§4. Real numbers $\mathbb{R}$ and completeness Axiom.

$\mathbb{R}$: the ordered field $F$ containing $\mathbb{Q}$, with no "gaps".

**Def.** Let $S \subseteq F$.

(a) If $S$ contains a largest element $s_0$ [i.e. $s_0 \in S$ and $s \leq s_0$ for all $s \in S$], call $s_0$ the maximum of $S$, and write $s_0 = \max S$.

(b) If $S$ contains a smallest element $s_0'$ [i.e. $s_0' \in S$ and $s \geq s_0'$ for all $s \in S$], call $s_0'$ the minimum of $S$, and write $s_0' = \min S$.

**Example:**

1. Every finite subset $S$ has a maximum $\max S$ and a minimum $\min S$.

   $\max \{1, 2, 3, 4\} = 4$, $\min \{1, 2, 3, 4\} = 1$.

2. Given $a, b \in F$, $a < b$.

   
   $[a, b] = \{x \in F : a \leq x \leq b\}$
   
   $[a, b) = \{x \in F : a \leq x < b\}$
   
   $(a, b] = \{x \in F : a < x \leq b\}$
   
   (closed interval, open interval, half-open interval).

   $\max [a, b] = b$, $\min (a, b] = a$.

   $(a, b)$ has no maximum in general.

3. $\{x \in \mathbb{R} : 0 \leq x \leq \sqrt{2}\}$ has no maximum.

**Def.** Let $S \subseteq F$, $S \neq \emptyset$. 


a) if \( M \in F \) and \( s \leq M \), then \( M \) is called an upper bound of \( S \) and \( S \) is said bounded from above.

b) if \( M \in F \) and \( m \leq S \), then \( m \) is called an lower bound of \( S \) and \( S \) is bounded from below.

c) \( S \) is said bounded if it is bounded above and below.

\[ S \leq [m, M] \] for \( m, M \in F \).

Example: 1. if \( S \) has \( \text{max} S \), then \( \text{max} S \) is an upper bound.

Similarly \( \text{min} S \) is a lower bound of \( S \).

2. \( b \) is an upper bound of \( (a, b) \), \( (a, b) \), \( [a, b) \) and \( a \) \( \text{b} \).

3. 2 is an upper bound of \( \{ r \in \mathbb{Q} : 0 \leq r \leq 5 \} \).

\( 5 \) is the least upper bound.

Def: \( S \subseteq F \), \( S \neq \emptyset \).

a) if \( S \) is bounded from above and \( S \) has a least upper bound, then we call it the supremum of \( S \) and denote by \( \sup S \).

b) if \( S \) is bounded from below and \( S \) has an infimum, then \( \inf S \).

Least upper bd: \( M = \sup S \) if and only if

i) \( \forall M \leq s \in S \).

ii) if \( M < M \), then \( \exists s \in S \) such that \( s > M \).
Example: a) if \( S \) has a maximum, then \( \max S = \sup S \).

"Similar for minimum & \( \inf \)"

b) \( \sup \{a, b\} = \sup \{a, b\} = \sup \{a, b\} = \sup \{a, b\} = b \)

3. \( A = \{ r \in \mathbb{Q} : 0 \leq r < b \} \). \( \sup A = \delta + \Delta \).

Note: least upper bd. may not belong to \( S \).

"Completeness Axiom":

every non-empty subset \( S \subset \mathbb{R} \) that is bounded above has a least upper bound \( \Leftarrow \sup S \) exists and is a real number.

Def: the set of real numbers \( \mathbb{R} \) is an ordered field containing \( \mathbb{Q} \) and satisfies "Completeness Axiom".

Example: \( A = \{ r \in \mathbb{Q} : 0 < r < \delta \} \) shows that \( \mathbb{Q} \) does not satisfies "Completeness Axioms".

Cor: every bounded non-empty subset \( S \subset \mathbb{R} \) bounded from below has a greatest lower bound.

Proof: let \( -S = \{ -s : s \in S \} \). Since \( S \) is bounded from below \( \exists m \in \mathbb{R} : m < s \quad \forall s \in S \). \( \Rightarrow -m > -s \quad \forall s \in S \).

Thus \( -S \) is bounded from above \( \Rightarrow \inf (-S) \in \mathbb{R} \).

let \( s_0 = \sup (-S) \). Claim: \( -s_0 = \inf (S) \).
Need to prove:

1. \(-S < S \iff -S \leq 0 \iff -S < 0 \iff S < 0\)
2. \(t \leq S \iff t < -S\)
   \(\iff -t > -S \iff -t + S \iff -t > 0 \iff t < 0\)

Thus (Archimedian Property).

If \(a > 0\) and \(b > 0\), then for some new \(n > 0\), \(na > b\).

Case 1: if \(a > 0\) then \(\frac{1}{n} < a\) for some positive integer \(n\).

Case 2: if \(b > 0\) then \(b < n\) for -

Let \(b = 1\). Let \(a = 1\).

Assume not. Let \(a > 0\) \(b > 0\) such that \(na < b\), a new.

So \(b\) is an upper bound for \(S = \{na: n \in \mathbb{N}\}\)

By the completeness axiom, \(S = \sup S \in \mathbb{R}\).

Since \(a > 0\), \(S < S + a\). So \(S - a < S\).

Since \(S = \sup S\) is the least upper bound, \(S - a < na\) for some \(n > 0\).

\(\Rightarrow S < na + a = (n+1)a \in S\).

Contradiction to "\(S\) is an upper bound for \(S\)".

Then (Density of \(\mathbb{R}\)).

If \(a, b \in \mathbb{R}\), \(a < b\), then \(\exists \, r, \, \epsilon > 0\).

If need to find \(a < \frac{m}{n} < b\) for some \(m, n \in \mathbb{Z}\).

\(\Rightarrow an < m < bn\).
Since $b-a > 0$, by Archimedean property, there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$.

By Archimedean property again:

2. $k \in \mathbb{N} > \max\{la, lb\}$.

i.e. $-k < na < nb < k$.

Consider \( \{ j \in \mathbb{Z}, -k < j < k, an < j \} \) is finite & non-empty at $m = \min\{j \in \mathbb{Z}, -k < j < k, an < j \}$.

Then $an < m$, but $an = m - 1$.

So $m = (an + 1) + 1 \leq an + 1 < an + (bn - an) = bn$. 

\[ an \]