Abstract. This is the set of lecture notes for Math 117 during Fall quarter of 2017 at UC Santa Barbara. The lectures follow closely the textbook \[1\].

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1. The set $\mathbb{N}$ of natural numbers

We introduce basic properties for the set of natural numbers.

- $\mathbb{N} = \{1, 2, 3, \cdots \}$ denotes the set of all natural numbers, or all positive integers;
- each $n \in \mathbb{N}$ has a successor $n + 1$. For example, the successor of 5 is 6.

**Proposition 1.1.** The set $\mathbb{N}$ satisfies the following properties:

- **N1.** $1 \in \mathbb{N}$;
- **N2.** if $n \in \mathbb{N}$, then its successor $n + 1 \in \mathbb{N}$;
- **N3.** 1 is not the successor of any element in $\mathbb{N}$;
- **N4.** if $n$ and $m$ have the same successor, then $n = m$;
- **N5.** suppose $S$ is a subset of $\mathbb{N}$ satisfying: 1 $\in \mathbb{N}$ and if $n \in S$ then $n+1 \in S$, then $S = \mathbb{N}$.

Property (N5) is called the Peano Axiom or Peano Postulate.

Property (N5) is the basis for mathematical induction:

- Let $P_1, P_2, \cdots$ be a list of statements of propositions that may or may not be true;
- The Principle of Mathematical Induction asserts that:
  - All of $P_1, P_2, \cdots$ are true provided:
    - $P_1$ is true; (this is called the basis of induction)
    - if $P_n$ is true, then $P_{n+1}$ is true for all $n \in \mathbb{N}$. (this is called induction step)

**Example 1.2.** All numbers of form $7^n - 2^n$ ($n \in \mathbb{N}$) is divisible by 5.

**Proof.** The $n$-th proposition is:

$$P_n: \text{“} 7^n - 2^n \text{ is divisible by 5”}.$$  

- **Basis of induction** $P_1$: $7^1 - 2^1 = 5$ is divisible by 5, so $P_1$ is true;
- **Induction step**: Suppose $P_n$ is true, then $7^n - 2^n = 5m$ for some $m \in \mathbb{N}$.
  To verify $P_{n+1}$, we have
  $$7^{n+1} - 2^{n+1} = 7^{n+1} - 2 \cdot 7^n + 2 \cdot 7^n - 2^{n+1}$$
  $$= 7^n(7 - 2) + 2(7^n - 2^n)$$
  $$= 5 \cdot 7^n + 2(5m) = 5(7^n + 2m).$$

1The symbol $n \in \mathbb{N}$ means “$n$ is an element of the set $\mathbb{N}$”
That is, $7^{n+1} - 2^{n+1}$ is also divisible by 5. Therefore $P_n$ implies $P_{n+1}$, and the induction step holds.

□

Example 1.3. Show that $|\sin(nx)| \leq n|\sin(x)|$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$.

Proof. The $n$-th proposition is:

$P_n$: “$|\sin(nx)| \leq n|\sin(x)|$ for all $x \in \mathbb{R}$”.

• Basis of induction: $|\sin(x)| = 1 \cdot |\sin(x)|$, hence $P_1$ is true;

• Induction step: Suppose $P_n$ is true, that is $|\sin(nx)| \leq n|\sin(x)|$ for all $x \in \mathbb{R}$. To verify $P_{n+1}$, we have

$$|\sin((n+1)x)| = |\sin(nx + x)| = |\sin(nx) \cos(x) + \sin(x) \cos(nx)|$$

$$\leq |\sin(nx) \cos(x)| + |\sin(x) \cos(nx)| - (|\sin(x)|, |\cos(nx)| \leq 1)$$

$$\leq |\sin(nx)| + |\sin(x)| - (by \ P_n)$$

$$\leq n|\sin(x)| + |\sin(x)| = (n + 1)|\sin(x)|.$$

That is, $P_{n+1}$ is true. Therefore $P_n$ implies $P_{n+1}$, and the induction step holds.

□
2. The set \( \mathbb{Q} \) of rational numbers

We introduce basic properties for the set of integers and rational numbers.

- \( \mathbb{Z} = \{0, -1, 1, -2, 2, \ldots\} \) denotes the set of all integers;
- \( \mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\} \) denotes the set of all rational numbers.
  - \( \mathbb{Q} \) is a good algebraic system: one can do basic operations like additions, multiplication, subtraction, division over \( \mathbb{Q} \);
  - \( \mathbb{Q} \) is inadequate when solving algebraic equations like: \( x^2 - 2 = 0 \).

Actually, the solutions of \( x^2 - 2 = 0 \) makes sense:

1. by the Pythagorean Theorem, if \( d \) is the length of the diagonal of a square of edge length 1, then \( d^2 = 1^2 + 1^2 = 2 \), and hence \( d \) is a solution;
2. the graph of the function \( y = x^2 - 2 \) will pass through the \( x \)-axis at two points which are both solutions of \( x^2 - 2 = 0 \).

This means that:

\[
\text{there are gaps in } \mathbb{Q}, \text{ as } \sqrt{2} \notin \mathbb{Q}.
\]

Moreover, there are more exotic numbers like “\( \pi \)”, and “\( e \)”:  
- “\( \pi \)” appears when studying the perimeters of circles and spheres; 
- “\( e \)” appears when studying the infinite sums: \( \sum_{n \in \mathbb{N}} \frac{1}{n!} \).

**Definition 2.1.** A number is called an algebraic number if it satisfies a polynomial equation:

\[
a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0,
\]

where \( a_i \in \mathbb{Z} \) for all \( i = 0, \cdots, n \), and \( a_n \neq 0 \).

**Example 2.2.** We know that:

- all rational numbers are algebraic numbers:
  if \( r = \frac{m}{n}, m, n \in \mathbb{Z}, n \neq 0 \), then it satisfies \( nx - m = 0 \);
- square root \( \sqrt{\gamma} \), cubic root \( \sqrt[3]{\gamma} \), etc, are algebraic numbers;
- ordinary algebraic operations (including addition, multiplication, subtraction, division, taking roots) of algebraic numbers are also algebraic numbers.

**Example 2.3.** \( 17^{1/3}, (2+5^{1/3})^{1/2}, [(4-2\cdot3^{1/2})/7]^{1/2} \) are all algebraic numbers.

**Proof.**
1. \( 17^{1/3} \) is a solution of \( x^3 - 17 = 0 \);
2. Let \( a = (2 + 5^{1/3})^{1/2}, \Rightarrow a^2 = 2 + 5^{1/3}, \Rightarrow (a^2 - 2)^3 = 5, \Rightarrow a^6 - 2a^4 + 4a^2 - 8 = 5, \Rightarrow a^6 - 2a^4 + 4a^2 - 13 = 0; \)
(3) Let \( b = \left[ (4 - 2 \cdot 3^{1/2})/7 \right]^{1/2} \), \( \Rightarrow 7b^2 = 4 - 2 \cdot 3^{1/2} \), \( \Rightarrow 7b^2 - 4 = -2 \cdot 3^{1/2} \), \( \Rightarrow (7b^2 - 4)^2 = 12 \), \( \Rightarrow 49b^4 - 56b^2 + 16 = 12 \), \( \Rightarrow 49b^4 - 56b^2 + 4 = 0 \). □

Recall that

- an integer \( k \) is a factor of another integer \( m \), or \( k \) divides \( m \), if \( m/k \) is also an integer;
- an integer \( p \geq 2 \) is a prime provided that the only positive factors of \( p \) is 1 and \( p \).

**Proposition 2.4.** \( \sqrt{2} \) is not a rational number.

*Proof.* We prove by contradiction argument. Suppose the statement is not true, that is \( \sqrt{2} \in \mathbb{Q} \), \( \Rightarrow \sqrt{2} = \frac{p}{q} \), where \( p, q \) are both integers with no common factor and \( q \neq 0 \). Then

\[
\frac{p^2}{q^2} = 2, \quad \Rightarrow p^2 = 2q^2.
\]

**Lemma 2.5.** If \( p^2 \) is an even number \((p \in \mathbb{Z})\), then \( p \) is also even.

*Proof.* Using contradiction argument again, suppose not, then \( p \) is odd, \( \Rightarrow p = 2m + 1 \) for some \( m \in \mathbb{Z} \). Then

\[
p^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2) + 1,
\]
so \( p^2 \) is odd, and this is a contradiction. □

Let us go back to the proof of Proposition 2.4. Since \( p^2 \) is even, by the above lemma \( p \) is also even, \( \Rightarrow p = 2m \), hence

\[
p^2 = 4m^2 = 2q^2, \quad \Rightarrow q^2 = 2m^2.
\]

So \( q^2 \) is even, and hence \( q \) is even by the above lemma. But this is a contradiction to the assumption that \( p, q \) has no common factor. □

**Theorem 2.6** (Rational Zero Theorem). Suppose \( a_0, a_1, \ldots, a_n \) are all integers, and \( r \) is a rational number satisfying the polynomial equation:

\[
a_nx^n + \cdots + a_1x + a_0 = 0,
\]
where \( n \geq 1 \), \( a_n \neq 0 \), \( a_0 \neq 0 \). Write

\[
r = \frac{p}{q},
\]
where \( p, q \in \mathbb{Z} \) have no common factor and \( q \neq 0 \). Then

\( q \) divides \( a_n \), and \( p \) divides \( a_0 \).
Proof. Plug in $r = \frac{p}{q}$ into the equation:

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0.$$  

Multiply by $q^n$, $\Rightarrow$

$$a_np^n + a_{n-1}p^{n-1}q + \cdots + a_1pq^{n-1} + a_0q^n = 0.$$  

Re-arrange the equation:

$$a_np^n = -q(a_{n-1}p^{n-1} + \cdots + a_1pq^{n-2} + a_0q^{n-1}),$$  

$\Rightarrow q$ divides $a_np^n$. Since $p, q$ has no common factor, $q$ must divide $a_n$.

We can prove $p$ divides $a_0$ in a similar manner. In particular,

$$a_0q^n = -p(a_np^{n-1} + a_{n-1}p^{n-2} + \cdots + a_1q^{n-1}),$$  

$\Rightarrow p$ divides $a_0q^n$, and hence $p$ must divide $a_0$. $\square$

Example 2.7. Use the above theorem to prove that $\sqrt{2}$ is not a rational number.

Proof. Suppose the equation $x^2 - 2 = 0$ has a rational root $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ has no common factor and $q \neq 0$. Then by Theorem 2.6

$$q \divides 1, \text{ and } p \divides -2.$$  

There $p$ can only be $\pm 1$ or $\pm 2$, and $q$ can only be $\pm 1$, $\Rightarrow \frac{p}{q}$ must be of the forms $\pm 1$, $\pm 2$. By substituting back to $x^2 - 2 = 0$, we can see that none of them is a solution. This is a contradiction. $\square$

Example 2.8. Prove that $a = (2 + 5^{1/3})^{1/2}$ is not a rational number.

Proof. By the above example, we know that $a$ satisfies:

$$a^6 - 2a^4 + 4a^2 - 13 = 0.$$  

If $a$ is a rational number, then write $a = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ has no common factor and $q \neq 0$. By the Rational Zero Theorem,

$q \divides 1, \text{ and } p \divides -13.$  

There $p$ can only be $\pm 1$ or $\pm 13$, and $q$ can only be $\pm 1$, $\Rightarrow \frac{p}{q}$ must be of the forms $\pm 1$, $\pm 13$. By substituting back, we can see that none of them is a solution. This is a contradiction. $\square$
3. Ordered fields

The basic algebraic operations on \( \mathbb{Q} \) are “+” and “.”. The following properties hold in this algebraic system \((\mathbb{Q}, +, \cdot)\):

Properties for “+”:

A1. \( a + (b + c) = (a + b) + c, \forall a, b, c \in \mathbb{Q} \) – associative law;
A2. \( a + b = b + a, \forall a, b \in \mathbb{Q} \) – commutative law;
A3. \( a + 0 = a, \forall a \in \mathbb{Q} \) – existence of 0;
A4. For each \( a \in \mathbb{Q} \), there exists an element \(-a \in \mathbb{Q}\), such that \( a + (-a) = 0 \) – existence of reverse;

Properties for “.”:

M1. \( a \cdot (b \cdot c) = (a \cdot b) \cdot c, \forall a, b, c \in \mathbb{Q} \) – associative law;
M2. \( a \cdot b = b \cdot a, \forall a, b \in \mathbb{Q} \) – commutative law;
M3. \( a \cdot 1 = a, \forall a \in \mathbb{Q} \) – existence of 1;
M4. For each \( a \in \mathbb{Q}, a \neq 0 \), there exists an element \( a^{-1} \in \mathbb{Q} \), such that \( a \cdot (a^{-1}) = 1 \) – existence of reciprocal;

A property relating “.” and “.”:

DL. \( a \cdot (b + c) = a \cdot b + a \cdot c, \forall a, b, c \in \mathbb{Q} \) – distributive law.

**Definition 3.1.** A set \( F \) is a **field** if it has two operations “+” and “.”, and elements 0 and 1 satisfying all the properties (A1)...(A4) (M1)...(M4) (DL) (with \( \mathbb{Q} \) changed to \( F \)).

**Example 3.2.** There are two examples which are not fields:

- \( \mathbb{N} \) (the natural numbers ) fails properties (A4) and (M4);
- \( \mathbb{Z} \) (the integers) fails (M4).

\( \mathbb{Q} \) also has an order structure “\( \leq \)” satisfying the properties:

O1. given \( a, b \in \mathbb{Q} \), either \( a \leq b \) or \( b \leq a \);
O2. if \( a \leq b \) and \( b \leq a \), then \( a = b \);
O3. if \( a \leq b \) and \( b \leq c \), then \( a \leq c \) – transitive law;
O4. if \( a \leq b \), then \( a + c \leq b + c \);
O5. if \( a \leq b \) and \( 0 \leq c \), then \( a \cdot c \leq b \cdot c \).

**Definition 3.3.** A field \( F \) is called an **ordered field** if it has an order structure satisfying properties (O1)...(O5).

\(^2\)Here the symbol \( \forall \) means “for all”.

Remark 3.4. \( a < b \) means that \( a \leq b \) and \( a \neq b \).

Properties of ordered fields: based on the defining properties, we can deduce more properties for a field or an ordered field.

**Theorem 3.5.** Suppose \((F, +, \cdot)\) is a field with elements 0, 1, and \(a, b, c\) denote arbitrary elements in \(F\), then:

1. \(a + c = b + c \Rightarrow a = b\);
2. \(a \cdot 0 = a, \forall a\);
3. \((-a) \cdot b = -(a \cdot b)\);
4. \((-a) \cdot (-b) = a \cdot b\);
5. \(a \cdot c = b \cdot c, \text{ and } c \neq 0 \Rightarrow a = b\);
6. \(a \cdot b = 0 \Rightarrow \text{ either } a = 0 \text{ or } b = 0\).

**Proof.**

(i) \((a + c) + (-c) = a + (c + (-c)) = a + 0 = a\)

\[= (b + c) + (-c) = b + (c + (-c)) = b + 0 = b.\]

(ii) \(a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0, \Rightarrow \) (by (i)) \(a \cdot 0 = 0\).

