Topic class on minimal surfaces—lectures by Rick Schoen

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Abstract

This series of lecture notes were taken for the topic class on minimal surfaces given by Professor Rick Schoen in the Winter quarter of 2012 at Stanford. We kept the pace of these lectures by dates.

These lectures start from basic materials on minimal surfaces, e.g. first and second variations, and monotonicity formulae, and then discuss several curvature estimates for minimal surfaces. Afterwards, the notes cover basic existence theory for minimal surfaces, e.g. the classical Plateau problem and the Sacks-Uhlenbeck theorem, and finally end up with a survey of the proof of the Willmore conjecture. The materials covered are very good examples for the application of methods from partial differential equations and calculus of variation.

It is likely that we have numerous typos and mistakes here and there, and would appreciate it if these are brought to our attention.

Contents

1 Introduction and calibrations (1/10/2012)

Let $\Sigma^k \subset (M^n, g)$ be a $k-$dimensional submanifold of an $n-$dimensional Riemannian manifold (M^n, g), where ∇ is the corresponding Riemannian connection. Let $g|_\Sigma$ be the induced metric. Given tangent vectors X, Y of Σ, the second fundamental form(abbreviated as **2nd f.f.** in the following), which is a vector valued symmetric 2-tensor on Σ, is defined as:

$$\vec{A}(X,Y) = (\nabla_X Y)^{\perp}. \tag{1}$$

Definition 1.1 *The **mean curvature** of Σ is defined as: $\vec{H} = Tr_g\vec{A} = \sum_{i,j=1}^{k} g^{ij}\vec{A}(e_i,e_j)$, where $\{e_i\}_{i=1}^k$ is a tangent basis of Σ.*

Definition 1.2 *Σ is called **minimal**, if $\vec{H} = 0$.*

1.1 Minimal surface equation:

Consider a hyper surface $\Sigma_u^{n-1} \subset \mathbb{R}^n$ as a graph $\Sigma_u = \{(x, u(x)) : x = (x_1, \cdots, x_{n-1}) \in \Omega\}$ of a function u, where $\Omega \subset \mathbb{R}^n$. Denote $F(x_1, \cdots, x_{n-1}) = (x_1, \cdots, x_{n-1}, u(x))$. Then the induced metric is given by $g_{ij} = F_{x_i} \cdot F_{x_j} = \delta_{ij} + u_i u_j$, where $u_i = \frac{\partial u}{\partial x_i}$. The matrix $(\delta_{ij} + u_i u_j)$ has $n-2$ multiple eigenvalues 1 with eigenspace $(\nabla u)^{\perp}$ and a single eigenvalue $1+|\nabla u|^2$ with eigenvector ∇u. If we think u as a function defined on $\mathbb{R}^n$, then the graph Σ_u is the level set given by $x_n - u(x) = 0$. So the unite normal of Σ_u is given by $\nu = \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}}$. Hence the inverse matrix for g_{ij} is given by $g^{ij} = \delta_{ij} - \nu_i \nu_j$, where $\nu_i = \frac{u_i}{\sqrt{1+|\nabla u|^2}}$ for $i = 1, \cdots, n-1$.

Now the volume form of Σ_u is given by $dv = \sqrt{det(g)}dx = \sqrt{1+|\nabla u|^2}dx$. So the volume of Σ_u is:

$$|\Sigma_u| = \int_\Omega \sqrt{1+|\nabla|^2}dx.$$

Now let us calculate the Euler-Lagrange equation for $|\Sigma_u|$. For any $\eta \in C_c^\infty(\Omega)$, suppose u is a critical point of $|\Sigma_u|$, then

$$\begin{aligned} \frac{d}{dt}|_{t=0}|\Sigma_{u+t\eta}| &= \int_\Omega \frac{\nabla u \cdot \nabla \eta}{\sqrt{1+|\nabla u|^2}}dx \\ &= -\int_\Omega \frac{\partial}{\partial x_i}\left(\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right)\eta dx = 0. \end{aligned} \tag{2}$$

So the divergence form of the minimal surface equation(abbreviated as **(MSE)** in the following)

$$(i): \qquad \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}\left(\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{3}$$

Expanding the above equation, we get:

$$\frac{u_{x_i x_i}}{\sqrt{1+|\nabla u|^2}} - \frac{u_{x_i} u_{x_j} u_{x_i x_j}}{(1+|\nabla u|^2)^{3/2}} = 0,$$

which can be rewritten as:

$$\frac{1}{\sqrt{1+|\nabla u|^2}}\left(\delta_{ij} - \frac{u_i}{\sqrt{1+|\nabla u|^2}}\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right)u_{x_i x_j} = 0. \tag{4}$$

So we get the non divergence form of (MSE):

$$(ii): \qquad \frac{1}{\sqrt{1+|\nabla u|^2}} \sum_{i,j=1}^{n-1} g^{ij} u_{x_i x_j} = 0, \tag{5}$$

where $g^{ij} = \delta_{ij} - \nu_i \nu_j$ is the inverse matrix for the induced metric.

Now let us calculate the mean curvature of Σ_u from definition. Firstly,

$$\vec{A}(\partial_{x_i}, \partial_{x_j}) = (F_{x_i x_j})^{\perp} = (\underbrace{F_{x_i x_j} \cdot \nu}_{h_{ij}})\nu.$$

It is easy to see that

$$F_{x_i x_j} = (0, \cdots, 0, u_{x_i x_j}),$$

so $h_{ij} = \frac{u_{x_i x_j}}{\sqrt{1+|\nabla u|^2}}$. So it is easy to see that equation 5 is equivalent with equation 4.

1.2 Calibrated property of minimal graphs

Now extend the unit normal vector field ν to $\tilde{\nu}$ in $\Omega \times \mathbb{R}$ in depend of x_n.

Claim: (MSE) $\Longrightarrow div_{\mathbb{R}^n}(\tilde{\nu}) = 0$.

This is because $div_{\mathbb{R}^n}(\tilde{\nu}) = -\sum_{i=1}^{n-1} \partial_{x_i} \nu_i + \underbrace{\partial_{x_n} \nu_n}_{\equiv 0} = 0$ by (MSE).

We can define an $n-1$ form ω as

$$\omega = (-1)^{n-1} \tilde{\nu} \rfloor dx,$$

where $dx = dx_1 \wedge \cdots \wedge dx_{n-1}$ is the volume form of $\mathbb{R}^n$.

Claim: $div_{\mathbb{R}^n}(\tilde{\nu}) = 0 \Longrightarrow d\omega = 0$.

Proof:

$$\omega = (-1)^{n-1} \tilde{\nu} \rfloor dx = (-1)^{n-1} \sum_{j=1}^{n} (-1)^j dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_n.$$

So $d\omega = (-1)^n div(\tilde{\nu}) dx = 0$.

□

Definition 1.3 *Given an $n-1$ dimensional plane $V^{n-1} \subset \mathbb{R}^n$ in $\mathbb{R}^n$ with an oriented orthonormal frame $e_1, \cdots, e_{n-1}$, we can define:*

$$\omega(V) = \omega(e_1, \cdots, e_{n-1}).$$

Properties of ω : (1) $d\omega = 0$; (2) $|\omega_{\tilde{x}}(V)| \leq 1$, $\forall \tilde{x} \in \Omega \times \mathbb{R}$ and $\forall V$, and $\omega_{\tilde{x}}(V) = 1$ only if $V = T_{\tilde{x}}\Sigma_{\tilde{x}}$.

Proof: (of Property (2)) This comes from the following equation and basic linear algebra.

$\tilde{\nu}(\tilde{x})\rfloor dx(e_1, \cdots, e_{n-1}) = dx(\tilde{\nu}, e_1, \cdots, e_{n-1}) = det(\tilde{\nu}(\tilde{x}), e_1, \cdots, e_{n-1})$.

□

Theorem 1.1 *(1) Σ is volume minimizing in $\Omega \times \mathbb{R}$;*
(2) If Ω is convex, then Σ is volume minimizing in $\mathbb{R}^n$.

Proof: (1) For any $n-1$ dimensional sub manifold $\Sigma_1^{n-1} \subset \Omega \times \mathbb{R}$, with $\partial\Sigma_1 = \partial\Sigma$, we can form the n dimensional chain U such that $\partial\Sigma_1 \cup \Sigma = \partial U$. Using the Stokes Theorem and property (2) as above:

$$\underbrace{\int_\Sigma \omega}_{=|\Sigma|} - \underbrace{\int_{\Sigma_1} \omega}_{\leq|\Sigma_1|} = \int_U d\omega = 0.$$

Here ω is called the **calibrated form**.

(2) Consider the nearest point projection map $F : \mathbb{R}^n \to \Omega \times \mathbb{R}$. F is distance decreasing. So we can firstly contract any Σ_1 with the same boundary as Σ to $\Omega \times \mathbb{R}$ which decreases the area and then use part (1).

