§1. The set $\mathbb{N}$ of natural numbers.

$\mathbb{N} = \{1, 2, 3, \ldots\}$ all natural numbers

or positive integers.

- each $n$ has a successor $n+1$

  e.g. the successor of $5$ is $6$.

Properties:
- $i \in \mathbb{N}$
- if $n \in \mathbb{N}$, the its successor $n+1 \in \mathbb{N}$
- $1$ is not the successor of any element in $\mathbb{N}$
- if $n$ and $m$ have the same successor then $n = m$
- A subset $S \subseteq \mathbb{N}$ s.t. (such that) $i \in \mathbb{N}$ and

  if $n \in S$ then $n+1 \in S \Rightarrow S = \mathbb{N}$

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Peano Axioms or Peano Postulates

$\mathbb{N}$ is the basis for mathematical induction.

Let $P_1, P_2, \ldots$ be a list of statements or propositions

that may or may not be true.

The principle of mathematical induction asserts that:

all $P_1, P_2, P_3, \ldots$ are true provided

$P_1$ is true $\Rightarrow$ basis of induction

$P_{n+1}$ is true if $P_n$ is true $\Rightarrow$ induction step.
Example: all numbers of form $7^n - 2^n$ are divisible by 5.

Proof: The $n$-th proposition is

$P_n$: "$7^n - 2^n$ is divisible by 5."

$P_1$: $7^1 - 2^1 = 5$ is true.

Suppose $P_n$ is true, i.e., $P_n = 7^n - 2^n = 5m$.

To verify $P_{n+1}$:

$$7^{n+1} - 2^{n+1} = 7^{n+1} - 2 \cdot 7^n + 2 \cdot 7^n - 2^{n+1} = 7^n (7 - 2) + 2 (7^n - 2^n)$$

$$= 5 \cdot 7^n + 2 \cdot 5m = 5 (7^n + 2m).$$

Therefore, $P_n$ implies $P_{n+1}$, so the induction step holds.

Example. Show $|\sin(nx)| \leq n |\sin x|$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$.

Proof: The $n$-th proposition is

$P_n$: "$|\sin nx| \leq n |\sin x|$".

• Basis of induction. $P_1$: $|\sin x| = |\sin x| = 1$ is true.

• Induction step. Suppose $P_n$ is true.

To verify $P_{n+1}$:

$$|\sin (n+1)x| = |\sin (nx + x)| = |\sin nx \cos x + \cos nx \sin x|$$

$$\leq |\sin nx| \cos x| + |\cos nx| \sin x|$$

$$\leq |\sin nx| + |\sin x|$$

$$\leq n |\sin x| + |\sin x| = (n+1) |\sin x|$$

By $P_n$. (Therefore, $P_n$ implies $P_{n+1}$, so we finish the proof.)
82. The set \( \mathbb{Q} \) of rational numbers.
- \( \mathbb{Z} = \{ 0, 1, -1, 2, -2, \ldots \} \) integers
- \( \mathbb{Q} = \{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \} \) all rational numbers

- good algebraic system, basis operations like addition, multiplication, subtraction, and division can be studied
- inadequate when solving algebraic equations like \( x^2 - 2 = 0 \)

\[
\begin{align*}
\sqrt{1+1} &= 2 \\
\text{Pythagorean Thm.}
\end{align*}
\]

- the graph of \( y = x^2 - 2 \) crosses the \( x \)-axis at solutions of \( x^2 - 2 = 0 \)
- there are gaps in \( \mathbb{Q} \)
- there are more exotic numbers such as \( \pi \) & \( e \)
  - \( \pi \) appears when studying circles and spheres
  - \( e \) appears when studying \( \sum \frac{1}{n!} \)

**Def:** a number is called an "algebraic number" if it satisfies a polynomial equation.
\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \]
where \( a_i \in \mathbb{Z} \), \( a_n \neq 0 \), \( n \geq 1 \)
rational numbers are always algebraic numbers if \( r = \frac{m}{n} \Rightarrow \) it satisfies \( nx - m = 0 \).