(iii) \((-a) \cdot b + a \cdot b = (-a + a) \cdot b = 0 \cdot b = 0\) (by (ii)), so \((-a) \cdot b = -(a \cdot b)\).

(iv) By (iii), \((-a) \cdot (-b) = -a \cdot (-b) = -(a \cdot b) = ab\).

(v) Homework.

(vi) Homework.

\(\square\)

**Theorem 3.6.** Suppose \(F\) is an ordered field with order structure “\(\leq\)”, and \(a, b, c\) denotes arbitrary elements in \(F\), then:

1. if \(a \leq b\), then \(-b \leq -a\);
2. if \(a \leq b\) and \(c \leq 0\), then \(bc \leq ac\);
3. if \(0 \leq a\) and \(0 \leq b\), then \(0 \leq ab\);
4. \(0 \leq a^2 = a \cdot a, \forall a\);
5. \(0 < 1\);
6. if \(0 < a\), then \(0 < a^{-1}\);
7. if \(0 < a < b\), then \(0 < b^{-1} < a^{-1}\).

**Proof.**

(i) By (O4), \(a + (-b) \leq b + (-b) \Rightarrow a + (-b) \leq 0\); by (O4), \(-a + [a + (-b)] \leq -a + 0 = -a\); by (A1), \(-b \leq -a\).
(ii) By (i), $0 \leq -c$, so by (O5), $a \cdot (-c) \leq b \cdot (-c)$; by Theorem 3.5(iii), $-ac \leq -bc$; by (i), $bc \leq ac$.

(iii) By(ii).

(iv) If $0 \leq a$, then it follows from (iii); if $a \leq 0$, then by (i), $0 \leq (-a)$; by (iii), $0 \leq (-a) \cdot (-a) = a^2$ (by Theorem 3.5(iv)).

(v) Since $1^2 = 1$, by (v), $0 \leq 1$; also as $0 \neq 1$, we have $0 < 1$.

(vi) Homework.

(vii) Homework.

□

**Definition 3.7.** Let $F$ be an ordered field. Given an element $a \in F$, we can define the absolute value $|a|$ of $a$ as:

$$|a| := a \text{ if } a \geq 0, \text{ and } |a| := -a \text{ if } a < 0.$$  

Here $a \geq 0$ is an equivalent way to write $0 \leq a$.

The absolute value gives rise to the notion of distance.

**Definition 3.8.** Given $a, b \in F$, the distance between them is defined as:

$$\text{dist}(a, b) := |a - b|.$$  

**Theorem 3.9.** Let $F$ be an ordered field. For all $a, b \in F$,

(i) $|a| \geq 0$;

(ii) $|a \cdot b| = |a| \cdot |b|$;

(iii) $|a + b| \leq |a| + |b|$.

**Proof.**

(i) If $a \geq 0$, then by definition $|a| = a \geq 0$; if $a < 0$, then $|a| = -a \geq 0$ by Theorem 3.6(i).

(ii) We can compare both sides in four cases: (1) $a \geq 0, b \geq 0$, (2) $a \geq 0, b < 0$, (3) $a < 0, b \geq 0$, (4) $a < 0, b < 0$.

For case (1): $a \cdot b \geq 0$ by Theorem 3.6(iii), so $|ab| = ab$; on the other hand, $|a| = a, |b| = b$, so $|a| \cdot |b| = ab$.

We will leave other cases as exercises.

(iii) By definition, since either $a = |a|$ or $a = -|a|$, we have

$$-|a| \leq a \leq |a|, \text{ and } -|b| \leq b \leq |b|.$$  

Using (O4),

$$-|a| + (-|b|) \leq -|a| + b \leq a + b \leq |a| + b \leq |a| + |b|.$$  

Therefore
\[-(|a| + |b|) \leq a + b \leq |a| + |b|\]
\[-(|a| + |b|) \leq -(a + b) \leq |a| + |b|\]
\[|a + b| \leq |a| + |b|.

\[\square\]

**Corollary 3.10.** We have the **triangle inequality**:
\[\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)\]

*Proof.*
\[\text{dist}(a, c) = |a - c| = |a - b + b - c| \leq |a - b| + |b - c| = \text{dist}(a, b) + \text{dist}(b, c)\]

\[\square\]
4. Real numbers \( \mathbb{R} \) and the Completeness Axiom

Heuristically, the set of real numbers \( \mathbb{R} \) is the order field \( \mathbb{F} \) containing \( \mathbb{Q} \) with “no gaps”. We will make rigorous what does “no gap” mean.

**Definition 4.1.** Let \( S \subset \mathbb{F} \) be a subset of an ordered field \( \mathbb{F} \), and \( S \neq \emptyset \).

(a) If \( S \) contains a largest element \( s_0 \), [that is to say, \( s_0 \in S \), and \( s \leq s_0, \forall s \in S \)], we call \( s_0 \) the **maximum** of \( S \), and write \( s_0 = \max S \);

(b) If \( S \) contains a smallest element \( s'_0 \), [that is to say, \( s'_0 \in S \), and \( s'_0 \leq s, \forall s \in S \)], we call \( s'_0 \) the **minimum** of \( S \), and write \( s'_0 = \min S \).

**Example 4.2.**

(1) Every finite subset of \( \mathbb{Q} \) has a maximum and a minimum, i.e.

\[
\max \{1, 2, 2.6, 3, 4\} = 4, \quad \text{and} \quad \min \{1, 2, 2.6, 3, 4\} = 1.
\]

(2) Given \( a, b \in \mathbb{F} \), with \( a < b \), we denote by

\[
[a, b] = \{x \in \mathbb{F} : a \leq x \leq b\} - \text{closed interval}, \\
(a, b) = \{x \in \mathbb{F} : a < x < b\} - \text{open interval}, \\
[a, b) = \{x \in \mathbb{F} : a \leq x < b\} - \text{half-open interval}, \\
(a, b] = \{x \in \mathbb{F} : a < x \leq b\} - \text{half-open interval}.
\]

Among them:

\[
\max [a, b] = b, \quad \min [a, b] = a,
\]

\((a, b)\) has no maximum and minimum in general.

(3) The set \( \{r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2}\} \) has no maximum.

These examples say that maximums and minimums may not always exist. Nevertheless, we can generate the concepts to upper and lower bounds.

**Definition 4.3.** Let \( S \subset \mathbb{F} \) be a subset and \( S \neq \emptyset \).

(a) If \( M \in \mathbb{F} \) and \( s \leq M, \forall s \in S \), then \( M \) is called an upper bound of \( S \), and \( S \) is said to be **bounded from above**;

(b) If \( m \in \mathbb{F} \) and \( m \leq s, \forall s \in S \), then \( m \) is called a lower bound of \( S \), and \( S \) is said to be **bounded from below**;

(c) \( S \) is said to be bounded if it is bounded above and below, and this is equivalent to say \( S \subset [m, M] \) for some \( m, M \in \mathbb{F} \).

**Example 4.4.**

(1) If \( S \) has the maximum \( \max S \), then \( \max S \) is an upper bound of \( S \); similarly \( \min S \) is a lower bound of \( S \) if it exists.

(2) \( b \) is an upper bound of the intervals \([a, b], (a, b), (a, b] \) and \([a, b)\).
(3) 2 is an upper bound of \( \{ r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2} \} \); \( \sqrt{2} \) is the least upper bound.

Now we summarize the definition for least upper bound and greatest lower bound.

**Definition 4.5.** Let \( S \subset F \) be a subset and \( S \neq \emptyset \).

(a) If \( S \) is bounded from above and \( S \) has a least upper bound, then we call it the supreme of \( S \) and denote it by \( \sup S \).

(b) If \( S \) is bounded from below and \( S \) has a greatest lower bound, then we call it the infimum of \( S \) and denote it by \( \inf S \).

We have the following criterion for least upper bound: \( M = \sup S \) if and only if

1. \( s \leq M, \forall s \in S \),
2. if \( M_1 < M \), then \( \exists s_1 \in S \), such that \( s_1 > M \).

**Example 4.6.**

(a) If \( S \) has a maximum, then \( \max S = \sup S \), and similarly for the minimum and infimum of \( S \).

(b) \( \sup[a,b] = \sup[a,b] = \sup(a,b) = \sup(a,b) = b \).

(c) Let \( A = \{ r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2} \} \), then \( \sup A = \sqrt{2} \notin A \).

Let us elaborate a bit about this example using decimal expansions: first it is obvious that \( \sqrt{2} \) is an upper bound, and we will explain why it is the least one; the decimal expansion for \( \sqrt{2} \) is 1.4142135623.....; if \( M_1 \) is any number less than \( \sqrt{2} \), even if it is very close to \( \sqrt{2} \), there should exist a first decimal of \( M_1 \) that is less than the corresponding one for \( \sqrt{2} \); for instance, \( M_1 = 1.4141 \), or \( M_1 = 1.414212 \), or \( M_1 = 1.41421355 \); and for these numbers we can find \( s_1 = 1.4142, s_1 = 1.414213, s_1 = 1.41421356 \) that is less than \( \sqrt{2} \) but greater than \( M_1 \).

**Remark 4.7.** By case (c) above, the least upper bound may not belong to \( S \). This is a way to say that there exist gaps in \( F \).

**Completeness Axiom:**

every nonempty subset \( S \subset \mathbb{R} \) that is bounded above has a least upper bound, \( \iff \sup S \) exists and is a real number.

**Definition 4.8.** The set of real numbers \( \mathbb{R} \) is an ordered field containing \( \mathbb{Q} \) that satisfies the “Completeness Axiom”.
Remark 4.9. The example \( A = \{ r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2} \} \) shows that \( \mathbb{Q} \) does not satisfy the “Completeness Axiom”.

**Corollary 4.10.** Every nonempty subset \( S \subseteq \mathbb{R} \) which is bounded from below has a greatest upper bound.

**Proof.** Consider the subset, denoted by \( -S = \{ -s : s \in S \} \). Since \( S \) is bounded from below, \( \exists m \in \mathbb{R}, \) such that \( m \leq s, \forall s \in S, \implies -s \leq -m, \forall -s \in -S. \) Thus \( -S \) is bounded from above.

By the “Completeness Axiom”, the supreme of \( -S \) exists and \( \sup(-S) \in \mathbb{R} \).

Let \( s_0 = \sup(-S) \), and we claim that \( -s_0 = \inf S \). In particular,

1. \( -s \leq s_0, \forall -s \in -S, \implies -s_0 \leq s, \forall s \in S, \) and hence \( -s_0 \) is a lower bound of \( S \);
2. To show that \( -s_0 \) is the greatest lower bound, consider any other lower bound \( t \in \mathbb{R}, \) such that \( t \leq s, \forall s \in S, \implies -s \leq -t, \forall -s \in -S; \) thus \( -t \) is an upper bound of \( -S, \) so \( s_0 \leq -t, \implies t \leq -s_0; \) therefore \( -s_0 \) is the greatest lower bound.

There are two important corollaries of the completeness axiom.

**Theorem 4.11** (Archimedean Property). Given two real numbers \( a, b \in \mathbb{R}, \) if \( a > 0 \) and \( b > 0, \) then for some positive integer \( n \in \mathbb{N}, \) we have \( na > b. \)

**Proof.** We use the contradiction argument. Suppose the conclusion is not true, which means that \( na \leq b, \forall n \in \mathbb{N}. \) Denote the set \( S \) as

\[
S = \{ na : n \in \mathbb{N} \}.
\]

Then \( b \) is an upper bound of \( S. \) By the completeness axiom: the supreme of \( S \) exists and \( s_0 = \sup S \in \mathbb{R}. \)

Since \( a > 0, \) then \( s_0 < s_0 + a, \) or equivalently \( s_0 - a < s_0. \) By the definition of supreme, we know that \( s_0 - a \) is no longer an upper bound of \( S, \) so there exists an element \( n_0 a, \) for some \( n_0 \in \mathbb{N}, \) such that

\[
s_0 - a < n_0 a.
\]

\[
\implies s_0 < n_0 a + a = (n_0 + 1)a \in S.
\]

There is a contradiction to the fact that \( s_0 \) is an upper bound of \( S. \)

**Corollary 4.12.**

1. if \( a > 0, \) then \( 1/n (= n^{-1}) < a \) for some positive integer \( n \in \mathbb{N}; \)
2. if \( b > 0, \) then \( b < n \) for some positive integer \( n \in \mathbb{N}. \)
Proof.

(1) follows from the Archimedean Property by letting $b = 1$, and (2) by letting $a = 1$. 

\[ \square \]

**Theorem 4.13** (Denseness of $\mathbb{Q}$). Given two real numbers $a, b \in \mathbb{R}$, if $a < b$, then there exists a rational number $r \in \mathbb{Q}$, such that $a < r < b$.

**Proof.** We need to find some integers $m, n \in \mathbb{Z}$, $n > 0$, such that $a < \frac{m}{n} < b$, and this is equivalent to

$$na < m < nb.$$ 

Since $a < b \Rightarrow (b - a) > 0$, by the Archimedean property, there exists $n \in \mathbb{N}$, such that

$$n(b - a) > 1.$$ 

By the Archimedean property again, there exists $k \in \mathbb{N}$, such that

$$k > \max\{|na|, |nb|\}.$$ 

Therefore,

$$-k < na < nb < k.$$ 

Consider the set \( \{ j \in \mathbb{Z} : -k < j \leq k, \text{ and } na < j \} \), and it is a finite and nonempty set (since $k$ belongs to it). As it is finite, we pick the minimum

$$= \min\{ j \in \mathbb{Z} : -k < j \leq k, \text{ and } na < j \}.$$ 

Then $na < m$, and moreover, $m - 1 \leq na$. Therefore

$$na < m = (m - 1) + 1 \leq na + 1 < na + (nb - na) = nb.$$ 

So the pair $(m, n)$ satisfies the requirement. 

\[ \square \]
5. $+\infty$ AND $-\infty$

We add in two elements $+\infty$ and $-\infty$ into $\mathbb{R}$ so that $\mathbb{R} \cup \{+\infty, -\infty\}$ is still a good ordered system, and it makes taking supreme and infimum easier in many cases.

- $+\infty$ and $-\infty$ denote plus infinity and minus infinity;
- we will write $+\infty$ as $\infty$;
- we can add the ordering to $\mathbb{R} \cup \{+\infty, -\infty\}$ so that $\forall a \in \mathbb{R} \cup \{+\infty, -\infty\}$,
\[-\infty \leq a \leq +\infty;
- this ordering system satisfies O1, O2, O3;
- $+\infty, -\infty$ do not represent any real numbers, and there is no algebraic structure like “+” and “.” on $\mathbb{R} \cup \{+\infty, -\infty\}$;
- we can introduce new notations:
\[
(a, \infty) = \{x \in \mathbb{R} : a < x\} - \text{unbounded open interval},
(b, \infty) = \{x \in \mathbb{R} : a \leq x\} - \text{unbounded closed interval},
(\infty, b] = \{x \in \mathbb{R} : x \leq b\} - \text{unbounded closed interval},
(\infty, b) = \{x \in \mathbb{R} : x < b\} - \text{unbounded open interval},
(\infty, +\infty) = \mathbb{R};
- given an nonempty subset $S \subset \mathbb{R}$, denote
\[
\sup S = +\infty, \quad \text{if } S \text{ is not bounded from above},
\inf S = -\infty, \quad \text{if } S \text{ is not bounded from below};
- therefore, if $S \subset \mathbb{R}$ and $S \neq \emptyset$, then
\[
\sup S \text{ and } \inf S \text{ always make sense in } \mathbb{R} \cup \{+\infty, -\infty\}.
6. LIMITS OF SEQUENCES

We introduce the concept of sequences and their limits in this section.