□

1.3 A general calibrated argument

Theorem 1.2 *Suppose Ω is an open region in an oriented Riemannian manifold M^n, and there exist a foliation of Ω by oriented minimal hyper surfaces, then every leaves of the foliation minimizes volume in Ω.*

Proof: Let $\nu(x)$ be the unit normal vector fields of the foliation. Then
Claim: $div\nu = 0$ in Ω if each Σ is minimal.
To prove this, take $\{e_1, \cdots, e_{n-1}\}$ to be tangent orthonormal frames of the foliation, then:

$$div\nu = \sum_{i=1}^{n-1} \langle \nabla_{e_i}\nu, e_i\rangle + \underbrace{\langle \nabla_\nu \nu, \nu\rangle}_{=0, \text{ as } \nu \; is \; unit.} = -H_{\Sigma_x} = 0.$$

Define

$$\omega = (-1)^{n-1}\nu\rfloor dv_M.$$

Using ω as a calibrated form and arguments above, we can show the minimizing property.

□

2 First variation and consequences (1/12/2012)

2.1 First Variation Formula

Consider $\Sigma^k \subset M^n$. Let X be a smooth v.f.(abbreviated for vector field in the following) on M with compact support. Let $F_t : M \to M$ be a family of diffeomorphisms such that $F_0 = id$ and $\frac{d}{dt}|_{t=0} F_t = X$. Then

$$\text{(First Variation Formula:)} \quad \underline{\delta\Sigma(X) = \int_\Sigma div_\Sigma(X) d\mu.} \tag{6}$$

Here $div_\Sigma(X) = Tr_g(\langle \nabla .X, \cdot \rangle) = \sum_{i=1}^k \langle \nabla_{e_i} X, e_i \rangle$, with $\{e_1, \cdots, e_k\}$ an o.n.(abbreviated for orthonormal in the following) basis for Σ.

When Σ is smooth, we can decompose $X = X^T + X^\perp$ to tangent X^T and normal $X^\perp$ parts. So

$$div_\Sigma(X) = div_\Sigma(X^T) + div_\Sigma(X^\perp),$$

where $div_\Sigma(X^\perp) = -\langle \vec{X}, \vec{H} \rangle$. Using the divergence theorem, for general X, we have

$$\delta\Sigma(X) = -\int_\Sigma \langle X, H \rangle d\mu + \int_{\partial\Sigma} \langle X, \eta \rangle d\sigma,$$

where η is the outer normal of $\partial\Sigma$. So we know that

$$\Sigma \text{ is minimal}(\vec{H} = 0) \Longleftrightarrow \delta\Sigma(X) = 0, \ \forall X \text{ of compact support.}$$

Proof: (of 1st Variation formula) Consider a local parametrization

$$F : \Sigma \times (-\epsilon, \epsilon) \to M,$$

where $F(x,t) = F_t(x)$ with F_t given to be the integration of X above. Let $\{x^1, \cdots, x^k\}$ be local coordinates of Σ, then

$$\frac{d}{dt}|_{t=0} |\Sigma_t| = \int \frac{d}{dt}|_{t=0} \sqrt{detg(t)} dx.$$

Now $\frac{d}{dt}|_{t=0} \sqrt{detg(t)} = g^{ij} \langle \nabla_{\partial_{x^i}} \dot{F}, \partial_{x^j} \rangle \sqrt{detg} = div_\Sigma X \sqrt{detg}$. So we are done.

□

2.2 Examples

2.3 Convex hull property

Consider $\Sigma^k \subset \mathbb{R}^n$. Let $x_1, \cdots, x_n$ be coordinates on $\mathbb{R}^n$, then we have:

Propostion 2.1

$$\vec{H} = 0 \Longleftrightarrow \triangle_\Sigma x_i = 0, \ \forall i.$$

Proof: Let $\{e_1, \cdots, e_k\}$ be a local o.n. basis for $T\Sigma$, then

$$\triangle_\Sigma x^i = \sum_{j=1}^{k}(e_j e_j x^i - (\nabla^\Sigma_{e_j} e_j)x^i) = \sum_{j=1}^{k}(\nabla^\perp_{e_j} e_j)x^i.$$

As $e_j \vec{x} = e_j$ and $\sum \nabla^\perp_{e_j} e_j = \vec{H}$,

$$\triangle_\Sigma \vec{x} = \vec{H},$$

where $\vec{x}$ is the position vector.

□

Corollary 2.1 *If $(\Sigma^k, \partial\Sigma)$ is compact minimal in $\mathbb{R}^n$, then $\Sigma \subset \mathcal{C}(\partial\Sigma)$, where $\mathcal{C}(A)$ is the convex hall of A.*

Proof: $\mathcal{C}(A) = \cap\{H : \text{ closed half spaces with } A \subset H\}$. Now $H = \{x : \ l(x) \leq a\}$, where l is some linear function and $a \in \mathbb{R}$.

$$\partial\Sigma \subset H \Longrightarrow l(x) \leq a, \ \forall x \in \partial\Sigma \ \& \ \triangle_\Sigma l(x) = 0, \ \forall x \in \Sigma,$$

$$\Longrightarrow l(x) \leq a, \ \forall x \in \Sigma \text{ (weak M.P.)} \Longrightarrow \Sigma \subset H.$$

□

2.4 Fluxes

In the case of a minimal sub manifold $\Sigma^k \subset \mathbb{R}^n$,

$$\triangle_\Sigma x_i = 0 \Longleftrightarrow div_\Sigma(\nabla x_i) = 0 \Longleftrightarrow *dx_i \text{ is closed.}$$

Hence $*dx_i$ defines a $(k-1)$ dimensional deRham cohomology class. So for any $k-1$ cycle $\Gamma^{k-1} \subset \Sigma^k$, we can define

$$F([\Gamma]) = \int_\Gamma *dx_i = \int_\Gamma \langle \frac{\partial}{\partial x_i}, \eta\rangle d\sigma,$$

where η is the unit outer normal of Γ. The second "$=$" follows from the fact that $*dx_i = i_{\frac{\partial}{\partial x_i}} dv_\Sigma = \langle \frac{\partial}{\partial x_i}, \eta\rangle d\sigma$, with $i.dv$ the inner multiplication. Hence we have a group homeomorphism:

$$F : H_{k-1}(\Sigma) \to \mathbb{R}.$$

For the general cases, if $\Sigma^k \subset M^n$ is a minimal sub manifold, i.e. $\vec{H} = 0$, and X a Killing vector field on M, then there exists a homeomorphism:

$$F_X : H_{k-1}(\Sigma, \mathbb{Z}) \to \mathbb{R}.$$

Proof: Let $V = X^T$ the tangential part of X on Σ, then

$$div_\Sigma(V) = \sum_{i=1}^{k} \langle \nabla_{e_i}(X - X^\perp), e_i \rangle = \sum_{i=1}^{k} \underbrace{\langle \nabla_{e_i} X, e_i \rangle}_{=0,\text{ as } \mathcal{L}_X g = 0} - \underbrace{\sum_{i=1}^{k} \langle \nabla_{e_i} X^\perp, e_i \rangle}_{=\vec{H}\cdot X = 0} = 0.$$

Let $\omega = V \rfloor dvol_\Sigma$, then ω is a closed $k-1$ form on Σ, i.e. $d\omega = 0$. Hence we can define the flux as above.

□

3 Monotonicity formula and 2-d Bernstein Theorem (1/17/2012)

3.1 Monotonicity formula

Theorem 3.1 *Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal surface, then*

$$\frac{|B_t(x_0) \cap \Sigma|}{t^k} - \frac{|B_s(x_0) \cap \Sigma|}{s^t} = \int_{\Sigma \cap (B_t(x_0) \setminus B_s(x_0))} \frac{|(x - x_0)^\perp|^2}{|x - x_0|^{k+2}} dv_\Sigma,$$

where $B_t(x_0)$ is a ball of radius t with center x_0 in $\mathbb{R}^n$; $|B_t(x_0) \cap \Sigma|$ is the volume in Σ; and $(x - x_0)^\perp$ is the projection to the normal part of Σ of $(x - x_0)^\perp$.

We need the co-area formula before the proof.