\( \sqrt{2}, \sqrt{5} \) are algebraic numbers.

ordinary algebraic operations on rational numbers.

\[ a = (2 + 5^{\frac{3}{2}})^{\frac{1}{2}}, \quad \left(4 - 2.3^{\frac{3}{2}}/7\right)^{\frac{1}{2}} \]

\[ (17)^{\frac{3}{2}}, \quad x^2 - 17 = 0. \]

\[ a = (2 + 5^{\frac{3}{2}})^{\frac{1}{2}}, \quad a^2 = 2 + 5^{\frac{3}{2}}, \quad \left(a^2 - 2\right)^{\frac{3}{2}} = 5 \]

\[ \Rightarrow \quad a^6 - 6a^4 + 12a^2 - 8 = 5 \Rightarrow a^6 - 6a^4 + 12a^2 - 13 = 0. \]

\[ b = \left(4 - 2.3^{\frac{3}{2}}/7\right)^{\frac{1}{2}}, \quad b^2 = 4 - 2.3^{\frac{3}{2}} \Rightarrow 2.3^{\frac{3}{2}} = 4 - b^2 \]

\[ \Rightarrow \quad 12 = (4 - b^2)^2 = 49b^4 - 56b^2 + 16 \]

\[ \Rightarrow \quad 49b^4 - 56b^2 + 4 = 0. \]

Recall: an integer \( k \) is a factor of an integer \( m \), or divides \( m \) if \( \frac{m}{k} \) is also an integer.

an integer \( p > 2 \) is a prime provided the only positive factors are 1 and \( p \).

Proposition: \( \sqrt{2} \) is not a rational number.
\[ p^2 = \frac{p^2}{q} \text{ where } p \text{ and } q \text{ are integers with no common factor and } q \neq 0. \]

then \[ \frac{p^2}{q} = 2 \implies p^2 = 2q^2. \]

**Lemma:** If \( p^2 \) is an even number, then \( p \).

**Proof:** Suppose not. \( p \) is odd. Then \( p = 2m + 1 \).

and \( p^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1 \)

is also odd.

By the lemma, \( p \) is even, i.e., \( p = 2m \).

\[ \implies p^2 = (2m)^2 = 4m^2 = 2q^2 \implies q^2 = 2m^2. \]

By the lemma again, \( q \) is even.

\[ \implies \text{ contradiction to the fact that } p \text{ and } q \text{ have no common factor} \]

**Rational Zero Theorem**

Suppose that \( a_n, a_{n-1}, \ldots, a_0 \) are all integers and \( r \) is a rational number satisfying the polynomial equation:

\[ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad (1) \]

where \( n \geq 1, \ a_n \neq 0, \ a_0 \neq 0 \). Write \( r = \frac{p}{q} \) where \( p, q \in \mathbb{Z} \) having no common factors and \( q \neq 0 \). Then \( q \) divides \( a_0 \) and \( p \) divides \( a_0 \).

\[ \implies a_n \left( \frac{p}{q} \right)^n + \cdots + a_0 = 0. \]
multiply $y^n$:

\[ \text{an} \; p^n + \text{an} \; p^{n-1} \; y + \text{an} \; p^{n-2} \; y^2 + \cdots + a_1 \; p \; y^{n-1} + a_0 \; y^n = 0 \]

\[ \Rightarrow \text{an} \; p^n = -p \left( \text{an} \; p^{n-1} + \text{an} \; p^{n-2} \; y + \cdots + a_1 \; p \; y^{n-1} \right) \]

\[ \Rightarrow p \; y \text{ divides } \text{an} \; p^n. \text{ But } p, q \text{ have no common factor} \]

\[ \Rightarrow q \text{ divides } \text{an} \; p^n. \text{ Similarly for } q. \]

\[ \text{an} \; q^n = -q \left( \text{an} \; q^{n-1} + \text{an} \; q^{n-2} \; y + \cdots + a_1 \; q \; y^{n-1} \right) \]

\[ \Rightarrow p \text{ divides } \text{an} \; q^n \Rightarrow p \text{ divides } a_0 \text{ and } q \text{ divides } a_0. \]

\[ \text{Use the aboveThm to show } \sqrt{2} \text{ is not rational.} \]

\[ \text{pf.} \text{ The only roots of } x^2 - 2 = 0 \text{ are } \pm \sqrt{2}, \text{ where } p \text{ divides } -2 \text{ and } q \text{ divides } 1 \]

\[ \text{i.e. } p = \pm 1, \pm 2; \quad q = \pm 1 \]

\[ \Rightarrow \text{ the rational solutions must be of form } \pm 2, \pm 1 \]

\[ \text{By substitution back to } x^2 - 2 = 0. \text{ None of these are solutions.} \]

\[ \text{Example. } a = (2 + 5^{1/2})^{1/2} \text{ is not a rational number.} \]

\[ \text{pf. } a \text{ is a soln of } x^6 - 6x^4 + 12x^2 - 13 = 0. \]

\[ \text{By Thm, the rational solutions are } \pm 1, \pm 13 \text{. \underline{The only possible}} \]

\[ \text{By substitution back shows none of them are real soln's.} \]