- A sequence is a function, with domain a set of natural numbers of the form \( \{ n \in \mathbb{Z} : n \geq m \} \), where \( m \) is usually chosen to be 1 or 0, with ranges in \( \mathbb{R} \).
- Denote a sequence by \((s_n)_{n=m}^{\infty}, s_n \in \mathbb{R}\), or \((s_m, s_{m+1}, s_{m+2}, \cdots)\).
- If \( m = 2 \), write \((s_n)_{n \in \mathbb{N}}\).

Example 6.1.  

a) Let \( s_n = \frac{1}{n^2} \), then the sequence is \((1, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \cdots) = (1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \cdots)\). The set of values is \(\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \cdots\}\).

b) Let \( a_n = (-1)^n \) for \( n \geq 0 \), then the sequence is \((1, -1, 1, -1, \cdots)\), and the set of values is \(\{1, -1\}\).

Remark 6.2. Note the difference between a sequence and its set of values: we use \((\cdot, \cdot, \cdots)\) to denote a sequence, and \(\{\cdot, \cdot, \cdots\}\) to denote the set of values. In particular, the sequence \((a_n = (-1)^n)\) has infinite number of terms, while its set of values \(\{(−1)^n\}\) consists of only two numbers.

c) Consider the sequence \((\cos(\frac{n\pi}{3}))_{n \in \mathbb{N}}\). It can be written as \((\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \cdots)\). The set of values is \(\{1, -1, \frac{1}{2}, \frac{1}{2}\}\).

d) Let \( s_n = n^{1/n} \) for \( n \in \mathbb{N} \), then the sequence is \((1, \sqrt{2}, 3^{1/3}, \cdots)\). We will show that \( n^{1/n} \) is close to 1 for very large \( n \).

e) Let \((s_n = (1 + \frac{1}{n})^n)_{n \in \mathbb{N}}\).

The limit of a sequence \((s_n)\) is intuitively a real number for which the values \(s_n\) accumulate to. To be more precise,

Definition 6.3. A sequence \((s_n)\) of real numbers is said to converge to the real number \( s \), provided that

\[
\text{(6.1)} \quad \text{for each } \epsilon > 0, \text{ there exists a number } N, \text{ such that } n > N \text{ implies } |s_n - s| < \epsilon.
\]

If \((s_n)\) converges to \( s \), we will write \( \lim_{n \to \infty} s_n = s \), or \( s_n \to s \). \( s \) is called the limit of the sequence \((s_n)\).

A sequence that does not converge to some real number is said to diverge.

Remark 6.4.  

(1) \( \epsilon, \delta \) represent small numbers;

(2) \( \text{(6.1)} \) is an infinite number of statements. We sometime denote \( N = N(\epsilon) \) to show the dependence of \( N \) on \( \epsilon \), and in principle \( N(\epsilon) \) is large when \( \epsilon \) is small;
(3) \( N \) can be any large real or rational numbers by the Archimedean property.

**Example 6.5.**

a) \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \), and this will be proved soon.

b) The sequence \( (a_n = (-1)^n) \) diverges:

**Proof.** Assume by contradiction that the sequence converges to some real number \( s \in \mathbb{R} \), \( \lim (-1)^n = s \). Since (6.1) is assumed to be true for any \( \epsilon > 0 \), we choose \( \epsilon = \frac{1}{2} \). By (6.1), there exists a number \( N > 0 \), such that when \( n > N \), we have

\[
|(-1)^n - s| < \frac{1}{2} \implies s - \frac{1}{2} < (-1)^n < s + \frac{1}{2}.
\]

In particular, we can let \( n = 2N \) or \( n = 2N + 1 \) which are both greater than \( n \), and then

\[
\begin{align*}
\text{When } n &= 2N, s - \frac{1}{2} < 1 < s + \frac{1}{2} \implies \frac{1}{2} < s < \frac{3}{2}, \\
\text{When } n &= 2N + 1, s - \frac{1}{2} < -1 < s + \frac{1}{2} \implies -\frac{3}{2} < s < -\frac{1}{2}.
\end{align*}
\]

This is a contradiction, so we finish the proof. \( \square \)

c) The sequence \( \left( \cos \left( \frac{2n\pi}{3} \right) \right)_{n \in \mathbb{N}} \) diverges, and this is an exercise.

d) \( \lim n^{\frac{1}{n}} = 1 \), and this will be proved later.

e) \( \lim s_n = (1 + \frac{1}{n})^n = e \), and this will be proved later.

Using the definition, we can show that the limit, if it exists, is uniquely determined.

**Proposition 6.6.** Let \( (s_n) \) be a sequence, and assume that \( \lim s_n = s \) and \( \lim s_n = t \) for some real numbers \( s, t \in \mathbb{R} \), then \( s = t \).

**Proof.** By (6.1) and the assumptions, for any \( \epsilon > 0 \), there exists \( N_1 > 0 \), such that

\[
n > N_1 \implies |s_n - s| < \frac{\epsilon}{2};
\]

and there exists \( N_2 > 0 \), such that

\[
n > N_2 \implies |s_n - t| < \frac{\epsilon}{2}.
\]

So for \( n > \max\{N_1, N_2\} \),

\[
|s - t| \leq |s - s_n| + |s_n - t| < \epsilon.
\]

Therefore \( s = t \). (If this were not true, then take \( \epsilon = \frac{1}{2}|s - t| \) to get a contradiction.) \( \square \)
7. Proofs

The main purpose of this section is to introduce how to write rigorous proofs for limits.

Example 7.1. \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \).

**Discussion:** given \( \epsilon > 0 \), we want to find \( N > 0 \), such that if \( n > N \), then
\[
\left| \frac{1}{n^2} - 0 \right| < \epsilon.
\]

**Algebra:**
\[
\left| \frac{1}{n^2} - 0 \right| < \epsilon\iff \frac{1}{n^2} < \epsilon \iff n^2 > \frac{1}{\epsilon}, \text{ or } n > \frac{1}{\sqrt{\epsilon}}.
\]
Take \( N = \frac{1}{\sqrt{\epsilon}} \), then it should serve our purpose.

**Proof.** For any \( \epsilon > 0 \), let \( N = \frac{1}{\sqrt{\epsilon}} \). If \( n > N \), then \( n > \frac{1}{\sqrt{\epsilon}} \), so \( n^2 > \frac{1}{\epsilon} \), hence \( \epsilon > \frac{1}{n^2} \).

Thus \( n > N \) implies \( \left| \frac{1}{n^2} - 0 \right| < \epsilon \), and this proves \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \). \( \Box \)

Example 7.2. \( \lim_{n \to \infty} \frac{3n+1}{7n-4} = \frac{3}{7} \).

**Discussion:** given \( \epsilon > 0 \), we want to find \( N > 0 \), such that if \( n > N \), then
\[
\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \epsilon.
\]

**Algebra:**
\[
\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| = \left| \frac{19}{7(7n-4)} \right|.
\]
\[
\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \epsilon\iff \frac{19}{7(7n-4)} < \epsilon \iff \frac{19}{7n-4} < \frac{1}{\epsilon} \iff \frac{19}{7n} < 7n - 4 \iff 7n > \frac{19}{7} + 4 \iff n > \frac{19}{49} + \frac{4}{7}.
\]
So put \( N = \frac{19}{49} + \frac{4}{7} \).

**Proof.** For any \( \epsilon > 0 \), let \( N = \frac{19}{49} + \frac{4}{7} \). If \( n > N \), then we can reverse all steps above to show that \( \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \epsilon \). \( \Box \)

Example 7.3. \( \lim_{n \to \infty} \frac{4n^3+3n}{n^3-6} = 4 \).

**Discussion:** given \( \epsilon > 0 \), we want to find \( N > 0 \), such that if \( n > N \), then
\[
\left| \frac{4n^3+3n}{n^3-6} - 4 \right| < \epsilon, \iff \left| \frac{3n+24}{n^3-6} \right| < \epsilon.
\]

There is no way to easily solve this inequality for \( n \) by \( \epsilon \), so we need to do some rough estimates:

1. \( 3n + 24 \leq 3n + 24n \leq 27n \);
2. \( n^3 - 6 > n^3/2 \) provide \( n \) is large, actually this is equivalent to \( n^3 > 12 \), which holds true when \( n > 2 \).
By what we have above, we have

\[ 7.1 \iff \frac{27n}{n^3/2} < \epsilon, \text{ and } n > 2; \]
\[ \iff \frac{54}{n^2} < \epsilon, \text{ and } n > 2; \]
\[ \iff n > \sqrt{\frac{54}{\epsilon}}, \text{ and } n > 2. \]

**Proof.** For any \( \epsilon > 0 \), let \( N = \max\{\sqrt{\frac{54}{\epsilon}}, 2\} \). If \( n > N \), then we can reverse all steps above to (7.1).

**Example 7.4.** Let \( (s_n) \) be a sequence of nonnegative numbers, i.e. \( s_n \geq 0 \). Suppose that \( \lim_{n \to \infty} s_n = s \). (Show that \( s \geq 0 \) as an exercise). Prove that \( \lim \sqrt{s_n} = \sqrt{s} \).

**Discussion:** given \( \epsilon > 0 \), we want to find \( N > 0 \), such that if \( n > N \), then

\[ |\sqrt{s_n} - \sqrt{s}| < \epsilon. \]

Note that

\[ \sqrt{s_n} - \sqrt{s} = \frac{s_n - s}{\sqrt{s_n} + \sqrt{s}}. \]

We will divide the discussion into two cases:

(1) if \( s > 0 \), then \( \sqrt{s_n} + \sqrt{s} \geq \sqrt{s} \), hence

\[ |\sqrt{s_n} - \sqrt{s}| \leq \frac{s_n - s}{\sqrt{s}}. \]

By the assumption \( \lim_{n \to \infty} s_n = s \), we can select \( N > 0 \), such that \( n > N \implies |s_n - s| < \sqrt{s} \epsilon \), and this will serve our purpose.

(2) The case when \( s = 0 \) will be left as homework.

**Proof.** We only discuss the case when \( s > 0 \) here. For any \( \epsilon > 0 \), since \( \lim_{n \to \infty} s_n = s \), there exists \( N > 0 \), such that \( n > N \implies |s_n - s| < \sqrt{s} \epsilon \). This implies that

\[ |\sqrt{s_n} - \sqrt{s}| \leq \frac{s_n - s}{\sqrt{s}} < \epsilon, \]

and this finishes the proof.

**Example 7.5.** Let \( (s_n)_{n \in \mathbb{N}} \) be a sequence of real numbers such that \( s_n \neq 0 \) for all \( n \in \mathbb{N} \) and \( \lim s_n = s \neq 0 \). Prove that

\[ \inf\{|s_n| : n \in \mathbb{N}\} > 0. \]
Discussion: Let $\epsilon = |s|/2$ in the definition (6.1). For $N = N(\epsilon)$ in (6.1), if $n > N$, then we have

$$|s_n - s| < \epsilon \implies s_n \in (s - \epsilon, s + \epsilon).$$

Proof. Let $\epsilon = |s|/2$. Since $\lim s_n = s$, there exists $N > 0$, such that

$$n > N \implies |s_n - s| < \epsilon = \frac{|s|}{2}.$$

So $n > N$ implies that

$$|s_n| = |s + s_n - s| \geq |s| - \frac{|s|}{2} = \frac{|s|}{2}.$$

Now set

$$m = \min\{\frac{|s|}{2}, |s_1|, |s_2|, \ldots, |s_N|\},$$

then $|s_n| \geq m$, for all $n \in \mathbb{N}$, and this finishes the proof. \qed
8. Problem session

(1) (4.15) Let $a, b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$.

Proof. Suppose not. Then $a > b$, or $a - b > 0$. By the Archimedean Property, there exists $n \in \mathbb{N}$, such that

$$\frac{1}{n} < a - b.$$ 

And this is equivalent to $a > b + \frac{1}{n}$, but this is a contradiction. \(\square\)

(2) Show that the Archimedean Property of the real numbers holds if and only if the set of natural numbers $\mathbb{N}$ is not bounded above.

Proof. First assume that the Archimedean Property holds. Then for any positive number $M > 0$, there exists a $n \in \mathbb{N}$, such that $M < n$. This means that $\mathbb{N}$ is not bounded above.

Now assume that $\mathbb{N}$ is not bounded above, and this means that for any $N > 0$, there exists $n \in \mathbb{N}$, such that $N < n$. Now given any two numbers $a, b \in \mathbb{R}$, $a > 0$, $b > 0$. Consider the number $N = b/a$, so there exists $n \in \mathbb{N}$, such that $N = b/a < n$, which is equivalent to $na > b$. \(\square\)

(3) (HW3.2) Show that if $S \subset \mathbb{Z}$ is bounded above then $S$ has a maximum, i.e., $\sup S \in S$.

(4) Let $x \in \mathbb{R}$. We define the Greatest Integer function of $x$ as the largest integer $m \in \mathbb{Z}$ such that $m \leq x$; it is denoted by $[x]$ and satisfies the inequality

$$[x] \leq x < [x] + 1.$$ 

For instance, $[0.1] = 0$, $[1.1] = 1$, $[-2.1] = -3$, $[5] = 5$, $[-4] = -4$, etc. Show that for any real number $x \in \mathbb{R}$, $[x]$ exists. Hint: Consider the set

$$S = \{n \in \mathbb{Z}: n \leq x\}.$$

Then consider two cases depending on whether $x \geq 0$ or $x < 0$. In the case when $x < 0$, use the Archimedean property of $\mathbb{R}$ to show that $S$ is nonempty. Then apply the completeness axiom and finally show that $\sup S = [x]$, i.e., show that $\sup S$ is an integer (this follows from Problem 3) and it satisfies

$$\sup S \leq x < \sup S + 1.$$
Proof. Let us only consider the case when $x \geq 0$, then $0 \in S$, hence $S \neq \emptyset$. Since $S$ is bounded from above, ($x$ is an upper bound), by the Completeness Axiom, sup $S$ exists. By Problem 3, sup $S \in S$, and sup $S = \max S$. Then the only thing we need to prove is

$$x < \text{sup} \ S + 1.$$ 

Assume by the contrary that $x \geq \text{sup} \ S + 1$. Then since sup $S$ is an integer, and hence sup $S + 1$ is an integer, so by the definition of $S$, we have

$$\text{sup} \ S + 1 \in S.$$ 

This is a contradiction to the fact that sup $S = \max S$. $\square$
9. Limit theorems for sequences

**Definition 9.1.** A sequence \((s_n)\) of real numbers is said to be bounded if the set of values \(\{s_n : n \in \mathbb{N}\}\) is bounded. This is equivalent to saying that:

\[\exists M \in \mathbb{R}, \text{ such that } |s_n| \leq M \text{ for all } n \in \mathbb{N}.\]

**Theorem 9.2.** Convergent sequences are bounded.

*Proof.* Let \((s_n)\) be a convergent sequence, and let \(s\) be the limit, or equivalently \(s = \lim s_n\).