Lemma 3.1 *(Co-area Formula) Let $h : \Sigma \to \mathbb{R}_+$ be a nonnegative Lipschitz function on a Riemannian manifold Σ, and proper i.e. $\{x \in \Sigma : \ h(x) \leq a\}$ is compact for all a. Given f integrable on Σ, then*

$$\int_{h \leq t} f |\nabla_\Sigma h| = \int_{-\infty}^{t} (\int_{h=t} f) d\tau.$$

Remark 3.1 *This follows heuristically from the fact that $dv_\Sigma = \frac{dt \wedge dv_{\{h=t\}}}{|\nabla_\Sigma h|}$ when t is a regular value.*

Proof: (Monotonicity formula) Take $h(x) = |x - x_0|$, then $\{h \leq t\} = B_t(x_0)$. Let $X = x - x_0$, then $div_\Sigma(X) = \sum_{i=1}^{k} \nabla_{e_i} X \cdot e_i = k$, where $\{e_1, \cdots, e_k\}$ is an o.n. basis on Σ. Then by the first variation formula 6,

$$\delta \Sigma_{B_r(x_0)}(X) = \int_{\Sigma \cap B_r(x_0)} div_{\Sigma_0}(X) = \int_{\Sigma \cap \partial B_r(x_0)} X \cdot \eta,$$

where η is the co-normal vector of $\Sigma \cap \partial B_r(x_0)$, and $\eta = \frac{\nabla^\Sigma |x - x_0|}{|\nabla^\Sigma |x - x_0||} = \frac{(x - x_0)^T}{|(x - x_0)^T|}$. Using the Co-area formular,

$$k|\Sigma \cap B_r(x_0)| = \int_{\Sigma \cap \{|x - x_0| = r\}} |(x - x_0)^T| = \frac{d}{dr} \int_{\Sigma \cap B_r(x_0)} |(x - x_0)^T| |\nabla^\Sigma |x - x_0||,$$

$$= \frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} \frac{|(x-x_0)^T|^2}{|x-x_0|} = r\frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} \frac{|(x-x_0)^T|^2}{|x-x_0|^2},$$

$$= r\frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} (1-\frac{|(x-x_0)^\perp|^2}{|x-x_0|^2}),$$

$$= r\frac{d}{dr}|\Sigma\cap B_r(x_0)| - r\frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^2}.$$

Multiplying the above by r^{-k-1}, we can re-write it as,

$$\frac{d}{dr}(r^{-k}|\Sigma\cap B_r(x_0)|) = r^{-k}\frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^2},$$

$$= \frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^{k+2}}.$$

In the last step, we can use the co-area formula again to absorb the factor r^{-k} into the integration. So we can get the monotonicity formula by integrating the above equation.

□

Corollary 3.1 *Let Σ^k be a smooth minimal surface in $\mathbb{R}^n$, with boundary $\Sigma\cap\partial B_R(0)$ inside the ball $B_R(0)$. If $x_0\in\Sigma\cap B_R(0)$, and $\sigma < R-|x_0|$, then*

$$\omega_k\sigma^k \le |\Sigma\cap B_\sigma(x_0)| \le \frac{\sigma^k}{(R-|x_0|^k)}||\Sigma\cap B_R(0),$$

where ω_k is the volume of unit ball $B_1^k(0)$ in $\mathbb{R}^k$.

Proof: The first " $\le$ " comes from the Monotonicity formula while comparing $B_\sigma(x_0)$ with an arbitrary small ball $B_r(x_0)$, with $\lim_{r\to 0} r^{-k}|\Sigma\cap B_r(x_0)| = \omega_k$, when $x_0\in\Sigma$ and Σ smooth. The second $\le$ is a direct consequence of the Monotonicity formula while comparing $B_\sigma(x_0)$ with a large ball $B_r(x_0)$ exhausting the whole $B_R(0)$.

□

Definition 3.1 *The **density** of Σ at x_0 is defined as:*

$$\Theta_{x_0} = \lim_{r\to 0}(\omega_k r^k)^{-1}|\Sigma\cap B_r(x_0)|.$$

3.2 Bernstein's theorem (n=2)

Theorem 3.2 S. Bernstein (1912) *Given a minimal graph* $\Sigma^2 \subset \mathbb{R}^3$, $\Sigma = \{(x, u(x)) : x \in \mathbb{R}^2\}$. *If* u *is defined on all of* $\mathbb{R}^2$, *then* u *is a linear function, and* Σ *is a plane.*

Theorem 3.3 Bernstein's Big Theorem: 1° *PDE version: let* $u \in C^2(\mathbb{R}^2)$ *and* $\sum_{i,j=1}^2 a_{ij} u_i u_j = 0$, *with* $(a_{ij}) > 0$. *If* u *is bounded, then* $u \equiv const$;
$2^\circ : \Sigma^2 - Graph_u$, *where* u *is defined on* R^2 *and bounded, if the Gaussian curvature* $K_\Sigma \leq 0$, *then* Σ *is a plane.*

Consider the Gauss Maps:

$$N : \Sigma^2 \to S^2,$$

where N maps a point to the unit normal vector at that point.

Lemma 3.2 *If* $\vec{H} = 0$, *then* N *is a conformal and orientation reversing map, i.e.* $\forall v, w \in T_x\Sigma$, *if* $v \cdot w = 0$ *and* $|v| = |w|$, *then* $\nabla_v N \cdot \nabla_w N = 0$, *and* $|\nabla_v N| = |\nabla_w N|$. *Furthermore* $|\nabla_v N| \leq \frac{1}{\sqrt{2}}|A||v|$, *and* $N^*(\omega_{S^2}) = K_\Sigma \omega_\Sigma = -\frac{1}{2}|A|^2 \omega_\Sigma$.

Proof: We only need to check that under a special o.n. basis. Take an o.n. principle basis $\{e_1, e_2\}$ for Σ, i.e. $\nabla_{e_1} N = -K_1 e_1$, $\nabla_{e_2} N = -K_2 e_2$. So $|\nabla_{e_1} N| = |K_1| = |K_2| = |\nabla_{e_2} N|$, by the minimality $H = K_1 + K_2 = 0$. Hence $|\nabla_v N| \leq |K||v| = \frac{1}{\sqrt{2}}|A|$. Furthermore, the Jacobian of N is $Jac(N) = K_1 K_2 = -\frac{1}{2}|A|^2$.

□

Propostion 3.1 *Given a minimal* $\Sigma^2 \subset \mathbb{R}^3$, *with the image of the Gauss Maps lying in the upper hemisphere* $N(\Sigma) \subset S^2_+$, *if* φ *has compact support on* Σ, *then there exists a constant* $C > 0$, *such that*

$$\int_\Sigma |A|^2 \varphi^2 \leq C \int_\Sigma |\nabla \varphi|^2.$$

Proof: Since S^2_+ is simply connected, the closed form $\omega_{S^2} = d\alpha$ is also exact. Hence

$$-\frac{|A|^2}{2}\omega_\Sigma = N^*\omega_{S^2} = d(N^*\alpha).$$

So

$$\int_\Sigma |A|^2 \varphi^2 \omega_\Sigma = -2\int_\Sigma \varphi^2 d(N^*\alpha) = 4\int_\Sigma \varphi d\varphi \wedge N^*\alpha$$
$$\leq 4\int_\Sigma |\varphi||\nabla\varphi||N^*\alpha|\omega_\Sigma.$$

Since $|N^*\alpha| \leq |A||\alpha| \leq C|A|$,

$$\int_\Sigma |A|^2\varphi^2 \leq C\int_\Sigma (|\varphi||A|)(|\nabla\varphi|) \leq \frac{\epsilon}{2}\int |A|^2\varphi^2 + \frac{C}{2\epsilon}\int |\nabla\varphi|^2.$$

So we get the inequality.

□

Propostion 3.2 *Let Σ be an entire minimal graph, then $|\Sigma \cap B_R(0)| \leq 4\pi R^2$, $\forall R > 0$.*

Proof: This comes from the area-minimizing property of minimal graphs. We can compare $|\Sigma \cap B_R(0)|$ with the large area of the truncated surfaces of $B_R(0)$ by Σ.

□

Lemma 3.3 *When $\Sigma = Graph_u$ and u is an entire function on $\mathbb{R}^2$, then we can choose a Lipschetz $\varphi = \varphi_R$, such that $\int_\Sigma |\nabla\varphi|^2 \to 0$ as $R \to \infty$.*

Proof: Now choose

$$\varphi_R(r) = \begin{cases} 1 & \text{if } r \leq R \\ 1 - \log(r/R)/\log R & \text{if } R < r < R^2 \\ 0 & \text{if } r \geq R^2 \end{cases}$$

where r is the distance function of $\mathbb{R}^3$. By discretize $B_{R^2} \setminus B_R = \cup_{k=1}^{\log R}(B_{e^k R} \setminus B_{e^{k-1}R})$, we have

$$\int_\Sigma |\nabla\varphi|^2\omega_\Sigma = \frac{1}{(\log R)^2}\int_{\Sigma\cap(B_{R^2}\setminus B_R)} \frac{1}{r^2}\omega_\Sigma = \frac{1}{(\log R)^2}\sum_{k=1}^{\log R}\int_{\Sigma\cap(B_{e^k R}\setminus B_{e^{k-1}R})}\frac{1}{r^2}\omega_\Sigma$$

$$\leq \frac{1}{(\log R)^2}\sum_{k=1}^{\log R}\frac{1}{(e^{k-1}R)^2}|\Sigma\cap(B_{e^k R}\setminus B_{e^{k-1}R})| \leq \frac{1}{(\log R)^2}\sum_{k=1}^{\log R}\frac{1}{(e^{k-1}R)^2}C(e^k R)^2$$

$$= \frac{C}{(\log R)^2}\sum_{k=1}^{\log R}\frac{1}{e^2} = \frac{C}{\log R} \to 0,$$

where in the second " $\leq$ ", we used the quadratic area bound Lemma above.

