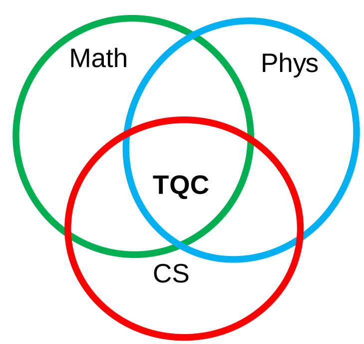
Willett reported data 09
Heiblum data on neutral mode
Spin polarization?

Challenging:

Little correlation between anyons and local measurement

Extreme conditions

Can we do better? We have to build a small topological quantum computer to confirm non-abelian anyons



Some References

- *Computing with Quantum Knots*, Graham P. Collins
  Scientific American **4**, 57 (2006).

- Fractional statistics and anyon superconductivity
  a book of classical papers on anyons---F. Wilczek

- Topological quantum computation---J. Preskill
  http://www.theory.caltech.edu/~preskill/ph219/

- Non-abelian anyons and topological quantum computation
  C. Nayak et al, Rev. Mod. Phys. 2008, Arxiv 0707.1889

- Topological quantum computation---CBMS monograph vol. 115 (Z.W.)