Apply the definition of convergence with \(\epsilon = 1\), then we get some large number \(N \in \mathbb{N}\), such that

\[n > N \text{, then } |s_n - s| < 1.\]

By the triangle inequality, \(|s_n - s| < 1 \implies |s_n| \leq |s + s_n - s| \leq |s| + |s_n - s| \leq |s| + 1.\]

Define

\[M = \max\{|s_1|, |s_2|, \ldots, |s_N|, |s| + 1\}.\]

Then we have

\[|s_n| \leq \begin{cases} M & \text{if } n \leq N \\ |s| + 1 & \text{if } n > N \end{cases}.\]

Therefore \(|s_n| \leq M\) for all \(n \in \mathbb{N}\), and so \(\{s_n\}\) is bounded. \(\square\)

**Remark 9.3.** In the proof above, we only apply the defining condition (6.1) for a single \(\epsilon = 1\). Actually, we can take \(\epsilon\) to be any finite positive number in the proof, and this is left as an exercise.

**Theorem 9.4.** If a sequence \((s_n)\) converges to \(s\), and \(k \in \mathbb{R}\), then the sequence \((ks_n)\) converges to \(ks\). That is,

\[\lim(ks_n) = k \lim s_n.\]

*Proof.* If \(k = 0\), the proof is trivial as \(ks_n \equiv 0\), and \(\lim(0 \cdot s_n) = 0 = 0 \cdot \lim s_n\).

Assume \(k \neq 0\) now. Let \(\epsilon > 0\) be any positive real number, and we need to show that

\[|ks_n - ks| < \epsilon, \text{ for large } n \in \mathbb{N}.\]

Note that the above inequality is equivalent to saying that

\[|s_n - s| < \frac{\epsilon}{|k|}, \text{ for large } n \in \mathbb{N}.\]
Since \( \lim s_n = s \), for \( \frac{\varepsilon}{|k|} > 0 \), by (6.1) there exists \( N > 0 \), such that
\[
\text{if } n \geq N, \text{ then } |s_n - s| < \frac{\varepsilon}{|k|}.
\]
Therefore for \( n > N \), we have \( |ks_n - ks| < \varepsilon \).

\[ \square \]

**Theorem 9.5.** If \( (s_n) \) converges to \( s \) and \( (t_n) \) converges to \( t \), then \( (s_n + t_n) \) converges to \( s + t \). That is to say,
\[
\lim (s_n + t_n) = \lim s_n + \lim t_n.
\]

**Proof.** Let \( \varepsilon > 0 \) be any positive real number, and we want to show that
\[
|(s_n + t_n) - (s + t)| < \varepsilon, \text{ for all large } n \in \mathbb{N}.
\]
Note that
\[
|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t|.
\]

(\textbf{Hint:} so we can try to show \( |s_n - s| < \varepsilon/2 \) and \( |t_n - t| < \varepsilon/2 \).)

Since \( \lim s_n = s \), \( \exists N_1 \in \mathbb{N} \), such that
\[
\text{if } n \geq N_1, \text{ then } |s_n - s| < \varepsilon/2.
\]
Similarly by \( \lim t_n = t \), \( \exists N_2 \in \mathbb{N} \), such that
\[
\text{if } n \geq N_2, \text{ then } |t_n - t| < \varepsilon/2.
\]

Let
\[
N = \max\{N_1, N_2\}.
\]
Then if \( n > N \), we have
\[
|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \varepsilon.
\]

\[ \square \]

**Theorem 9.6.** If \( (s_n) \) converges to \( s \) and \( (t_n) \) converges to \( t \), then \( (s_n \cdot t_n) \) converges to \( s \cdot t \). That is to say,
\[
\lim (s_n \cdot t_n) = (\lim s_n) \cdot (\lim t_n).
\]

**Discussion:**
\[
|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \leq |s_n||t_n - t| + |t||s_n - s| \leq M|t_n - t| + |t||s_n - s|.
\]

- In the above inequality, \( |t_n - t| \) and \( |s_n - s| \) can be very small for large \( n \);
- \( |s_n| \) is bounded by Theorem 9.2 i.e. \( |s_n| \leq M \).
Proof. Let $\epsilon > 0$ be any positive real number. By Theorem 9.2, $\exists M > 0$, such that $|s_n| \leq M$, for all $n \in \mathbb{N}$.

Since $\lim t_n = t$, $\exists N_1 \in \mathbb{N}$, such that

if $n \geq N_1$, then $|t_n - t| < \epsilon/(2M)$.

Similarly by $\lim s_n = s$, $\exists N_2 \in \mathbb{N}$, such that

if $n \geq N_2$, then $|s_n - s| < \frac{\epsilon}{2(|t| + 1)}$.

Let $N = \max\{N_1, N_2\}$.

Then if $n > N$, we have

$$|s_n t_n - st| \leq |s_n||t_n - t| + |t||s_n - s|$$

$$\leq M \cdot \frac{\epsilon}{2M} + |t| \cdot \frac{\epsilon}{2(|t| + 1)} < \epsilon.$$

\[ \square \]

Lemma 9.7. If $(s_n)$ converges to $s$ and if $s_n \neq 0$ for all $n \in \mathbb{N}$, and if $s \neq 0$, then

$$\lim \frac{1}{s_n} = \frac{1}{s}.$$

Discussion:

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s_n - s}{s_n s} \right| \leq \frac{|s_n - s|}{m|s|}.$$ 

We know that $|s_n - s|$ is very small for $n$ very large. To show the right hand side is small, we need the denominator $|s_n s|$ to be bounded away from 0. By Example 7.5, we do have that

$$m = \inf \{|s_n| : n \in \mathbb{N}\} > 0.$$ 

Proof. Let $\epsilon > 0$ be any positive real number, and $m$ be defined as above. Since $\lim s_n = s$, there exists $N > 0$, such that

if $n \geq N$, then $|s_n - s| < \epsilon \cdot m \cdot |s|$.

Therefore if $n \geq N$, then

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| \leq \frac{\epsilon \cdot m \cdot |s|}{|s_n| \cdot |s|} < \epsilon.$$

\[ \square \]
Theorem 9.8. Suppose \( \lim s_n = s \) and \( \lim t_n = t \). If \( s \neq 0 \) and \( s_n \neq 0 \) for all \( n \in \mathbb{N} \), then
\[
\lim \left( \frac{t_n}{s_n} \right) = \frac{t}{s}.
\]

Proof. By the previous lemma, \( \lim \frac{1}{s_n} = \frac{1}{s} \). Then
\[
\lim \frac{t_n}{s_n} = (\lim t_n)(\lim \frac{1}{s_n}) = \frac{t}{s}.
\]

\[\square\]

A few basic examples:

a) \( \lim \frac{1}{n^p} = 0 \) for \( p > 0 \) (assume \( p \in \mathbb{N} \) at this moment);

b) \( \lim a^n = 0 \) if \( |a| < 1 \);

c) \( \lim n^{1/n} = 1 \);

d) \( \lim a^{1/n} = 1 \) for \( a > 0 \).

Proof. a). Let \( \epsilon > 0 \) be any positive real number. (We want to show \( \frac{1}{n^p} < \epsilon \), which is equivalent to \( n > (\frac{1}{\epsilon})^{1/p} \).)

Let \( N = (\frac{1}{\epsilon})^{1/p} \),

(Exercise: use the completeness axiom to show that: “for any \( n \in \mathbb{N} \), and \( a > 0 \), there exists \( x > 0 \), such that \( x^n = a \), and we write \( x = a^{1/n} \).”)

then if \( n > N \), we have \( \frac{1}{n^p} < \epsilon \). Therefore
\[
\left| \frac{1}{n^p} - 0 \right| = \frac{1}{n^p} < \epsilon.
\]

b). If \( a = 0 \), then \( a^n = 0 \), and the proof is trivial.

If \( a \neq 0 \) and \( |a| < 1 \), then \( |a| = \frac{1}{1+b} \) for some \( b > 0 \). Using the binomial expansion,
\[
(1 + b)^n \geq 1 + nb > nb.
\]

So
\[
|a^n = 0| = |a|^n = \frac{1}{(1+b)^n} < \frac{1}{nb}.
\]

Let \( \epsilon > 0 \) be any positive real number, and let \( N = \frac{1}{\epsilon b} \). Then if \( n > N \), we have
\[
|a^n = 0| < \frac{1}{nb} < \epsilon.
\]

c). Let \( s_n = n^{1/n} - 1 \). Note that \( s_n \geq 0 \). It suffices to show \( \lim s_n = 0 \). Note that
\[
1 + s_n = n^{1/n} \iff (1 + s_n)^n = n.
\]
By the binomial expansion:
\[ n = (1 + s_n)^n = 1 + n \cdot s_n + \frac{1}{2}n(n + 1)s_n^2 > \frac{1}{2}n(n + 1)s_n^2. \]
So
\[ s_n^2 < \frac{2}{n - 1}, \implies s_n < \sqrt{\frac{2}{n - 1}}. \]

Exercise: use the above formula to show \( \lim s_n = 0. \)

d). First assume that \( a \geq 1. \) Then for \( n \geq a, \) we have
\[ 1 \leq a^{1/n} \leq n^{1/n}. \]

Exercise: use the fact \( \lim n^{1/n} = 1 \) to prove \( \lim a^{1/n} = 1. \)

(Hint: let \( \epsilon > 0 \) be any positive real number, there exists \( N > 0, \) such that
\[ n > N \implies n^{1/n} - 1 < \epsilon \implies a^{1/n} - 1 < \epsilon.) \]

Suppose \( 0 < a < 1, \) then \( \frac{1}{a} > 1. \) So \( \lim \left(\frac{1}{a}\right)^{1/n} = 1. \) By Lemma 9.7, we can get \( \lim a^{1/n} = 1. \)

Definition 9.9. A sequence \( (s_n) \) is said to diverges to \( +\infty, \) provided:
\[ (9.1) \quad \text{for each } M > 0, \exists N > 0, \text{ such that } n > N, \implies s_n > M. \]

We write \( \lim s_n = +\infty. \)

Similarly we write \( \lim s_n = -\infty, \) provided
\[ (9.2) \quad \text{for each } M < 0, \exists N > 0, \text{ such that } n > N, \implies s_n < M. \]

Example 9.10. Prove that \( \lim (\sqrt{n} + 5) = +\infty. \)

Discussion: consider an arbitrarily large \( M > 0, \) we need to find \( N > 0, \) such that
\[ n > N \implies \sqrt{n} + 5 > M \iff n > (M - 5)^2. \]

Proof. Given \( M > 0, \) let \( N = (M - 5)^2, \) then \( n > N \) implies \( \sqrt{n} + 5 > M. \)

Theorem 9.11. Assume that \( \lim s_n = +\infty \) and \( \lim t_n > 0, \) then
\[ \lim (t_n \cdot s_n) = +\infty. \]

Discussion: consider an arbitrarily large \( M > 0, \) we want \( (s_n \cdot t_n) > M. \)
- Since \( \lim s_n = +\infty, \) \( s_n \) can be as large as we want when \( n \) is large, so we need to show that \( t_n \)'s are bounded away from 0;
- Choose \( 0 < m < \lim t_n, \) and observe that \( t_n > m \) for \( n \) large;
- Then we only need \( s_n > M/m \) for large \( n. \)
Proof. Let \( M > 0 \) be an arbitrarily large number. Select \( m > 0 \), such that
\[
0 < m < \lim t_n.
\]
By Problem 8.10 in textbook, \( \exists N_n > 0 \), such that
\[
n > N_1 \implies t_n > m.
\]
Since \( \lim s_n = +\infty \), \( \exists N_2 > 0 \), such that
\[
n > N_2 \implies s_n > \frac{M}{m}.
\]
Let \( N = \max\{N_1, N_2\} \), then
\[
n > N \implies s_n \cdot t_n > \frac{M}{m} \cdot m = M.
\]
\( \square \)

**Theorem 9.12.** Let \( (s_n) \) be a sequence of positive numbers, then
\[
\lim s_n = +\infty \iff \lim \left( \frac{1}{s_n} \right) = 0.
\]

*Proof.*

1. Assume \( \lim s_n = +\infty \). Given any \( \epsilon > 0 \), let \( M = \frac{1}{\epsilon} > 0 \), then \( \exists N_1 \), such that
\[
n > N_1 \implies s_n > M = \frac{1}{\epsilon}, \implies \frac{1}{s_n} < \epsilon.
\]

2. Assume \( \lim \left( \frac{1}{s_n} \right) = 0 \). Given \( M > 0 \), let \( \epsilon = \frac{1}{M} \), then \( \exists N_2 \), such that
\[
n > N_2 \implies \frac{1}{s_n} > M = \frac{1}{\epsilon}, \implies s_n > M.
\]
\( \square \)
10. Monotone sequences and Cauchy sequences

**Definition 10.1.** Let \((s_n)\) be a sequence.

- \((s_n)\) is called a **non-decreasing sequence** if \(s_n \leq s_{n+1}\) for all \(n \in \mathbb{N}\).
- \((s_n)\) is called a **non-increasing sequence** if \(s_n \geq s_{n+1}\) for all \(n \in \mathbb{N}\).

Such sequences are called **monotone sequences**.

**Example 10.2.**

- \(a_n = 1 - \frac{1}{n}\), \(b_n = n^3\), \(c_n = (1 + \frac{1}{n})^n\) are non-decreasing sequences;
- \(d_n = \frac{1}{n^2}\) is a non-increasing sequence;
- \(s_n = (-1)^n\), \(t_n = \cos\left(\frac{n\pi}{3}\right)\), \(u_n = (-1)^n\cdot n\), \(v_n = \frac{(-1)^n}{n}\) are not monotone sequences;
- \(x_n = n^{1/n}\) is not monotone.

**Theorem 10.3.** All bounded monotone sequences converge.

**Proof.** Let \((s_n)\) be a bounded non-decreasing sequence. Consider the set of values:

\[ S = \{ s_n : n \in \mathbb{N} \}. \]

Since \((s_n)\) is bounded, the set \(S\) is also bounded. By the Completeness Axiom, we can take

\[ u = \sup S. \]

**Claim:** \(\lim s_n = u\).

Given \(\epsilon > 0\), since \(u\) is the supremum, there exists \(N \in \mathbb{N}\), such that

\[ s_N > u - \epsilon. \]

Since \((s_n)\) is non-decreasing,

\[ \forall n > N, \quad s_n \geq s_N > u - \epsilon. \]

On the other hand, since \(u\) is an upper bound for \(S\), we have

\[ \forall n \in \mathbb{N}, \quad s_n \leq u. \]

Combining the two inequalities above, \(\forall n > N,\)

\[ u - \epsilon < s_n \leq u \implies |s_n - u| < \epsilon. \]

If \((s_n)\) is a bounded non-increasing sequence, the proof is similar and will be left as an exercise. (Hint: let \(u = \inf S\).) \(\square\)

**Decimals:**
Different decimal expansions can represent the same real number;
Focus on non-negative decimals expansions and non-negative real numbers.