□

Proof: (Bernstein's Theorem) When $\Sigma = Graph_u$ is an entire graph, the image of the Gauss Maps $N(\Sigma)$ lies in an hemisphere, so we get $\int_\Sigma |A|^2\varphi^2 \leq C\int_\Sigma |\nabla\varphi|^2$. Then if we take the φ_R in the above Lemma, and let $R \to \infty$, we see that $\int_\Sigma |A|^2 \to 0$. So $A = 0$, and Σ is a plane.

□

4 Second variation and Stability (1/19/2012)

4.1 Second Variation of Volume

Consider a minimal $\Sigma^k \subset M^n$, i.e. $\vec{H} = 0$. Given a vector field X on Σ, let $F(x,t) : \Sigma\times[-\epsilon,\epsilon] \to M$ be a one parameter family of variations, with $\dot{F}(x,0) = X$ and denote $\Sigma_t = F_t(\Sigma)$.

Theorem 4.1 *The* *Second Variation Formula* *is:*

$$\delta^2\Sigma(X,X) \equiv \frac{d^2}{dt^2}|_{t=0}|\Sigma_t| = \int_\Sigma [|\nabla^\perp X|^2 - |\langle \vec{A}, X\rangle|^2 - \sum_{i=1}^k R^M(e_i, X, e_i, X)], \tag{7}$$

where $\{e_1,\cdots,e_k\}$ *is an o.n. basis tangent to* Σ*, and* X *is compact supported and normal on* Si*.*

Theorem 4.2 *In the case* $\Sigma^{n-1}\subset M^n$ *is a hyper surface and 2-sided(*$\exists\nu$ *unit normal),* $X=\varphi\nu$*, with* φ *a function with compact support, then*

$$\delta^2\Sigma(\varphi,\varphi) = \int_\Sigma [|\nabla\varphi|^2 - (|A|^2 + Ric^M(\nu,\nu))\varphi^2] \tag{8}$$

Proof: (of Second Variation Formula)

- $F:\Sigma\times(-\epsilon,\epsilon)\to M$, with $\{x^1,\cdots,x^k\}$ local coordinates on Σ. Then $dv_t=\sqrt{detg(t)}dx$, where $g_{ij}(t)=\langle\frac{\partial F}{\partial x^i},\frac{\partial F}{\partial x^j}\rangle$.
- $$\frac{d}{dt}\sqrt{detg(t)} = \frac{1}{2}g^{ij}\dot{g}_{ij}\sqrt{detg(t)}$$
- $$\frac{d^2}{dt^2}\sqrt{detg(t)} = \frac{1}{4}(g^{ij}\dot{g}_{ij})^2\sqrt{detg} + \frac{1}{2}g^{ij}\ddot{g}_{ij}\sqrt{detg} + \frac{1}{2}(\dot{g}^{ij}\dot{g}_{ij})\sqrt{detg},$$
  where $\dot{g}^{ij}=-g^{ik}g^{jl}\dot{g}_{kl}$.
- $$\begin{aligned}\dot{g}_{ij} &= \langle\nabla_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial x^i},\frac{\partial F}{\partial x^j}\rangle + \langle\frac{\partial F}{\partial x^j},\nabla_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial x^i}\rangle\\ &= |_{t=0}\langle\vec{A}(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}),X\rangle.\end{aligned}$$

  So

  $$(g^{ij}\dot{g}_{ij})|_{t=0} = -2\langle\vec{H},X\rangle = 0,$$

  and

  $$(\dot{g}^{ij}\dot{g}_{ij})|_{t=0} = -4|\langle\vec{A},X\rangle|^2.$$
- $$\begin{aligned}\ddot{g}_{ij} &= \langle\nabla_{\frac{\partial F}{\partial t}}\nabla_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial t},\frac{\partial F}{\partial x^j}\rangle + \langle\nabla_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial t},\nabla_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial x^j}\rangle + i,j \text{ reversed terms}\\ &= \langle R^M(\frac{\partial F}{\partial t},\frac{\partial F}{\partial x^i})\frac{\partial F}{\partial t},\frac{\partial F}{\partial x^j}\rangle + \langle\nabla_{\frac{\partial F}{\partial x^i}}\ddot{F},\frac{\partial F}{\partial x^j}\rangle + \langle\nabla_{\frac{\partial F}{\partial x^i}}X,\nabla_{\frac{\partial F}{\partial x^j}}X\rangle\\ &\qquad + i,j \text{ reversed terms}\\ &= |_{t=0} - R^M(X,\partial x^i,X,\partial x^j) + \langle\nabla_{\partial x^i}\ddot{F},\partial x^j\rangle + \langle\nabla_{\partial x^i}X,\nabla_{\partial x^j}X\rangle\\ &\qquad + i,j \text{ reversed terms}\end{aligned}$$

  So

  $$(g^{ij}\ddot{g}_{ij})|_{t=0} = -2g^{ij}R^M(X,\partial x^i,X,\partial x^j) + 2div_\Sigma\ddot{F} + 2|\nabla^\perp X|^2 + \underbrace{2g^{ij}\langle\nabla^T_{\partial x^i}X,\nabla^T_{\partial x^j}X\rangle}_{=2|\langle\vec{A},X\rangle|^2}$$

- Combining all the above,

$$\frac{d^2}{dt^2}|_{t=0}\sqrt{detg(t)} = div_\Sigma \ddot{F} + |\nabla^\perp X|^2 - |\langle \vec{A}, X\rangle|^2 - g^{ij}R^M(X, \partial x^i, X, \partial x^j).$$

An integration on Σ finishes the proof.

□

4.2 Jacobi operator and Stability

- Hypersurface Case $k = n-1$: $X = \varphi\nu$,

$$I(\varphi,\varphi) \equiv \delta^2\Sigma(\varphi,\varphi) = -\int_\Sigma \varphi L\varphi,$$

where the Jacobi operator is

$$L\varphi = \triangle\varphi + (\underbrace{|A|^2 + Ric(\nu,\nu)}_{Q})\varphi. \tag{9}$$

- When boundary exists $(\Sigma, \partial\Sigma)$, L has discrete eigenvalues λ_j and eigenfunctions u_j, i.e. $Lu_j + \lambda_j u_j = 0$ in Σ with $u_j = 0$ on $\partial\Sigma$, and

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots,$$

with $\lambda_n \to +\infty$.

Definition 4.1 *Σ is stable if $\lambda_1 \geq 0$, i.e. $I(\varphi,\varphi) \geq 0$, $\forall\varphi$, with $\varphi = 0$ on $\partial\Sigma$.*

- Morse Index=# of negative eigenvalues counted with multiplicity.
- (Properties of λ_1) λ_1 has multiplicity 1;
- If u_1 is an eigenfunction of λ_1, u_1 does not change sign.

Proof: By the variational characterization, u_1 minimizes $I(\varphi,\varphi)$ among all φ with $\varphi \equiv 0$ on $\partial\Sigma$ and $\int_\Sigma \varphi^2 = 1$. Since $|(|\varphi|, |\varphi|) = I(\varphi,\varphi)$, if u_1 is the first eigenfunction, so is $|u_1|$. So $u_1 = |u_1|$, or there is a contradiction to $u_1 \in C^\infty$. The fact that u_1 does not change sign shows that the dimension of the eigen space of λ_1 is 1, or we can always form some eigenfunction changing sign.

□

- When Σ is non-compact, then

$$Ind(\Sigma) = \lim_{i\to\infty} Ind(\Omega_i),$$

where $\{\Omega_i\}_{i=1}^\infty$ is an open exhaustion of Σ, i.e. $\Sigma = \cup_{i=1}^\infty \Omega_i$, $\overline{\Omega_i} \subset \Omega_{i+1}$, with $\partial\Omega_i$ smooth and compact.

Remark 4.1 *In fact, the definition is independent of the exhaustion, say $\{\Omega_i\}$ and $\{\tilde{\Omega}_i\}$. Since $Ind(\Omega)$ is non decreasing when Ω is expanding, so we can always embed $\Omega_i \subset \tilde{\Omega}_{i'}$ for $i' \gg i$, so $Ind(\Omega_i) \leq Ind(\tilde{\Omega})_{i'}$, and $\lim_{i\to\infty} Ind(\Omega_i) \leq \lim_{i'\to\infty} Ind(\tilde{\Omega}_{i'})$, and vise versa.*

- When Σ is open:

$$\lambda_1(\Sigma) = \lim_{i\to} \lambda_1(\Omega_i) \in [-\infty, \lambda_1(\Omega_1)),$$

Σ is stable if $\lambda_1(\Sigma) \geq 0$, or equivalently $\lambda_1(\Omega) \geq 0$ for all $\overline{\Omega} \subset \Sigma$.