Given a decimal:
\[ k.d_1d_2d_3d_4\cdots, k \in \mathbb{Z}_+, d_j \in \{0, 1, 2, \cdots, 9\} \]

Let
\[ s_n = k + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}, \]
then \((s_n)\) is a non-decreasing sequence, and \((s_n)\) is bounded from above by \(k + 1\). By Theorem \[10.3\] \((s_n)\) converges and we denote the limit by
\[ s = k.d_1d_2d_3d_4\cdots. \]

**Theorem 10.4.**

1. If \((s_n)\) is an unbounded non-decreasing sequence, then \(\lim s_n = +\infty\);
2. If \((s_n)\) is an unbounded non-increasing sequence, then \(\lim s_n = -\infty\).

**Proof.**

1. Since \((s_n)\) is unbounded, and \(s_n \geq s_1\), \((s_n)\) is unbounded from above. This implies that:
   
   for any \(M > 0\), \(\exists N > 0\), such that \(s_N > M\).

   Since \((s_n)\) is non-decreasing,
   
   \[ \forall n > N, \quad s_n \geq s_N > M. \]

2. Exercise.

**Corollary 10.5.** If \((s_n)\) is a monotone sequence, then it either converges, diverges to \(+\infty\), or diverges to \(-\infty\). Therefore \(\lim s_n\) always is meaningful.

**lim sup and lim inf:**

Let \((s_n)\) be a bounded sequence. It may or may not converge. The limiting behavior of \((s_n)\) depends only on the terms \(\{s_n : n > N\}\) for \(N\) large. Let \(N \in \mathbb{N}\) be an arbitrary natural number:

**Claim 1:** if \(\lim s_n\) exists, then \(\lim s_n \in [u_N, v_N]\), where
\[ u_N = \inf\{s_n : n > N\}, \quad v_N = \sup\{s_n : s > N\}. \]
Claim 2: we have:
\[ u_1 \leq u_2 \leq u_3 \leq \cdots; \]
\[ v_1 \geq v_2 \geq v_3 \geq \cdots. \]

Note that as \( N \) increases, the sets \( S_N = \{s_n : n > N\} \) get smaller:
\[ S_1 \supset S_2 \supset S_3 \supset \cdots. \]

The above inequalities follow by Problem 4.7 in textbook.

By Theorem 10.3 the following limits exist:
\[ u = \lim u_N, \quad v = \lim v_N, \quad \text{and} \quad u \leq v. \]

Moreover,
\[ \text{if } \lim s_n \text{ exists, } \longrightarrow \ u \leq \lim s_n \leq v. \]

**Definition 10.6.**
\[
\limsup s_n := \lim_{n \to \infty} \left( \sup \{s_n : n > N\} \right);
\]
\[
\liminf s_n := \lim_{n \to \infty} \left( \inf \{s_n : n > N\} \right).
\]

**Remark 10.7.** In the above definition, the sequence \((s_n)\) is not restricted to be bounded.

- If \( \sup\{s_n : n > N\} = +\infty \), then \( \limsup s_n = +\infty \);
- If \( \inf\{s_n : n > N\} = -\infty \), then \( \liminf s_n = -\infty \).

**Theorem 10.8.** Let \((s_n)\) be a sequence.

(i) If \( \lim s_n \) exists \((in \ \mathbb{R}, \ or \ is \ +\infty, \ or \ -\infty)\), then
\[
\liminf s_n = \lim s_n = \limsup s_n.
\]

(ii) If \( \liminf s_n = \limsup s_n \), then \( \lim s_n \) exists, and
\[
\lim s_n = \liminf s_n = \limsup s_n.
\]

**Proof.** As in the definition, denote
\[
u_N = \inf\{s_n : n > N\}, \quad v_N = \sup\{s_n : s > N\},
\]
and \( u = \lim \inf s_n = \lim u_N, \quad v = \lim \sup s_n = \lim v_N. \)
(i) **Case 1**: \( \lim s_n = +\infty \). Then \( \forall M > 0, \exists N > 0 \), such that
\[
n > N, \implies s_n > M.
\]
Then \( u_N \geq M \), \( \implies \) if \( m > N \), then \( u_m \geq u_N \geq M \), so \( \lim u_N = +\infty \).
Therefore
\[
\lim sup s_n = \lim v_N \geq \lim u_N = +\infty.
\]
**Case 2**: \( \lim s_n = -\infty \) can be proven similarly.
**Case 3**: assume \( \lim s_n = s \). Then \( \forall \epsilon > 0, \exists N > 0 \), such that
\[
|s_n - s| < \epsilon, \quad \text{for } n > N.
\]
We have
\[
\begin{align*}
\implies s_n &< s + \epsilon, \quad \text{for } n > N; \\
\implies v_N &\leq \sup\{s_n : n > N\} \leq s + \epsilon; \\
\text{if } m > N, \implies &u_m \leq v_N \leq s + \epsilon; \\
\implies \lim sup s_n &= \lim v_N \leq s.
\end{align*}
\]
Similarly, we have
\[
\begin{align*}
\implies s_n &> s - \epsilon, \quad \text{for } n > N; \\
\implies u_N &\geq \inf\{s_n : n > N\} \leq s - \epsilon; \\
\text{if } m > N, \implies &u_m \geq u_N \geq s - \epsilon; \\
\implies \lim inf s_n &= \lim u_m \geq s.
\end{align*}
\]
Since we know \( \lim inf s_n \leq \lim sup s_n \), they must all equal to \( s \).
(ii) The proof when \( \lim sup s_n = \lim inf s_n = \pm \infty \) is left as exercise.

Assume \( \lim sup s_n = \lim inf s_n = s \). Given \( \epsilon > 0 \), since \( \lim v_N = s \), there exists \( N_1 > 0 \), such that if \( n > N_1 \)
\[
|v_N - s| = |\sup\{s_n : n > N\} - s| < \epsilon.
\]
Note that \( \{v_N\} \) is monotone non-increasing, so we have \( v_N \geq s \), and hence
\[
\sup\{s_n : n > N\} - s < \epsilon,
\]
\[
\implies s_n < s + \epsilon, \quad \text{if } n > N_1.
\]
Similarly, since since \( \lim u_N = s \), there exists \( N_2 > 0 \), such that if \( n > N_2 \)
\[
|u_N - s| = |\inf\{s_n : n > N\} - s| < \epsilon.
\]
Note that \( \{u_N\} \) is monotone non-decreasing, so we have \( u_N \leq s \), and hence
\[
s - \inf\{s_n : n > N\} < \epsilon,
\]
\[ s_n > s - \epsilon, \quad \text{if } n > N_2. \]

Let \( N = \max\{N_1, N_2\} \), then if \( n > N \),

\[ s - \epsilon < s_n < s + \epsilon. \]

This implies that \( \lim s_n = s \).

\[ \square \]

**Definition 10.9.** A sequence \((s_n)\) is called a **Cauchy sequence** if

\[ \forall \epsilon > 0, \exists N > 0, \text{ such that } m, n > M, \implies |s_m - s_n| < \epsilon. \]  

(10.1)

**Lemma 10.10.** Convergent sequences are Cauchy sequences.

*Proof.* Assume that \( \lim s_n = s \). Let \( \epsilon > 0 \), then there exists \( N > 0 \), such that

\[ n > N, \implies |s_n - s| < \frac{\epsilon}{2}. \]

Therefore, if \( n, m > N \),

\[ |s_n - s_m| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]

\[ \square \]

**Lemma 10.11.** Cauchy sequences are bounded.

*Proof.* Apply (10.1) with \( \epsilon = 1 \). Then there exists \( N > 0 \), such that

\[ m, n > N, \implies |s_n - s_m| < 1. \]

In particular, \( |s_n - s_{N+1}| < 1 \), and so

\[ |s_n| < |s_{N+1}| + 1. \]

Let

\[ M = \max\{|s_{N+1}| + 1, |s_1|, \ldots, |s_N|\}. \]

Then \( |s_n| \leq M \).

\[ \square \]

**Theorem 10.12.** A sequence converges if and only if it is a Cauchy sequence.

*Proof.* According Lemma \[10.10\] we only need to show that Cauchy sequence must converge. Assume \((s_n)\) is Cauchy. By Lemma \[10.11\] \((s_n)\) is bounded, so by Theorem \[10.8\] we only need to show that

\[ \liminf s_n = \limsup s_n. \]

Given \( \epsilon > 0 \), there exists \( N > 0 \), such that

\[ \text{if } m, n > N, \implies |s_m - s_n| < \epsilon. \]
\[ s_n < s_m + \epsilon; \]
\[ v_N = \sup s_n : n > N \leq s_m + \epsilon, \forall m > N; \]
\[ v_N - \epsilon \leq s_m, \forall m > N; \]
\[ v_N - \epsilon \leq \inf \{ s_m : m > N \} = u_N; \]
\[ \limsup s_n \leq v_N \leq u_N + \epsilon \leq \liminf s_N + \epsilon. \]
11. Subsequences

**Definition 11.1.** Let \((s_n)\) be a sequence. A subsequence of \((s_n)\) is a sequence \((t_k)_{k \in \mathbb{N}}\), such that \(t_k = s_{n_k}\), with
\[
n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots.
\]

**Example 11.2.** Consider \(s_n = n^2(-1)^n\). The positive terms form a subsequence:
\[
(t_k) = (4, 16, 36, \cdots).
\]

Here \(t_k = s_{2k}\).

**Example 11.3.** Consider \(a_n = \sin\left(\frac{n\pi}{3}\right)\). The nonnegative terms are
\[
\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \cdots\right)
\]

**Example 11.4.** \(\mathbb{Q}\) can be listed as a sequence \((r_n)\).

**Proposition 11.5.** Given any real number \(a\), there exists a subsequence \((r_{n_k})\) of \((r_n)\) that converges to \(a\).

**Discussion:** we want to construct \(r_{n_k}\), such that
\[
|r_{n_k} - a| < \frac{1}{k}, \quad \text{for all } k \in \mathbb{N}.
\]

**Proof.** We use “Definition/construction by induction”.

1. Select \(n_1\), such that \(|r_{n_1} - a| < 1\), and this follows from the Denseness of \(\mathbb{Q}\);
2. Suppose \(n_1, \cdots, n_k\) have been selected such that
\[
n_1 < n_2 < \cdots < n_k,
\]
and \(|r_{n_j} - a| < \frac{1}{j}\) for \(j = 1, 2, \cdots, k\).

We want to find \(r_{n_{k+1}}\), with \(n_{k+1} > n_k\) and \(|r_{n_{k+1}} - a| < \frac{1}{k+1}\), which is equivalent to \(r_{n_{k+1}} \in \left(a - \frac{1}{k+1}, a + \frac{1}{k+1}\right)\).

By Problem 4.11 in the textbook, there are infinitely many rationals in \(\left(a - \frac{1}{k+1}, a + \frac{1}{k+1}\right)\). Therefore, there exists \(n_{k+1} > n_k\), such that \(r_{n_{k+1}} \in \left(a - \frac{1}{k+1}, a + \frac{1}{k+1}\right)\).

By Mathematical Induction, there exists a subsequence \((r_{n_k})\), with \(\lim r_{n_k} = a\). □
Example 11.6. Let \((s_n)\) be a sequence of positive numbers, i.e. \(s_n > 0\), such that 
\[
\inf\{s_n : n \in \mathbb{N}\} = 0.
\]
Prove that a subsequence of \((s_n)\) converges monotonically to 0.

Discussion: we want to construct \(s_{n_k}\), such that 
\[
s_1 > s_2 > \cdots > s_k \cdots; \quad \text{and} \quad s_k < \frac{1}{k}, \quad \text{for all } k \in \mathbb{N}.
\]

Proof. We use “Definition/construction by induction”.

(1) Since \(\inf\{s_n : n \in \mathbb{N}\} = 0\), there exists \(n_1 \in \mathbb{N}\), such that \(s_1 < 1\);
(2) Suppose \(n_1, \cdots, n_k\) have been selected such that 
\[
n_1 < n_2 < \cdots < n_k,
\]
and
\[
s_{n_{j+1}} < \min\{s_{n_j}, \frac{1}{j+1}\}, \quad \text{for } j = 1, 2, \cdots, k - 1.
\]

We want to find \(s_{n_{k+1}}\), such that \(s_{n_{k+1}} < \min\{s_{n_k}, \frac{1}{k+1}\}\). Note that 
\[
\min\{s_n : 1 \leq n \leq n_k\} > 0, \implies \inf\{s_n : n > n_k\} = 0.
\]
Therefore, there exists \(n_{k+1} > n_k\), such that 
\[
s_{n_{k+1}} < \min\{s_{n_k}, \frac{1}{k+1}\}.
\]

By Mathematical Induction, there exists a decreasing subsequence \((s_{n_k})\), with \(\lim s_{n_k} = 0\).

Theorem 11.7. If a sequence \((s_n)\) converges, then every subsequence converges to the same limit.

Proof. Let \(\lim s_n = s\), and \((s_{n_k})\) be a subsequence of \((s_n)\). Note that \(n_k \geq k\) for all \(k\).

Given \(\epsilon > 0\), there exists \(N > 0\), such that \(n > N, \implies |s_n - s| < \epsilon\).
If \(k > N\), then \(n_k > N\), so \(|s_{n_k} - s| < \epsilon\). This implies that \(\lim s_{n_k} = s\).

Theorem 11.8. Every sequence \((s_n)\) has a monotone subsequence.

Proof. Say that “the \(n\)-term \(s_n\) is dominant if \(s_n > s_m\) for all \(m > n\).”

We divide the proof into two cases:
Case 1: there exists infinitely many dominant terms \((s_{n_k})\), then \(s_{n_k+1} < s_{n_k}\), so \((s_{n_k})\) is a decreasing sequence.

Case 2: suppose there are only finitely many dominant terms, and we only focus on this case in the following.

Select \(n_1\), such that \(s_{n_1}\) is larger than all dominant terms. Then

\[(11.1) \quad \text{For any } N \geq n_1, \text{ there exists } m > N, \text{ such that } s_m \geq s_N.\]

We are going to use “inductive construction” to find a monotone sequence.

1. Applying \((11.1)\) with \(N = n_1\) gives \(n_2 > n_1\), such that \(s_{n_2} \geq s_{n_1}\);
2. Suppose \(n_1, \ldots, n_k\) have been selected such that \(n_1 < n_2 < \cdots < n_k\), and \(s_{n_1} \leq s_{n_2} \leq \cdots \leq s_{n_k}\).