Remark 4.2 *By the variational characterization, $\lambda_1(\Omega)$ is strictly decreasing as Ω is expanding, so we can argue as above to show the well-definedness of $\lambda_1(\Sigma)$.*

5 Criterion for stability (1/24/2012)

Theorem 5.1 *Assume $\Sigma^{n-1} \subset M^n$ is a 2-sided minimal hypersurface.*
1° *Σ is stable $\Longleftrightarrow$ $\exists u > 0$, s.t. $Lu \leq 0$;*
2° *Σ(non-compact) is stable $\Longleftrightarrow$ $\exists u > 0$, s.t. $Lu = 0$.*

Remark 5.1 *This can be viewed as an infinitesimal version of the Calibration argument i.e. using foliation of minimal surfaces.*

Proof: $1^\circ + 2^\circ \Longleftarrow$: Since $Lu = \triangle_\Sigma u + Qu \leq 0$, let $w = \log u (u > 0)$, we have

$$\triangle w = \frac{\triangle u}{u} - |\nabla w|^2 \leq -Q - |\nabla w|^2.$$

Then $\forall \varphi$ compactly supported,

$$\int_\Sigma \varphi^2(\triangle w + Q) \leq -\int_\Sigma \varphi^2 |\nabla w|^2.$$

Using integration by part formula,

$$\int_\Sigma Q\varphi^2 \leq \int 2\varphi\langle\nabla\varphi, \nabla w\rangle - |\nabla w|^2\varphi^2 \leq \int 2|\varphi||\nabla\varphi||\nabla w| - |\nabla w|^2\varphi^2$$

$$\leq \int |\nabla\varphi|^2 + \varphi^2|\nabla w|^2 - |\nabla w|^2\varphi^2 \leq \int_\Sigma |\nabla\varphi|^2.$$

Hence we have the stability inequality for Σ.
$1^\circ \Longrightarrow$: Assume Σ is compact, then $\exists u > 0$, which is the first eigenfunction, such that $\lambda_1(\Sigma) \geq 0$, so

$$Lu = -\lambda_1 u \leq 0.$$

$2^\circ \Longrightarrow$: Assume Σ is non-compact and stable, then Σ has an exhaustion $\Sigma = \cup_{i=1}^\infty \Omega_i$, and $\lambda_1(\Omega_i) > 0$ for all i. Now by elementary elliptic PDE, $\forall\psi$ on $\partial\Omega_i$, $\exists! u$ in Ω_i, such that

$$Lu = 0 \text{ in } \Omega_i, \text{ with } u = \psi \text{ on } \partial\Omega_i.$$

Claim: $\lambda_1(\Omega_i) > 0 \Longrightarrow$ if $\psi > 0$, then $u > 0$.
(If $u \le 0$, then $\Omega_{\{u\le 0\}}$ has eigenvalue equals 0, since u is then a Dirichlet eigenfunction on $\Omega_{\{u\le 0\}}$ with 0 boundary values, which is a contradiction to $\lambda_1(\Omega) > 0$.)
Now we can solve the boundary value problem for $u_i > 0$:

$$Lu_i = 0 \text{ in } \Omega_i, \text{ with } u = 1 \text{ on } \partial\Omega_i.$$

Consider the normalized sequence $\{\frac{u_i}{u_i(0)}\}_{i=1}^{\infty}$,
Claim: there exists a subsequence $i' \to \infty$, such that

$$\frac{u_{i'}}{u_{i'}(0)} \to u \text{ in } C^2 \text{ on compact subset of } \Sigma.$$

Pf:

- C^0 bound and elliptic theory of $L \Longrightarrow C^3$ bound for $\frac{u_{i'}}{u_{i'}(0)}$ on compact set;
- Harnack inequality $\Longrightarrow C^0$ bound.
- Harnack inequality: If $u > 0$ is a positive solution of $Lu = 0$ on $\Omega^{open} \subset M$, and if $\overline{\Omega}_1^{compact} \subset \Omega$, then $\exists c = c(L, \Omega_1) > 0$, such that

$$\max_{\overline{\Omega}_1} u \le c \min_{\overline{\Omega}_1} u.$$

The limit u is a positive solution of $Lu = 0$.

□

Theorem 5.2 *If Σ is a stable 2-sided minimal hyper surface in M, and $\hat{\Sigma}$ is any covering of Σ, then $\hat{\Sigma}$ is also a stable minimal hyper surface in M.*

Remark 5.2 *Σ can be viewed as a minimal immersion $i : \Sigma \to M$, so if $\pi : \hat{\Sigma} \to \Sigma$ is the covering map, then $\hat{\Sigma}$ is viewed as a minimal immersion $i \circ \pi : \hat{\Sigma} \to \Sigma$.*

The 2-sided property is essential here. A counterexample is $\mathbb{RP}^1 \subset \mathbb{RP}^2$, while the covering space S^1 is not stable.

Proof:

- Σ is 2-sided and stable $\Longrightarrow \exists u > 0$ on Σ with $Lu \le u$.
- Let $\pi : \hat{\Sigma} \to \Sigma$ be the covering map, then $u \circ \pi > 0$ on $\hat{\Sigma}$ and $L(u \circ \pi) \le 0$, hence $\hat{\Sigma}$ is stable.

□

Propostion 5.1 1°. *Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be a 2-sided minimal surface, and if the Gauss image $G(\Sigma) \subset S_+^{n-1}$, then Σ is stable;*
2°. *Let $\Sigma^2 \subset \mathbb{R}^3$ be a 2-sided minimal surface, and if $G(\Sigma) \subset U^{open} \subset S^2$, with $\mu_1(U) \ge 1$, where $\mu_1(U)$ is the Dirichlet eigenvalue of $\triangle_{S^2}$ on U, then Σ is stable. In perticular, $\mu_1(U) \ge 1$ is true if the area $|U| \le 2\pi$.*

Proof: 1°. Let $e \in \mathbb{R}^n$ be the direction vector to the north pole, and let $u = e \cdot \nu$, where ν is the normal vector field of Σ, since the parallel translation in the e direction does not change the area of Σ, we have $\underline{Lu = 0}$. Since $G(\Sigma) \subset S_+^{n-1} \Longleftrightarrow e \cdot \nu > 0$, so $u > 0$, hence Σ is stable.
2°. $\mu_1(U) \geq 1 \Longrightarrow \exists v > 0$ on U such that

$$\begin{cases} \triangle_{S^2} v = \mu_i(U) v \leq -v, & \text{in } U, \\ v = 0, & \text{on } U. \end{cases}$$

Let $u = v \circ G$, where G is the Gauss Map. By Lemma 3.2, $G : \Sigma \to S^2$ is a conformal map, so

$$\triangle_\Sigma u = |A|^2 (\triangle_{S^2} v) \circ G \leq -|A|^2 u,$$

i.e. $Lu \leq 0$, hence Σ is stable. (In fact, on 2-dimension, the Jacobi operator $L = G^*(\triangle_{S^2} + 1)$.)

□

6 Bochner formula and 2-d stable minimal surface (1/26/2012)

6.1 Bochner Formula

Let (Σ^k, g) be a Riemannian manifold, and $\{e_1, \cdots, e_k\}$ an o.n. frame, with $\{\theta^1, \cdots, \theta^k\}$ the dual frame. Denote

$$(\nabla_{e_j} \alpha) = \sum_i \alpha_{i,j} \theta^i,$$

$$\nabla_{e_i}(\nabla \alpha) = \sum_{i,j} \alpha_{i,jk} \theta^i \otimes \theta^j;$$

then

$$\nabla \alpha = \sum_{i,j} \alpha_{i,j} \theta^i \otimes \theta^j, \quad \nabla^2 \alpha = \sum_{i,j,k} \alpha_{i,jk} \theta^i \otimes \theta^j \otimes \theta^k.$$

Ricci Formula:

$$\boxed{\alpha_{i,jk} - \alpha_{i,kj} = \textstyle\sum_p \alpha_p R^\Sigma_{pijk}.}$$

Definition 6.1 *α is harmonic if $d\alpha = 0$ and $\delta\alpha = 0$(i.e. $\alpha_{i,j} = \alpha_{j,i}$ and $\sum_i \alpha_{i,i} = 0$).*

Bochner Formula: If α is harmonic, then

$$\triangle \alpha = Ric(\alpha^\sharp, \cdot),$$

where $\alpha^\sharp$ the vector field dual to α, and $\triangle \alpha = \sum_{i,j} \alpha_{i,jj} \theta^i$ is the rough laplacian.
Proof:

$$\sum_j \alpha_{i,jj} = \sum_j \alpha_{j,ij} = \underbrace{\sum_j \alpha_{j,ji}}_{=0} + \sum_{p,j} \alpha_p R^\Sigma_{pjij} = \sum_p \alpha_p Ric^\Sigma_{pi}.$$