Applying \((11.1)\) with \(N = n_k\) gives \(n_{k+1} > n_k\), such that \(s_{n_{k+1}} \geq s_{n_k}\).

By Mathematical Induction, \((s_{n_k})\) is a nondecreasing subsequence. \(\blacksquare\)

**Corollary 11.9.** Let \((s_n)\) be a sequence, then there exists

- a monotone subsequence with limit \(\limsup s_n\), and
- a monotone subsequence with limit \(\liminf s_n\).

*Proof.* We only prove the first statement for \(\limsup s_n\).

Let \(v_N = \sup\{s_n : s > N\}\), \(N \in \mathbb{N}\), then \(v = \lim v_N = \limsup s_n\).

If \(v = -\infty\), then \(\lim s_n = -\infty\), so any monotone sequence will converge to \(\limsup s_n\), and we finished.

If \(v \neq -\infty\), select an arbitrary monotone increasing sequence \((t_N)\), such that \(\lim t_N = v\):

- if \(v\) is finite, let \(t_N = v - \frac{1}{N}\);
- if \(v = +\infty\), let \(t_N = N\).

Now we discuss the two cases as in the proof of the above theorem.

Case 1: by assumption, \(s_{n_k} = \sup\{s_m : m \geq n_k\} = v_{n_k-1}\), so \(\{s_{n_k}\}\) is decreasing and

\[\lim_{k \to \infty} s_{n_k} = \lim v_N = \limsup s_n.\]

Case 2: Given \(N > n_1\) as above, by \((11.1)\), there exists \(m_1 > N\), such that \(s_{m_1} \geq s_N\). Since

\[t_N < v \leq v_N = \sup\{s_m : m > N\},\]
there exists $m_2 > N$, such that $s_{m_2} > t_N$.

\[
\begin{cases}
\text{either} & s_{m_1} \geq s_{m_2}, \implies s_{m_1} \geq s_N \text{ and } s_{m_1} > t_N; \\
\text{or} & s_{m_2} \geq s_{m_1}, \implies s_{m_2} \geq s_N \text{ and } s_{m_2} > t_N.
\end{cases}
\]

Then we have

(11.2) Given $N \geq n_1$, there exists $m > N$, such that $s_m \geq s_N$ and $s_m > t_N$.

We are going to use “inductive construction” to find the monotone subsequence.

1. Applying (11.2) with $N = n_1$ gives $n_2 > n_1$, such that $s_{n_2} \geq s_{n_1}$ and $s_{n_2} > t_{n_1}$.
2. Suppose $n_1, \cdots, n_k$ have been selected such that

\[
\begin{align*}
n_1 &< n_2 < \cdots < n_k, \\
s_{n_1} &\leq s_{n_2} \leq \cdots \leq s_{n_k},
\end{align*}
\]

and

\[
s_{n_{j+1}} > t_{n_j}, \quad \text{for all } 1 \leq j \leq k - 1.
\]

Applying (11.2) with $N = n_k$ gives $n_{k+1} > n_k$, such that $s_{n_{k+1}} \geq s_{n_k}$ and $s_{n_{k+1}} > t_{n_k}$. Moreover, $s_{n_{k+1}} \leq \sup \{s_m : m > n_k\} = v_{n_k}$, so

\[
t_{n_k} < s_{n_{k+1}} \leq v_{n_k}.
\]

By Mathematical Induction, $(s_{n_k})$ is a nondecreasing sequence and $\lim s_{n_k} = \limsup s_n$. \qed

**Definition 11.10.** Let $(s_n)$ be a sequence. A subsequential limit is any real number, or $+\infty$, or $-\infty$, that is the limit of some subsequence.

**Example 11.11.** If $\lim s_n = s$, then the set of subsequential limit is $\{s\}$.

**Example 11.12.** Let $(r_n)$ be the list of rational numbers, then the set of subsequential limits are $\mathbb{R} \cup \{+\infty, -\infty\}$.

**Theorem 11.13.** Let $(s_n)$ be a sequence, and $S$ be the set of subsequential limits of $(s_n)$. Then

(i) $S \neq \emptyset$;
(ii) $\sup S = \limsup s_n$, $\inf S = \liminf s_n$;
(iii) $\lim s_n$ exists if and only if $S$ consists of only one element, i.e. $S = \{s\}$. 

Proof. (i) follows from the above corollary since \( \lim \sup s_n, \lim \inf s_n \in S \), and (iii) follows from (ii).

To prove (ii), let \( t = \lim s_{n_k} \) for some subsequence \( (s_{n_k}) \), then
\[
t = \lim \sup s_{n_k} = \lim \inf s_{n_k}.
\]
Note that
\[
\{s_{n_k} : k > N\} \subset \{s_n : n > N\}.
\]
So
\[
\sup\{s_{n_k} : k > N\} \leq \sup\{s_n : n > N\};
\]
\[
\inf\{s_{n_k} : k > N\} \geq \inf\{s_n : n > N\}.
\]
Therefore,
\[
\lim \inf s_n \leq \lim \inf s_{n_k} = t = \lim \sup s_{n_k} \leq \lim \sup s_n.
\]
So
\[
\lim \inf s_n \leq \inf S \leq \sup S \leq \lim \sup s_n.
\]
Also \( \lim \sup s_n, \lim \inf s_n \in S \), so we finish the proof.

Theorem 11.14. Let \( S \) be the set of subsequential limits of \( (s_n) \). If \( t_n \in S \cap \mathbb{R} \) and \( \lim t_n = t \), then \( t \in S \).

Proof. We use “construction by induction”.

1. Since \( t_1 = \lim s_{n_k} \) for some subsequence \( (s_{n_k}) \), there exists \( n_1 \), such that
\[
|s_{n_1} - t_1| < 1;
\]
2. Assume \( n_1 < \cdots < n_k \) have been selected with
\[
|s_{n_j} - t_j| < \frac{1}{j}, \text{ for } j = 1, \cdots, k.
\]
Since \( t_{k+1} = \lim s_n \) for some other subsequence \( (s_n) \), there exists \( n_{k+1} > n_k \), such that
\[
|s_{n_{k+1}} - t_{k+1}| < \frac{1}{k + 1}.
\]
By induction, we found a subsequence \( (s_{n_k}) \) such that \( |s_{n_k} - t_k| < \frac{1}{k} \). It is then an easy exercise to show that \( t = \lim s_{n_k} \).

Definition 11.15. A subset \( S \subset \mathbb{R} \) is called closed, if for any convergent sequence \( (t_n) \) where \( t_n \in S \), then the limit \( t = \lim t_n \in S \).
Theorem 12.1. If \((s_n)\) is a sequence with \(\lim s_n = s\) and \(s > 0\), and \((t_n)\) is an arbitrary sequence, then
\[
\lim sup(s_n \cdot t_n) = s \cdot \lim sup t_n.
\]

Proof. First we will show \(\lim sup(s_n \cdot t_n) \geq s \cdot \lim sup t_n\). Denote \(\beta = \lim sup t_n\).

Case 1: \(\beta \in \mathbb{R}\). There exists a subsequence \((t_{n_k})\) of \((t_n)\), such that \(\lim t_{n_k} = \lim sup t_n = \beta\). Then
\[
\lim (s_{n_k} t_{n_k}) = s \beta \leq \lim sup(s_n t_n).
\]

Case 2: \(\beta = +\infty\). There exists a subsequence \((t_{n_k})\) of \((t_n)\), such that \(\lim t_{n_k} = +\infty\). Since \(\lim s_{n_k} = s > 0\), we have \(\lim(s_{n_k} t_{n_k}) = +\infty\) by Theorem 9.11. Hence
\[
\lim sup(s_n t_n) = +\infty.
\]

Case 3: \(\beta = -\infty\). Exercise.

To show the reverse direction: assume \(s_n \neq 0\) by ignore the first few terms. Then
\[
\lim sup t_n = \lim sup \left( \frac{1}{s_n} \cdot (s_n t_n) \right) \geq \lim \left( \frac{1}{s_n} \right) \cdot \lim sup(s_n t_n) = \frac{1}{2} \lim sup(s_n t_n).
\]
This finishes the proof. \(\square\)

Theorem 12.2. Let \((s_n)\) be a sequence with \(s_n \neq 0\). Then
\[
\lim inf \left| \frac{s_{n+1}}{s_n} \right| \leq \lim inf |s_n|^{1/n} \leq \lim sup |s_n|^{1/n} \leq \lim sup \left| \frac{s_{n+1}}{s_n} \right|.
\]

Proof. Let \(\alpha = \lim sup |s_n|^{1/n}\), and \(L = \lim sup \left| \frac{s_{n+1}}{s_n} \right|\). If \(L = +\infty\), then we are done.

Assume \(L < +\infty\). It suffices to show \(\alpha \leq L_1\) for any \(L_1 > L\). Since
\[
L = \lim_{N \to \infty} \left( \sup \left\{ \left| \frac{s_{n+1}}{s_n} \right| : n > N \right\} \right) < L_1,
\]
there exists \(N > 0\), such that
\[
\sup \left\{ \left| \frac{s_{n+1}}{s_n} \right| : n \geq N \right\} < L_1.
\]
Then by iteration, for $n > N$,
\[
|s_n| = \left| \frac{s_n}{s_{n-1}} \right| \cdot \frac{|s_{n-1} - s_{n-2}|}{|s_{n-2}|} \cdot \frac{|s_{n-2} - s_{n-3}|}{|s_{n-3}|} \cdot \cdots \cdot \frac{|s_{N+1} - s_N|}{|s_N|} \cdot |s_N| < L_1^{n-N} |s_N| = L_1^n \cdot \frac{|s_N|}{L_1^n}.
\]

Let $a = L_1^{-N} |s_N| > 0$, then
\[
|s_n| < a \cdot L_1^n \implies |s_n|^{1/n} < L_1 a^{1/n}.
\]

As $\lim a^{1/n} = 1$, we have
\[
\alpha = \limsup |s_n|^{1/n} \leq L_1.
\]

The other part $\liminf \left| \frac{s_{n+1}}{s_n} \right| \leq \liminf |s_n|^{1/n}$ is left as an exercise. \qed
13. Series

Denote
\[ \sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n. \]

We are interested in infinite series \( \sum_{n=m}^{\infty} a_n \). Let
\[ s_n = a_m + a_{m+1} + \cdots + a_n = \sum_{k=m}^{n} a_k. \]

**Definition 13.1.**

1. The infinite series \( \sum_{n=m}^{\infty} a_n \) is said to converge provided the sequence of partial sums \( (s_n) \) converges to a real number \( s \in \mathbb{R} \), and we define
\[ \sum_{n=m}^{\infty} a_n = s. \]
2. A series that does not converge is said to diverge.
3. Say \( \sum_{n=m}^{\infty} a_n = \text{diverges to } +\infty \), and write \( \sum_{n=m}^{\infty} a_n = +\infty \), provided \( \lim s_n = +\infty \). Similarly we can define \( \sum_{n=m}^{\infty} a_n = -\infty \).
4. If we do not care the initial index \( m \), just write \( \sum a_n \).
5. If \( a_n \geq 0 \), then \( (s_n) \) is nondecreasing, so \( \sum a_n \) either converges or diverges to \( +\infty \). Hence for any series \( \sum a_n \), the sum of absolute values \( \sum |a_n| \) is meaningful.
6. \( \sum a_n \) is said to converge absolutely or be absolutely convergent if \( \sum |a_n| \) converges.

**Example 13.2.** A series \( \sum_{n=0}^{\infty} ar^n \) for constants \( a \) and \( r \) is called a geometric series. For \( r \neq 1 \),
\[ s_n = \sum_{k=0}^{n} ar^k = a \frac{1 - r^{n+1}}{1 - r}. \]

For \( |r| < 1 \), \( \lim_{n \to \infty} r^{n+1} = 0 \), we have
\[ \lim_{n \to \infty} s_n = \frac{a}{1 - r}. \]

And
\[ \sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}. \]

If \( a \neq 0 \), and \( |r| \geq 1 \), the series does not converge.
Example 13.3. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. (This will be proved later.)

Definition 13.4. We say a series $\sum a_n$ satisfies the “Cauchy criterion” if the sequence of partial sums $(s_n)$ is a Cauchy sequence:

$$(13.1) \quad \text{for each } \epsilon > 0, \exists N > 0, \text{ such that if } n, m > N, \implies |s_n - s_m| < \epsilon.$$ 

And this is equivalent to

$$(13.2) \quad \text{for each } \epsilon > 0, \exists N > 0, \text{ such that } n \geq m > N, \implies |s_n - s_{m-1}| < \epsilon.$$ 

This is equivalent to

$$(13.3) \quad \text{for each } \epsilon > 0, \exists N > 0, \text{ such that } n \geq m > N, \implies \left| \sum_{k=m}^{n} a_k \right| < \epsilon.$$ 

As a direct corollary of Theorem 10.12, we have,

Theorem 13.5. A series converges if and only if it satisfies the Cauchy criterion.

Corollary 13.6. If $\sum a_n$ converges, then $\lim a_n = 0$.

Proof. For any $\epsilon > 0$, by the Cauchy criterion, there exists $N > 0$, such that if $n \geq m > N$, then $|\sum_{k=m}^{n} a_k| < \epsilon$. We can take $n = m$, then we have $|a_m| < \epsilon$. This implies that $\lim a_m = 0$. \qed

Remark 13.7. The converse of the above result is not true. For instance, $\sum \frac{1}{n} = +\infty$, although $\lim \frac{1}{n} = 0$.

Theorem 13.8 (Comparison Test). Consider a series $\sum a_n$, with $a_n \geq 0$ for all $n \in \mathbb{N}$.

(i) If $\sum a_n$ converges, and $|b_n| \leq a_n$, then $\sum b_n$ converges;

(ii) If $\sum a_n = +\infty$, and $b_n \geq a_n$, then $\sum b_n = +\infty$.

Proof.

(i) For any $n \geq m$, by the triangle inequality,

$$\left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k.$$ 

Therefore $\sum a_n$ satisfies the Cauchy criterion implies that $\sum b_k$ also satisfies the Cauchy criterion.

(ii) Denote $t_n = \sum_{k=0}^{n} b_k$ and $s_n = \sum_{k=0}^{n} a_k$. Then $t_n \geq s_n$. So $\lim s_n = +\infty$ implies $\lim t_n = +\infty$. 

Corollary 13.9. Absolutely convergent series are convergent.

Proof. This is an easy corollary of the comparison test. □

Theorem 13.10 (Ratio Test). Consider a series \( \sum a_n \) of nonzero terms \( a_n \neq 0 \) for all \( n \in \mathbb{N} \).

(i) \( \sum a_n \) converges absolutely, if \( \limsup \left| \frac{a_{n+1}}{a_n} \right| < 1 \);
(ii) \( \sum a_n \) diverges if \( \liminf \left| \frac{a_{n+1}}{a_n} \right| > 1 \);
(iii) otherwise if \( \liminf \left| \frac{a_{n+1}}{a_n} \right| \leq 1 \leq \limsup \left| \frac{a_{n+1}}{a_n} \right| \), then the test gives no information.

We will use the following root test to prove the ratio test.

Theorem 13.11 (Root Test). Let \( \sum a_n \) be a series, and \( \alpha = \limsup |a_n|^{1/n} \).

The series \( \sum a_n \)

(i) converges absolutely, if \( \alpha < 1 \);
(ii) diverges if \( \alpha > 1 \);
(iii) otherwise if \( \alpha = 1 \), then the test gives no information.

Proof.
(i) Suppose \( \alpha < 1 \), and select \( \beta \) such that \( \alpha < \beta < 1 \).

By the definition of \( \limsup \), there exists \( N > 0 \), such that

\[
\sup \{|a_n|^{1/n} : n > N\} < \beta.
\]

In particular, \( |a_n|^{1/n} < \beta \) for all \( n > N \), and hence

\[
|a_n| < \beta^n, \text{ for all } n > N.
\]

Since \( 0 < \beta < 1 \), the series \( \sum \beta^n \) converges. By the comparison test, \( \sum |a_n| \) converges.

(ii) If \( \alpha > 1 \), then a subsequence of \( |a_n|^{1/n} \) will converge to \( \alpha > 1 \), hence

\[
|a_n| > 1 \text{ for infinitely many choices of } n.
\]

By Corollary [13.6] \( \sum a_n \) cannot converge.

(iii) For \( \sum \frac{1}{n} \) and \( \sum \frac{1}{n^2} \), \( \alpha \) turns out to be 1, but \( \sum \frac{1}{n} \) diverges, and \( \sum \frac{1}{n^2} \) converges.

\[ \square \]

Proof of Ratio Test. Let \( \alpha = \limsup |a_n|^{1/n} \). By Theorem 12.2.
• if \( \limsup \left| \frac{a_{n+1}}{a_n} \right| < 1 \), then \( \alpha < 1 \), so \( \sum a_n \) converges absolutely by Root Test;
• if \( \liminf \left| \frac{a_{n+1}}{a_n} \right| > 1 \), then \( \alpha > 1 \), so \( \sum a_n \) diverges by Root Test;
• if \( \liminf \left| \frac{a_{n+1}}{a_n} \right| \leq 1 \leq \limsup \left| \frac{a_{n+1}}{a_n} \right| \), nothing is known by checking the examples: \( \sum \frac{1}{n} \) and \( \sum \frac{1}{n^2} \).

□

Example 13.12. Given \( \sum_{n=2}^{\infty} \left( -\frac{1}{3} \right)^n = \frac{1}{9} \sum_{n=0}^{\infty} \left( -\frac{1}{3} \right)^n \), does it converge or not? If converge, calculate the limit.

Solution. Since this is a geometric series with \( a = \frac{1}{9} \) and \( r = \frac{1}{3} \), so

\[
\frac{1}{9} \sum_{n=0}^{\infty} \left( -\frac{1}{3} \right)^n = \frac{1/9}{1 - (-1/3)} = \frac{1}{12}.
\]

□

Example 13.13. Given \( \sum_{n=0}^{\infty} \frac{n}{n^2 + 3} \), does it converge or not?

Solution.

• If denote \( a_n = \frac{n}{n^2 + 3} \), then

\[
\frac{a_{n+1}}{a_n} = \frac{n + 1}{n} \cdot \frac{n^2 + 3}{n^2 + 2n + 4},
\]

so \( \lim \left| \frac{a_{n+1}}{a_n} \right| = 1 \). Neither Ratio nor Root Test works.

• As \( a_n \) approaches \( \frac{1}{n} \) for \( n \) large, we have

\[
a_n = \frac{n}{n^2 + 3} \geq \frac{n}{n^2 + 3n^2} = \frac{1}{4n}.
\]

Since \( \sum \frac{1}{n} \) diverges, \( \sum \frac{1}{4n} \) also diverges. By Comparison Test, \( \sum_{n=0}^{\infty} \frac{n}{n^2 + 3} \) diverges.

□

Example 13.14. Given \( \sum \frac{1}{n^2 + 1} \), does it converge or not?

Solution.

• Neither Ratio nor Root Test works;

• Since \( \frac{1}{n^2 + 1} \leq \frac{1}{n^2} \) and \( \sum \frac{1}{n^2} \) converges, so by Comparison Test, \( \sum \frac{1}{n^2 + 1} \) converges.

□
Example 13.15. Given $\sum \frac{n}{3^n}$, does it converge or not?

Solution. Let $a_n = \frac{n}{3^n}$.

1. $\frac{a_{n+1}}{a_n} = \frac{n+1}{3n}$, so $\lim |\frac{a_{n+1}}{a_n}| = \frac{1}{3} < 1$. Then $\sum \frac{n}{3^n}$ converges by Ratio Test;

2. $\lim |a_n|^{1/n} = \lim \frac{n^{1/n}}{3} = \frac{1}{3} < 1$, so Root Test also works.

\[\square\]

Example 13.16. Given $\sum a_n$ where $a_n = \left[\frac{2}{(-1)^n - 3}\right]^n$, does it converge or not?

Solution.

- When $n$ is even, $a_n = (-1)^n = 1$, so $|a_n|^{1/n} = 1$; when $n$ is odd, $a_n = (-\frac{1}{2})^n$, so $|a_n|^{1/n} = \frac{1}{2}$. (As an exercise):

\[
\alpha = \lim \sup |a_n|^{1/n} = 1.
\]

So Root Test does not work.

- Since $a_n = 1$ for all even $n$, $(a_n)$ cannot converge to 0, so $\sum a_n$ diverges by Corollary 13.6.

\[\square\]

Example 13.17. Given $\sum_{n=0}^{\infty} 2(-1)^n - n$, does it converge or not?

Solution. Let $a_n = 2(-1)^n - n$.

- $a_n \leq \frac{1}{2^{n+1}}$, so $\sum a_n$ converges by Comparison Test;

- $\frac{a_{n+1}}{a_n} = \frac{1}{8}$ for even $n$, and $\frac{a_{n+1}}{a_n} = 2$ for odd $n$. Ratio Test fails since

\[
\frac{1}{8} = \lim \inf |a_{n+1}| \leq |a_n|^{1/n} < 1 < \lim \sup |a_{n+1}| = 2;
\]

- $|a_n|^{1/n} = 2^{\frac{1}{n}} - 1$ for $n$ even, and $|a_n|^{1/n} = 2^{-\frac{1}{n}} - 1$ for $n$ odd, so

\[
\lim |a_n|^{1/n} = \frac{1}{2},
\]

and Root Test implies that $\sum a_n$ converges.

\[\square\]
14. Alternating series and Integral Tests

Example 14.1. Show that $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$.

Solution. Consider $f(x) = \frac{1}{x}$:

$$\sum_{k=1}^{n+1} \frac{1}{k} = \text{sum of the area of the rectangles with base } [i, i + 1] \text{ and height } \frac{1}{i}$$

for $1 \leq i \leq n$

$$\geq \text{area under the curve } f(x) = \frac{1}{x} \text{ between } 1 \text{ and } n + 1$$

$$= \int_{1}^{n+1} \frac{1}{x} = \log(n+1).$$

Since $\log(n+1) \to +\infty$ as $n \to \infty$, we have $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$. $\square$

Example 14.2. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \text{ converges.}$

Solution. Consider $f(x) = \frac{1}{x^2}$:

$$\sum_{k=1}^{n+1} \frac{1}{k^2} = \text{sum of the area of the rectangles with base } [i - 1, i] \text{ and height } \frac{1}{i^2}$$

for $1 \leq i \leq n$

$$\leq 1 + \text{area under the curve } f(x) = \frac{1}{x^2} \text{ between } 1 \text{ and } n$$

$$= 1 + \int_{1}^{n} \frac{1}{x^2} = 2 - \frac{1}{n} \leq 2.$$

Therefore the sequence of partial sums $(s_n)$ is non-decreasing and bounded from above by 2, so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. $\square$

Theorem 14.3. $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 0, \text{ converges if and only if } p > 1$.

Proof. If $p > 1$, then by similar comparison argument with areas under the graph $f(x) = \frac{1}{x^p}$ as Example 14.2 we have

$$\sum_{k=1}^{n+1} \frac{1}{k^p} \leq 1 + \int_{1}^{n} \frac{1}{x^p} = 1 + \frac{1}{p-1} \left(1 - \frac{1}{n^{p-1}}\right)$$

$$< 1 + \frac{1}{p-1} = \frac{p}{p-1}.$$

Therefore, the sequence of partial sums $(s_n)$ is non-decreasing and bounded from above by $\frac{p}{p-1}$, so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.
If \( p \leq 1 \), then \( \frac{1}{n^p} \geq \frac{1}{n} \) for all \( n \in \mathbb{N} \). Since \( \sum \frac{1}{n} \) diverges to \(+\infty\) by Example \( 14.1 \), we have \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) diverges to \(+\infty\) by the Comparison Test (Theorem 13.8). \( \square \)

**Theorem 14.4** (Alternating Series Theorem). If \( (a_n) \) is a non-increasing sequence of nonnegative numbers, i.e.\[ a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0, \]
and if \( \lim a_n = 0 \), then the alternating series
\[ \sum_{n=1}^{\infty} (-1)^n a_n \]
converges.

**Proof.** First we introduce the following observation: given \( n \geq m \), denote \( A = a_m - a_{m+1} + a_{m+1} - a_{m+2} \pm a_n \), where the sign of \( a_n \) depends on whether \( n - m \) is even or odd. In particular, we have
\[ \sum_{k=m}^{n} (-1)^k a_k = (-1)^k A. \]

(1) If \( n - m \) is odd, then the last term in \( A \) is \(-a_n\), so
\[ A = [a_m - a_{m+1}] + [a_{m+2} - a_{m+3}] + \cdots + [a_{n-1} - a_n] \geq 0, \]
because each \([\cdot]\)-term is nonnegative by the monotonicity; moreover
\[ A = a_m - [a_{m+1} - a_{m+2}] - \cdots - [a_{n-2} - a_{n-1}] - a_n \leq a_m, \]
because each \([\cdot]\)-term is nonnegative and \( a_n \) is nonnegative.

(2) If \( n - m \) is even, then the last term in \( A \) is \( a_n \), so
\[ A = [a_m - a_{m+1}] + \cdots + [a_{n-2} - a_{n-1}] + a_n \geq 0, \]
because each \([\cdot]\)-term is nonnegative and \( a_n \) is nonnegative; and
\[ A = a_m - [a_{m+1} - a_{m+2}] - \cdots - [a_{n-1} - a_n] \leq a_m, \]
because each \([\cdot]\)-term is nonnegative.

Therefore, we proved:
\[ 0 \leq A \leq a_m. \]

Using the fact that \( \lim a_m = 0 \), for any \( \epsilon > 0 \), there exists \( N > 0 \), such that \( m > N \) implies \( a_m < \epsilon \). Therefore if \( n \geq m > N \), we have
\[ \left| \sum_{k=m}^{n} (-1)^k a_k \right| = |A| = A \leq a_m < \epsilon. \]
The Cauchy Criterion (Definition 13.4) implies that \( \sum_{n=1}^{\infty} (-1)^n a_n \) converges.

\[ \square \]

**Example 14.5.** Prove that \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) converges.

**Proof.**

- Since \( \lim \sqrt{\frac{n}{n+1}} = 1 \), the Ratio test does not apply. Similarly, since \( \lim \left| \frac{1/n}{1/n+1} \right| = 1 \), the Root test does not apply.
- Since \( \sum \frac{1}{\sqrt{n}} = +\infty \), the Comparison test does not apply.
- This is a standard situation to apply the Alternating Series Theorem. We can let \( a_n = \frac{1}{\sqrt{n}} \), and it is easy to check that this series satisfies the requirement of Theorem 14.4.

\[ \square \]
15. Continuous functions

Let $f$ be a real-valued function. The **defining domain** of $f$, denoted as $\text{dom}(f) = \text{the subset (in } \mathbb{R})$ where $f$ is defined.

**Example 15.1.** Let $f(x) = \sqrt{4 - x^2}$, then $\text{dom}(f) = [-2, 2]$.

**Definition 15.2.** Let $f$ be a function.

(i) Given $x_0 \in \text{dom}(f)$, $f$ is said to be **continuous at** $x_0$ if for any sequence $(x_n)$ in $\text{dom}(f)$ converging to $x_0$, we have

$$\lim_{n \to \infty} f(x_n) = f(x_0).$$

(ii) If $f$ is continuous at each point of a subset $S \subset \text{dom}(f)$, then we say $f$ is **continuous on** $S$.

(iii) $f$ is said to be **continuous** if $f$ is continuous on $\text{dom}(f)$.

**Theorem 15.3.** Let $f$ be a real valued function whose domain is a subset of $\mathbb{R}$. Then $f$ is continuous at $x_0 \in \text{dom}(f)$ if and only if

$$\text{for each } \epsilon > 0, \text{ there exists a number } \delta > 0, \text{ such that}$$

if $x \in \text{dom}(f)$, and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

**Proof.**

**Step 1:** Suppose (15.2) holds true. Consider a sequence $(x_n)$, $x_n \in \text{dom}(f)$, such that $\lim x_n = x_0$. We want to show $\lim f(x_n) = f(x_0)$.

Let $\epsilon > 0$, then there exists $\delta > 0$ satisfying the requirement of (15.2). Since $\lim x_n = x_0$, there exists $N > 0$, such that if $n > N$, then $|x_n - x_0| < \delta$. According to (15.2), we have $|f(x_n) - f(x_0)| < \epsilon$. This implies $\lim f(x_n) = f(x_0)$.

**Step 2:** Assume $f$ is continuous at $x_0$. If (15.2) fails, then there exists $\epsilon > 0$, such that for any $\delta > 0$, there exists $x \in \text{dom}(f)$, $|x - x_0| < \delta$, but $|f(x) - f(x_0)| \geq \epsilon$. In particular, let $\delta = \frac{1}{n}$, then there exists $x_n \in \text{dom}(f)$, $|x_n - x_0| < \frac{1}{n}$, but $|f(x_n) - f(x_0)| \geq \epsilon$.

However, we thus get a sequence $(x_n)$, $x_n \in \text{dom}(f)$, such that $\lim x_n = x_0$, but we cannot have $\lim f(x_n) = f(x_0)$, hence a contradiction. □

**Example 15.4.** Let $f(x) = 2x^2 + 1$, $x \in \mathbb{R}$. Prove that $f$ is continuous on $\mathbb{R}$, by

(a) using the definition;

(b) using the $\epsilon - \delta$ property.
Proof.

(a) Suppose \((x_n)\) is a sequence and \(\lim x_n = x_0\). Then we have
\[
\lim f(x_n) = \lim [2x_n^2 + 1] = 2(\lim x_n)^2 + 1 = 2x_0^2 + 1 = f(x_0).
\]

Therefore \(f\) is continuous at \(x_0\).