□

Hence we have:

$$\boxed{\tfrac{1}{2}\triangle|\alpha|^2 = \langle \alpha, \triangle\alpha\rangle + |\nabla\alpha|^2 = Ric(\alpha^\sharp, \alpha^\sharp) + |\nabla\alpha|^2}.$$

In the case $\Sigma^{n-1} \subset \mathbb{R}^n$ is minimal, $R^{\Sigma}_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}$ under the o.n. frame $\{e_i\}$ by the Gauss equation, hence

$$Ric^{\Sigma}_{ik} = \sum_j R^{\Sigma}_{ijkj} = -\sum_j h_{ij}h_{jk}, \quad (\sum_j h_{jj} = 0).$$

$$\Longrightarrow \quad \frac{1}{2}\triangle|\alpha|^2 = |\nabla\alpha|^2 + \sum_{ij} Ric^{\Sigma}_{ij}\alpha_i\alpha_j = |\nabla\alpha|^2 - \sum_i(\sum_j h_{ij}\alpha_j)^2 \geq |\nabla\alpha|^2 - |A|^2|\alpha|^2.$$

Plug in $\frac{1}{2}\triangle|\alpha|^2 = |\alpha|\triangle|\alpha| + \big|\nabla|\alpha|\big|^2$,

$$|\alpha|(\underbrace{\triangle|\alpha| + |A|^2|\alpha|}_{L|\alpha|}) \geq |\nabla\alpha|^2 - \big|\nabla|\alpha|\big|^2 \geq c(n)\big|\nabla|\alpha|\big|^2,$$

where $Lu = \triangle u + |A|^2 u$ is the stability operator, and $c(n)$ a constant depending only on n.

In general, choose the o.n. basis $\{e_1, \cdots, e_k\}$ such that under this basis $\alpha_1 = |\alpha|$ and $\alpha_j = 0$ for $j = 2, \cdots, k$, then

$$|\nabla\alpha|^2 - \big|\nabla|\alpha|\big|^2 = \sum_{ij}\alpha_{i,j}^2 - \frac{\sum_j(\sum_i \alpha_i\alpha_{i,j})^2}{|\alpha|^2} = \sum_{i,j}\alpha_{i,j}^2 - \sum_j \alpha_{1,j}^2$$

$$= \sum_{i>1,j}\alpha_{i,j}^2 \geq \sum_{i=2}^k \alpha_{i,i}^2 + \sum_{i=2}^k \alpha_{i,1}^2 \geq \frac{1}{k-1}(\underbrace{\sum_{i=2}^k \alpha_{i,i}}_{=-\alpha_{1,1}})^2 + \sum_{i=2}^k \alpha_{1,i}^2$$

$$\geq \frac{1}{k-1}[\alpha_{1,1}^2 + \sum_{i=2}^k \alpha_{1,i}^2] = \frac{1}{k-1}\big|\nabla|\alpha|\big|^2.$$

Theorem 6.1 *If $\Sigma^{n-1} \subset \mathbb{R}^n$ is a complete, stable and 2-sided minimal surface, then any L^2 harmonic 1-form on Σ vanishes.*

Proof: 2-sided and stability means that $-\int_\Sigma \varphi L\varphi \geq 0$ for any φ compactly supported. So $\forall\varphi$ compactly supported

$$-\int_\Sigma \varphi|\alpha|L(\varphi|\alpha|) \geq 0,$$

i.e.

$$\int_\Sigma \varphi|\alpha|(\underbrace{\triangle(\varphi|\alpha|)}_{I} + |A|^2\varphi|\alpha|) \leq 0,$$

where

$$I = \int_\Sigma \varphi|\alpha|(\varphi\triangle|\alpha| + 2\langle\nabla\varphi, \nabla|\alpha|\rangle + |\alpha|\triangle\varphi) = \int_\Sigma \varphi^2|\alpha|\triangle|\alpha| + \frac{1}{2}\langle\nabla\varphi^2, \nabla|\alpha|^2\rangle + |\alpha|^2\varphi\triangle\varphi$$
$$\leq \int_\Sigma \varphi^2|\alpha|\triangle|\alpha| - \int_\Sigma \frac{1}{2}\triangle(\varphi^2)|\alpha|^2 + \int_\Sigma |\alpha|^2\varphi\triangle\varphi$$
$$= \int_\Sigma \varphi^2|\alpha|\triangle|\alpha| - (\varphi\triangle\varphi + |\nabla\varphi|^2)|\alpha|^2 + |\alpha|^2\varphi\triangle\varphi$$
$$= \int_\Sigma \varphi^2|\alpha|\triangle|\alpha| - |\nabla\varphi|^2|\alpha|^2.$$

Plug into the above

$$\int_\Sigma \varphi^2|\alpha|L(|\alpha|) \leq \int_\Sigma |\nabla\varphi|^2|\alpha|^2.$$

Now by taking $\varphi = \varphi_R$ to be cutoff functions on geodesic disk, and letting $R \to \infty$, the righthand side of the above inequality is zero, hence by $|\alpha|L(|\alpha|) \geq c(n)\big|\nabla|\alpha|\big|$ proved above,

$$c(n)\int_\Sigma \big|\nabla|\alpha|\big|^2 \leq \int_\Sigma |\alpha|L(|\alpha|) = 0,$$

which means that $|\alpha|$ is a constant, and hence is 0 since the area of Σ is ∞ by the monotonicity $|B_\sigma(p)| \geq w_k\sigma^k$.

□

6.2 Continuity of Section 5

Theorem 6.2 *Any complete 2-sided stable minimal immersion $\Sigma^2 \subset \mathbb{R}^3$ is a plane.*

Proof: (Σ, g) is an oriented Riemann surface, where g is the restriction metric. If $z = x + iy$ then $g = \lambda^2(dx^2 + dy^2)$ locally. So Σ has a complex striation. Let $\hat{\Sigma}$ be the universal cover of Σ, then $\hat{\Sigma}$ is a simply connected non-compact Riemann surface, hence

$$\hat{\Sigma} \simeq \begin{cases} \mathbb{C}, & \text{the complex plane,} \\ D, & \text{the unit disk.} \end{cases}$$

Case 1: $\hat{\Sigma} \simeq \mathbb{C}$, then let $F : \mathbb{C} \to \mathbb{R}^3$, where $F = i \circ \pi$ is given by the composition of the minimal immersion $i : \Sigma \to \mathbb{R}^3$ with the covering map $\pi : \mathbb{C} \simeq \hat{\Sigma} \to \Sigma$. Since i is harmonic, and the harmonic property is preserved under the conformal change $\hat{\Sigma} \simeq \mathbb{C}$, we know that F is both conformal and harmonic, i.e. $\triangle_{\mathbb{C}}F = 0$. Since Σ is stable and 2-sided, $\hat{\Sigma}$ is also stable and 2-sided, $\Longrightarrow \exists u > 0$, such that $Lu = \triangle_{\hat{\Sigma}}u + |\hat{A}|^2u = 0$ on $\hat{\Sigma}$. So $\triangle_{\hat{\Sigma}}u \leq 0$, hence $\triangle_{\mathbb{C}}(u \circ F) \leq 0$. So $u \circ F$ is a super-harmonic function. Since $\mathbb{C}$ has quadratic area growth,together with the fact that $u \circ F > 0$, we know that $u \circ F = 0$, and hence $|\hat{A}|^2 = 0$ by the following Proposition.
Case 2: $\hat{\Sigma} \simeq D$.

Claim: the L^2 norm of p-forms on an n-dimensional manifold is a conformal invariant when $p = \frac{n}{2}$.
This is because $|\alpha|_g^2 = \underbrace{g^{i_1 j_1} \cdots g^{i_p j_p}}_{p \text{ copies}} \underbrace{\alpha_{i_1 j_1} \cdots \alpha_{i_p j_p}}_{p \text{ copies}}$, so if $\tilde{g} = \lambda^2 g$, then

$$\int |\alpha|_{\tilde{g}}^2 \sqrt{det\tilde{g}} dx = \int \lambda^{-2p} |\alpha|_g^2 \lambda^n \sqrt{detg} dx = \int |\alpha|_g^2 \sqrt{detg} dx.$$

Furthermore, harmonic $p-$forms change to harmonic $p-$forms under conformal change. This is because $d\lambda = 0$ does not change, while $\delta_{\tilde{g}} \alpha = \lambda^{-2} \delta \alpha = 0$ when $p = \frac{n}{2}$ (see Page 59 in [1]).

So L^2 harmonic 1-forms on D corresponds to L^2-harmonic 1-forms on $\tilde{\Sigma}$. Since there are many harmonic 1-forms on D by just taking dx where x is harmonic functions, so it is a contradiction to the Theorem we proved in the above section.

□

Definition 6.2 *A Riemannian manifold Σ^k is called* <u>*parabolic*</u> *if every positive super-harmonic function is constant.*

Propostion 6.1 *If $h : \Sigma \to \mathbb{R}_+^1$ is a proper Lipschitz function $|\nabla h| \leq c$, and if $|\Sigma_a| \geq c a^a$ for some $c > 0$, where $\Sigma_a = \{p \in \Sigma : \ h(p) \leq a\}$, then Σ is parabolic.*

Proof: Take a positive super-harmonic function u, i.e. $\triangle u \leq 0$ and $u > 0$. Take $w = \log u$, then

$$\triangle w = \frac{\triangle u}{u} - |\nabla w|^2 \leq -|\nabla w|^2.$$

Take φ a compactly supported function,

$$\int_\Sigma \varphi^2 |\nabla w|^2 \leq - \int_\Sigma \triangle w \varphi^2 = \int 2 \langle \nabla w, \varphi \rangle \varphi$$

$$\leq 2 \int |\varphi| |\nabla w| |\nabla \varphi| \leq \epsilon \int \varphi^2 |\nabla w|^2 + \frac{1}{\epsilon} \int |\nabla \varphi|^2.$$

Taking $\epsilon = \frac{1}{2}$, then

$$\int_\Sigma \varphi^2 |\nabla w|^2 \leq 4 \int_\Sigma |\nabla \varphi|^2.$$

By taking $h = dist_\Sigma(\cdot, p)$, we know that Σ has more than quadratic area growth, so we can take $\varphi = \varphi_R$ as in Lemma 3.3, and use the same logarithmic cut-off trick, to get $\int_\Sigma |\nabla \varphi_R|^2 \to 0$, and $\varphi_R \to 1$. So $|\nabla w| = 0$, and w hence u is a constant.

□

7 Weierstrass representation and Simons Identity(1/31/2012)

7.1 Weierstrass representation

Let $F : \Omega \to \mathbb{R}^n$ be a minimal immersion, where Ω is a Riemann surface with complex coordinates $z = x + iy$, then F is conformal(i.e. $\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \rangle = 0$ and $|\frac{\partial F}{\partial x}| = |\frac{\partial F}{\partial y}|$) and harmonic(i.e. $\triangle F = 0$). Define

$$\psi dz = \frac{\partial F}{\partial z} dz = (\frac{\partial F}{\partial x} - i \frac{\partial F}{\partial y})(dx + idy).$$

Lemma 7.1 *The complex vector* $\psi = \frac{\partial F}{\partial z}$ *is holomorphic, i.e.* $\frac{\partial \psi}{\partial \bar{z}} = 0$*, and isotropic, i.e.* $\sum_{j=1}^n \psi_j^2 = 0$.

Proof: Since F is harmonic,

$$\frac{\partial \psi}{\partial \bar{z}} = \triangle F = 0.$$

Since F is conformal,

$$\sum_{j=1}^n \psi_j^2 = |\frac{\partial F}{\partial x}|^2 - |\frac{\partial F}{\partial y}|^2 + 2i \langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \rangle = 0.$$

□

Conversely, we would like to represent F as $F(z) = Re \int^z \psi(s) ds$.

When $\underline{n = 3}$, given a meromorphic function g and a holomorphic one form φdz on Ω, we can take

$$\psi dz = \big(\frac{1}{2}(g^{-1} - g), \frac{1}{2}(g^{-1} + g, 1)\big)\varphi dz,$$

and get a minimal immersion $F : \Omega \to \mathbb{R}^3$ by

$$F(z) = Re \int_{z_0}^z \psi(s) ds.$$

If $N : \Omega \to S^2$ is the Gauss Map, then $g = \pi \circ N$, with $\pi : S^2 \to \mathbb{R}^2$ the stereographic projection from $(0, 0, 1)$.

Examples:

- Catenoid: $\Omega = \mathbb{C} \setminus \{0\}$, $g(z) = z$ and $\varphi(z) = \frac{dz}{dz}$;
- Helicoid: (simply connected $\pi_1 = 0$) $\Omega = \mathbb{C}$, $g(z) = e^{iz}$ and $\varphi(z) = dz$. If we denote the Helicoid by Σ, then $\epsilon\Sigma$ converges to a Foliation by plane $x_3 = c$, where $|A|^2 \to +\infty$ on the axis, but $|A|^2 \to 0$ everywhere away from the axis;
- Hoffman-White examples.

7.2 Simons Identity

Consider a minimal hypersurface $\Sigma^{n-1} \subset \mathbb{R}^n$. Let $\{e_1, \cdots, e_{n-1}\}$ be local o.n. frames on Σ, and denote $h_{ij,klm}$ by the covariant derivatives of the second fundamental form h on Σ. The rough laplacian for h is defined as

$$\triangle h_{ij} = \sum_{k=1}^{n-1} h_{ij,kk}.$$

Propostion 7.1

$$\triangle h_{ij} + |A|^2 h_{ij} = 0, \quad 0 \leq i,j \leq n-1 \tag{10}$$

Proof: Firstly we have the Ricci identity:

$$h_{ij,kl} - h_{ij,lk} = \sum_p h_{pj} R_{pikl} + \sum_p h_{ip} R_{pjkl},$$

Gauss Equation:

$$R^{\Sigma}_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk},$$

and Codazzi equation:

$$h_{ij,k} = h_{ik,j}.$$

Using the Eistein summation, we have

$$\begin{aligned}\triangle h_{ij} = h_{ij,kk} = h_{ik,jk} = & \underbrace{h_{ik,kj}}_{=h_{kk,ij}=0} + h_{pk} R^{\Sigma}_{pijk} + h_{ip} R^{\Sigma}_{pkjk} \\ &= h_{pk}(h_{pj}h_{ik} - h_{pk}h_{ij}) + h_{ip}(h_{pj} \underbrace{h_{kk}}_{=0} - h_{pk}h_{kj}) \\ &= -|A|^2 h_{ij} + \underbrace{(h_{ik}h_{kp}h_{pj} - h_{ip}H_{pk}h_{kj})}_{=0}.\end{aligned}$$

So we finished the proof.

□

Now recall that the stability operator is $L\varphi = \triangle\varphi + |A|^2\varphi$.

Propostion 7.2

$$|A|(L|A|) \geq \frac{2}{n-1}\big|\nabla|A|\big|^2. \tag{11}$$

Proof: By the Bochner Formula,

$$\frac{1}{2}\triangle|A|^2 = |\nabla A|^2 + \langle A, \triangle A\rangle = |\nabla A|^2 - |A|^4.$$

While $\frac{1}{2}\triangle|A|^2 = |A|\triangle|A| + |\nabla|A||^2$,

$$|A|L(|A|) = |\nabla A|^2 - \big|\nabla|A|\big|^2 = \sum_{i,j,k} h_{ij,k}^2 - \frac{\sum_k(\sum_{ij} h_{ij}h_{ij,k})^2}{|A|^2}.$$

In an o.n. eigenbasis $\{e_1, \cdots, e_{n-1}\}$ of h, $h_{ij} = \lambda_i \delta_{ij}$, so

$$\big|\nabla|A|\big|^2 = \frac{\sum_k(\sum_i \lambda_i h_{ii,k})^2}{|A|^2} \leq \sum_{i,k} h_{ii,k}^2 = \sum_{i\neq k} h_{ii,k}^2 + \sum_i h_{ii,i}^2$$

$$= \sum_{i\neq k} h_{ii,k}^2 + \sum_i(-\sum_{j\neq i} h_{jj,i})^2 \leq \sum_{i\neq k} h_{ii,k}^2 + (n-2)\sum_{i\neq j} h_{jj,i}^2$$

$$= (n-1)\sum_{i\neq k} h_{ii,k}^2 = \frac{n-1}{2}(\sum_{i\neq k} h_{ik,i}^2 + \sum_{i\neq k} h_{ki,i}).$$

So

$$(1+\frac{2}{n-1})\big|\nabla|A|\big|^2 \leq \sum_{i,k} h_{ii,k}^2 + \sum_{i\neq k} h_{ik,i}^2 + \sum_{i\neq k} h_{ki,i}^2 \leq \sum_{i,j,k} h_{ij,k}^2 = |\nabla A|^2.$$

So we finished the proof.

□

8 Curvature estimates (2/2/2012)

Curvature Estimates:

Let $\mathcal{C}_{r_0} = \{\Sigma^{n-1} \subset \mathbb{R}^n :\ H_\Sigma = 0,\ 0 \in \Sigma,\ \&\ \partial\Sigma \cap B_{r_0}(0) = \emptyset\}$.