(b) Given \(x_0 \in \mathbb{R}, \) let \(\epsilon > 0,\) and we want to show
\[
|f(x) - f(x_0)| < \epsilon, \text{ provided } |x - x_0| < \delta.
\]

We have
- \(|f(x) - f(x_0)| = |2x^2 - 2x_0^2| = 2|x + x_0| \cdot |x - x_0|;
- Need to bound \(|x + x_0|\) from above;
- If \(|x - x_0| < 1,\) then \(|x| < |x_0| + 1,\) and \(|x + x_0| \leq |x| + |x_0| < 2|x_0| + 1.\)

Thus:
\[
|f(x) - f(x_0)| < 2|2|x_0| + 1| \cdot |x - x_0|, \text{ provided } |x - x_0| < 1.
\]

Let
\[
\delta = \min \left\{ 1, \frac{\epsilon}{2(2|x_0| + 1)} \right\}.
\]

Then \(|x - x_0| < \delta\) implies \(|f(x) - f(x_0)| < \epsilon.\)

\(\square\)

**Example 15.5.** Let \(f(x) = x^2 \sin \left(\frac{1}{x}\right)\) for \(x \neq 0\) and \(f(0) = 0.\) Prove that \(f\) is continuous at 0.

**Proof.** Let \(\epsilon > 0.\) We have
\[
|f(x) - f(0)| = x^2 \left|\sin \left(\frac{1}{x}\right)\right| \leq x^2.
\]

Set \(\delta = \sqrt{\epsilon}.\) Then if \(|x - 0| < \delta, \implies x^2 < \delta^2 = \epsilon,\) so
\[
|f(x) - f(0)| \leq x^2 < \epsilon.
\]

\(\square\)

**Example 15.6.** Let \(f(x) = \frac{1}{x} \sin \left(\frac{1}{x^2}\right)\) for \(x \neq 0,\) and \(f(0) = 0.\) Show that \(f\) is discontinuous at 0.

**Proof.** If suffices to find a sequence \((x_n)\) such that \(\lim x_n = 0,\) but
\[
(f(x_n)) \text{ does not converge to } f(0) = 0.
\]
Since \( f(x_n) = \frac{1}{x_n^2} \sin\left(\frac{1}{x_n^2}\right) \), we can find \( x_n \to 0 \), such that \( \sin\left(\frac{1}{x_n^2}\right) = 1 \). That is we can let
\[
\frac{1}{x_n^2} = \frac{\pi}{2} + 2n\pi, \quad \text{that is:} \quad x_n = \frac{1}{\sqrt{\frac{\pi}{2} + 2n\pi}}.
\]
Then \( \lim x_n = 0 \), but
\[
\lim f(x_n) = \lim \frac{1}{x_n} = +\infty.
\]

\[\square\]

**Algebraic operations on functions.**

Let \( f \) be a real-valued function. For any \( k \in \mathbb{R} \), we can define
\[
(kf)(x) = kf(x), \quad \text{for} \quad x \in \text{dom}(f);
\]
\[
|f|(x) = |f(x)|, \quad \text{for} \quad x \in \text{dom}(f).
\]

**Example 15.7.** Let \( f(x) = \sqrt{x} - 4 \) for \( x \geq 0 \), then \( 3f = 3\sqrt{x} - 12 \), and \( |f| = |\sqrt{x} - 4| \).

Given \( f, g \) two real-valued functions: we can define
\[
(1) \quad (f + g)(x) = f(x) + g(x);
\]
\[
(2) \quad (fg)(x) = f(x)g(x);
\]
\[
(3) \quad (f/g)(x) = f(x)/g(x);
\]
\[
(4) \quad g \circ f(x) = g(f(x));
\]
\[
(5) \quad \max\{f, g\}(x) = \max\{f(x), g(x)\};
\]
\[
(6) \quad \min\{f, g\}(x) = \min\{f(x), g(x)\}.
\]

The defining domains are:
- For (1)(2)(5)(6), the defining domain is \( \text{dom}(f) \cap \text{dom}(g) \);
- For (3), \( \text{dom}(f/g) = \text{dom}(f) \cap \{x \in \text{dom}(g) : g(x) \neq 0\} \);
- For (4), \( \text{dom}(g \circ f) = \{x \in \text{dom}(f) : f(x) \in \text{dom}(g)\} \).

**Theorem 15.8.** Let \( f \) be a real-valued function. If \( f \) is continuous at \( x_0 \in \text{dom}(f) \), then \( |f|, kf \) (for \( k \in \mathbb{R} \)) are continuous at \( x_0 \).

**Proof.** Consider a sequence \( (x_n), x_n \in \text{dom}(f) \), such that \( \lim x_n = x_0 \), then we have
\[
\lim f(x_n) = f(x_0).
\]
Therefore
Consider the function $kf$:

$$
\lim ((kf)(x_n)) = \lim k(f(x_n)) = k \lim f(x_n) = kf(x_0) = (kf)(x_0).
$$

This proved that $kf$ is continuous at $x_0$.

To show $|f|$ is continuous at $x_0$, we need to prove

$$
\lim |f(x_n)| = |f(x_0)|.
$$

By triangle inequality,

$$
| |f(x_n)| - |f(x_0)| | \leq |f(x_n) - f(x_0)|.
$$

For any $\epsilon > 0$, since $\lim f(x_n) = f(x_0)$, then there exists $N > 0$, such that if $n > N$, we have $|f(x_n) - f(x_0)| < \epsilon$, so

$$
| |f(x_n)| - |f(x_0)| | < \epsilon.
$$

This implies $\lim |f(x_n)| = |f(x_0)|$.

\[ \square \]

**Theorem 15.9.** Let $f, g$ be real-valued functions that are continuous at $x_0 \in \text{dom}(f) \cap \text{dom}(g)$. Then

(i) $f + g$ is continuous at $x_0$;

(ii) $fg$ is continuous at $x_0$;

(iii) $f/g$ is continuous at $x_0$ if $g(x_0) \neq 0$.

**Proof.** These follow by Limit Theorems (Section 9). \[ \square \]

**Theorem 15.10.** If $f$ is continuous at $x_0 \in \text{dom}(f)$, and $g$ is continuous at $f(x_0) \in \text{dom}(g)$, then $g \circ f$ is continuous at $x_0$.

**Proof.** Let $(x_n)$ be a sequence, such that $x_n \in \text{dom}(f)$, $f(x_n) \in \text{dom}(g)$, and

$$
\lim x_n = x_0.
$$

Since $f$ is continuous at $x_0$, we have $\lim f(x_n) = f(x_0)$, and as $g$ is continuous at $f(x_0)$, we have

$$
\lim g \circ f(x_n) = \lim g(f(x_n)) = g(f(x_0)) = g \circ f(x_0).
$$

This implies that $g \circ f$ is continuous at $x_0$. \[ \square \]

**Example 15.11.** Let $f, g$ be continuous at $x_0 \in \text{dom}(f) \cap \text{dom}(g)$. Prove that $\max\{f, g\}$ is continuous at $x_0$. 

Proof. We have the following equation:

$$\max\{f, g\} = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|,$$

since $\max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$.

We know that $f + g, f - g$ are both continuous at $x_0$, so is $|f - g|$. Therefore $\frac{1}{2}(f + g), \frac{1}{2}|f - g|$ are continuous at $x_0$. The conclusion then follows. \qed
16. Properties of continuous functions

**Definition 16.1.** Let \( f \) be a real-valued function. \( f \) is said to be bounded if \( \{ f(x) : x \in \text{dom}(f) \} \) is a bounded set: that is to say, there exists \( M \in \mathbb{R} \), such that \( |f(x)| \leq M \) for all \( x \in \text{dom}(f) \).

**Theorem 16.2.** Let \( f \) be a continuous function on a closed interval \([a, b]\), then \( f \) is bounded.

**Proof.** Assume by contradiction that \( f \) is not bounded on \([a, b]\). Then to each \( n \in \mathbb{N} \), the set of values \( \{ |f(x)| : x \in [a, b] \} \) is not bounded by \( n \): that is to say, there exists an element in \([a, b]\), denoted as \( x_n \), such that \( |f(x_n)| > n \).

For considering all \( n \in \mathbb{N} \), we obtain a sequence \((x_n), x_n \in [a, b]\), such that \( |f(x_n)| > n \).

By the Bolzano-Weierstrass Theorem, \((x_n)\) has a subsequence \((x_{n_k})\), such that \( \lim_{k \to \infty} x_{n_k} = x_0 \). Since \( a \leq x_n \leq b \), we have \( x_0 \in [a, b] \). By the continuity of \( f \) on \([a, b]\), we have

\[
\lim_{k \to \infty} f(x_{n_k}) = f(x_0).
\]

This is a contradiction to \( \lim_{k \to \infty} |f(x_{n_k})| = +\infty \). \( \square \)

**Theorem 16.3.** Let \( f \) be a continuous function on a closed interval \([a, b]\). Then \( f \) assumes it maximum and minimum on \([a, b]\): that is to say, there exist \( x_0, y_0 \in [a, b] \), such that \( f(x_0) \leq f(x) \leq f(y_0) \) for all \( x \in [a, b] \).

**Proof.** Since the set of values \( \{ f(x) : x \in [a, b] \} \) is bounded, the supremum exists by the Completeness Axiom:

\[
M = \sup \{ f(x) : x \in [a, b] \}.
\]

For each \( n \in \mathbb{N} \), by the definition of supremum, there exists an element in \([a, b]\), denoted as \( y_n \), such that

\[
M - \frac{1}{n} < f(y_n) \leq M.
\]

We obtain a sequence \((y_n), y_n \in [a, b]\), by considering all \( n \in \mathbb{N} \), and

\[
\lim_{n \to \infty} f(y_n) = M.
\]

By the Bolzano-Weierstrass Theorem, there exists a subsequence \((y_{n_k})\) of \((y_n)\), such that

\[
\lim_{k \to \infty} y_k = y_0.
\]
Since \( a \leq y_n \leq b \), we have \( y_0 \in [a, b] \). By the continuity of \( f \) on \([a, b]\), we have
\[
\lim_{k \to \infty} f(y_{n_k}) = f(y_0) = M.
\]
So \( M \) is a maximum of \( \{f(x) : x \in [a, b]\} \).

The case for \( \inf \{f(x) : x \in [a, b]\} \) is left as an exercise. \( \Box \)

**Remark 16.4.** The theorems fail if we change the defining domain \([a, b]\) to an open interval \((a, b)\). A counter-example is:
\[
f(x) = \frac{1}{x} : \quad x \in (0, 1).
\]

\( f(x) \) is continuous on \((0, 1)\), but \( f \) is not bounded on \((0, 1)\).

**Theorem 16.5** (Intermediate Value Theorem). If \( f \) is continuous on on interval \( I \), and when \( a, b \in I \), \( a < b \), and
\[
f(a) < y < f(b), \quad \text{or} \quad f(b) < y < f(a),
\]
there exists at least one \( x \in (a, b) \), such that \( f(x) = y \).

**Proof.** Assume that \( f(a) < y < f(b) \), (the other case is left as an exercise).

Let
\[
S = \{x \in [a, b] : f(x) < y\}.
\]

Since \( f(a) < y \), \( a \in S \) and \( S \neq \emptyset \).

Let \( x_0 = \sup S \), then \( x_0 \in [a, b] \).

For each \( n \in \mathbb{N} \), there exists an element in \( S \), denoted as \( s_n \), such that
\[
x_0 - \frac{1}{n} < s_n \leq x_0.
\]

Therefore, \( \lim s_n = x_0 \).

Since \( s_n \in S \), we have \( f(s_n) < y \), and hence by continuity of \( f \)
\[
f(x_0) = \lim f(s_n) \leq y.
\]

On the other hand, let
\[
t_n = \min\{b, x_0 + \frac{1}{n}\},
\]
then \( x_0 \leq t_n \leq x_0 + \frac{1}{n} \), hence
\[
\lim t_n = x_0.
\]

However, since \( b \in S \) (as \( f(b) > y \)), and \( x_0 + \frac{1}{n} \in S \), we have \( t_n \notin S \), and hence
\[
f(t_n) \geq y, \quad \text{for all} \quad n \in \mathbb{N}.
\]
Using continuity of \( f \) again,
\[
f(x_0) = \lim_{t \to x_0} f(t) \geq y.
\]
Therefore,
\[
f(x_0) = y.
\]
Since \( y \neq f(a), f(b) \), we deduce that \( x_0 \neq a, b \), and hence \( x_0 \in (a, b) \).

**Corollary 16.6.** Let \( f \) be a continuous function on a closed interval \( I = [a, b] \), then the image
\[
f(I) = \{ f(x) : x \in I \}
\]
is a closed interval or a point.

**Proof.** Using the Intermediate Value Theorem, we are easily show that \( J = f(I) \) has the following property:

\[\text{if } y_0, y_1 \in J, \ y_0 < y < y_1, \ \text{then } y \in J.\]

By Theorem \( \text{16.3} \), \( J \) has maximum \( \max J \) and minimum \( \min J \), hence
\[
J \subset [\min J, \max J].
\]
moreover, there exists \( y_0, y_1 \in I \), such that
\[
\max J = f(y_0), \ \min J = f(y_1).
\]
If \( \inf J < \sup J \), then \( J \) must be an interval. In fact, if \( \inf J < y < \sup J \), then \( f(y_0) < y < f(y_1) \), and by the Intermediate Value Theorem, there exists \( x \in (y_0, y_1) \) or \( x \in (y_1, y_0) \), such that \( f(x) = y \in J \), hence
\[
(\inf J, \sup J) \subset J.
\]
And hence
\[
J = [\min J, \max J].
\]

**Example 16.7.** Let \( f : [0, 1] \to [0, 1] \) be a continuous function defined on \([0, 1]\). Prove that there exists \( x_0 \in [0, 1] \), such that \( f(x_0) = x_0 \).

**Proof.** Consider the function:
\[
g(x) = f(x) - x.
\]
It is continuous on \([0, 1]\) by Theorem \( \text{15.9} \).

Notice that
\[
g(0) = f(0) - 0 \geq 0;
\]
\[
g(1) = f(1) - 1 \leq 0.
\]
Actually if either \( g(0) = 0 \) or \( g(1) = 0 \), then we can let \( x_0 = 0 \), or \( x_0 = 1 \). Otherwise
\[
g(0) = f(0) - 0 > 0; \\
g(1) = f(1) - 1 < 0.
\]
By the Intermediate Value Theorem, there exists \( x_0 \in (0,1) \), such that \( g(x_0) = 0 \), and this is
\[
f(x_0) = x_0.
\]
\[\square\]

**Example 16.8.** If \( y > 0 \) is a positive real number, and \( m \in \mathbb{N} \) is an integer, then \( y \) has a positive \( m \)-th root.

**Proof.** Consider the continuous function \( f(x) = x^m \) defined on \( \mathbb{R} \).

First we claim that there exists \( b > 0 \), such that \( y < b^m \). In fact
- if \( y \leq 1 \), let \( b = 2 \);
- if \( y > 1 \), let \( b = y \).

Then we have that
\[
f(0) = 0 < y < b^m = f(b).
\]
By the Intermediate Value Theorem, there exists \( x \in (0,b) \), such that
\[
f(x) = x^m = y.
\]
\[\square\]
References