Question: For which $\mathcal{C}$ is it true that

$$\sup_{\Sigma\cap B_{r_0/2}} |A|^2 \leq C r_0^{-2},$$

where C is independent of $\Sigma \in \mathcal{C}$.

Similar Questions: replace B_{r_0} in the above question by geodesic balls $B_{r_0}^\Sigma = \{x \in \Sigma :\ d_\Sigma(s,0) < r_0\}$.

Note: $B_{r_0}^\Sigma \subset B_{r_0}$.

Remark 8.1 1°. *By scaling, it suffices to assume that* $r_0 = 1$, *since* $\frac{1}{r_0}\Sigma \in \mathcal{C}_1$, *while* $|A_{\frac{1}{r_0}\Sigma}|^2 = r_0^2|A|_\Sigma^2$.

2°. *Curvature estimates* $\Longrightarrow$ *Bernstein Theorem. If* Σ *is complete and* $\Sigma \in \mathcal{C}_{r_0}$, $\forall r_0 > 0$, *then* Σ *is a plane by taking* $r_0 \to \infty$.

For which $\mathcal{C}_{r_0}$ is it True?

1. $\underline{n=3}$, $\int_{\Sigma\cap B_{r_0}} |A|^2 < \epsilon$, for some fixed small $\epsilon \Longrightarrow$ Curvature Estimates (see Theorem 8.2).

   (Counter Example, Down-scaled Catenoid, where $\int_{\Sigma\cap B_{r_0}} |A|^2$ is not small.)
2. $\underline{n=3}$, Σ^2 embedded and area minimizing. (e.g. Σ is a graph over a convex region.)
3. $\underline{n=3}$, Σ^2 embedded, simply connected and $|\Sigma| \leq A_0 r_0^2 \Longrightarrow$ Curvature Estimates (see Theorem 9.2).

   Ex: 1) down-scaled Helicoid, 2) Enneper Surface, immersed $|\Sigma \cap B_{r_0}| \leq c r_0^2$.
4. $\underline{n=3}$, Σ^2 is stable and 2-sided $\Longrightarrow$ Curvature Estimates. (see Corollary 9.2).
5. $\underline{3 \leq n \leq 6}$, $\Sigma^{n-1} \subset \mathbb{R}^n$ stable and 2-sided and $|\Sigma \cap B_{r_0}| \leq c r_0^{n-1} \Longrightarrow$ Curvature Estimates (see Theorem 10.1).
6. $\underline{3 \leq n \leq 7}$, Σ^{n-1} embedded and absolutely volume minimizing $\Longrightarrow$ Curvature Estimates. (False for $n-8$)

   The Simons cone $C = \{x \in \mathbb{R}^8 : \sum_{i=1}^4 x_i^2 = \sum_{i=5}^8 x_i^2\}$ is a minimal cone. In fact, it is absolutely volume minimizing, since there exists foliation of $\mathbb{R}^8$ by minimal hypersurfaces asymptotical to C, where the curvature blows up on C.

7 Σ^{n-1} with small access: $\Theta_0(r_0) - 1 < \epsilon$, $\Longrightarrow$ Curvature Estimates (see Theorem 9.3).

   The access for a $k-$dimensional sub manifold Σ^k at $0 \in \Sigma^k$ of radius r is defined as:

$$\Theta_0(r) = \frac{|\Sigma \cap B_r|}{\omega_h r^k}.$$

Basic idea: For some small $\epsilon > 0$, assume the scaling invariant inequality:

$$\int_{\Sigma\cap B_{r_0}} |A|^2 < \epsilon.$$

Suppose

$$|A|^2(x_0) = \max_{\Sigma\cap B_{r_0}} |A|^2, \text{ at some } x_0 \in B_{r_0/2}.$$

Rescale Σ by the factor $\delta = 1/|A|(x_0)$, then

$$\Sigma_\delta = \frac{1}{\delta}(\Sigma - x_0) \sim \begin{cases} \circ\ |A_{\Sigma_\delta}(x)|^2 = \delta^2 |A_\Sigma|^2 \leq 1, \ \forall x \in B_1(0), \\ \circ\ |A_{\Sigma_\delta}(0)|^2 = 1, \\ \circ\ \int_{B_1\cap\Sigma_\delta} |A_{\Sigma_\delta}|^2 < \epsilon. \end{cases}$$

Now $|A_{\Sigma_\delta}|(x) \leq c = 1 \Longrightarrow$ locally Σ_δ is a graph $Graph_u$ of some function $u \Longrightarrow C^3$ estimates of u, which will form a contradiction with $\int_{B_1\cap\Sigma_\delta} |A_{\Sigma_\delta}|^2 < \epsilon$.

Theorem 8.1 *(Choi-Schoen [4]) Suppose $\Sigma^2 \subset M^3$ is a minimal surface. Assume $0 \in \Sigma^2$, and $\partial\Sigma \cap B_{r_0}(0) = \emptyset$. Then there exists $\epsilon, \rho > 0$ (depending only on M), such that if $r_0 \leq \rho$, $\delta \in (0,1)$ and $\int_{\Sigma\cap B_{r_0}} |A|^2 < \delta\epsilon$, then*

$$|A|^2(y) \leq \delta\sigma^{-2}, \quad \textit{for } y \in B_{r_0-\sigma}(0).$$

Proof: Let us give a proof when $M^3 = \mathbb{R}^3$, and the general cases follow by the fact that M^3 is locally near $\mathbb{R}^3$ when ρ is small enough. Assume $\delta = 1$ and

$$F(y) = \big(r_0 - r(y)\big)^2 |A|^2(y),$$

where $r(y) = d(y, 0)$. Since $F|_{\partial B_{r_0}} = 0$, then $\exists y_0 \in B_{r_0}$, such that $F(y_0) = \max_{B_{r_0}} F(y)$.
Need to show: $F(y) \leq 1 (\Longrightarrow 1 \geq F(y) \geq \sigma^2 |A|^2(y)$, if $r(y) < r_0 - \delta)$.
Suppose $F(y_0) > 1$, let $\delta = \frac{r_0 - r(y_0)}{2}$, then

- $\sup_{B_{\delta(y)}} |A|^2 \leq 4|A|^2(y_0)$.
  This is because $(r_0 - r(y))^2 |A|^2(y) \leq (r_0 - r(y_0))^2 |A|^2(y_0)$, hence $|A|^2(y) \leq (\frac{r_0 - r(y_0)}{r_0 - r(y)})^2 |A|^2(y_0) \leq 4|A|^2(y_0)$, where $r_0 - r(y) \geq r_0 - (r(y_0) + \delta) \geq \frac{1}{2}(r_0 - r(y_0))$.
- $(2\delta)^2 |A|^2(y_0) = F(y_0) > 1 \Longrightarrow \delta^2 |A|^2(y_0) > 1/4$.

Let $\delta_0 = \frac{1}{|A|(y_0)}$, hence $\delta^2 \geq \frac{1}{4}\delta_0^2 \Longrightarrow \delta_0/2 < \delta$. So $B_{\delta_0/2}(y_0) \subset B_\delta(y_0)$. Let

$$\Sigma_{\delta_0} = \frac{2}{\delta_0}(\Sigma - y_0),$$

$$\Longrightarrow \begin{cases} \sup_{B_1} |A_{\Sigma_{\delta_0}}|^2 = 4|A_{\Sigma_{\delta_0}}|^2 = \delta_0^2 |A|^2(y_0) = 1, \\ \int_{B_1} |A_{\Sigma_{\delta_0}}|^2 \leq \epsilon. \end{cases}$$

So it forms a contradiction when ϵ is too small by the argument discussed in the Basic idea. So $F(y_0) \leq 1$.

□

Theorem 8.2 *Assume $\Sigma^2 \subset \mathbb{R}^3$ is stable and 2-sided with quadratic area growth, i.e. $|\Sigma \cap B_{r_0}| \leq c r_0^2$, then*

$$\sup_{\Sigma \cap B_{r_0/2}} |A|^2 \leq c r_0^{-2}.$$

Proof: By stability, we have $\int_\Sigma |A|^2 \varphi^2 \leq \int_\Sigma |\nabla \varphi|^2$. Since Σ has quadratic area growth, we can use the logarithmic cutoff trick to get,

$$\int_{\Sigma \cap B_{r_0/k}} |A|^2 \leq \frac{C}{\log k}, \quad k \gg 1.$$

So for k large enough, we have $\int_{\Sigma \cap B_{r_1}(y)} |A|^2 < \epsilon$, where $r_1 = r_0/k$, hence

$$|A|^2(y) \leq c r_1^{-2} \leq c' r_0^{-2}, \quad c' = kc.$$

□

9 More curvature estimates in 2-d (2/7/2012)

Let us firstly give a technical lemma used in the argument of the above section.

Lemma 9.1 *$\Sigma^2 \subset \mathbb{R}^n$ is minimal. Assume that $s^2 \sup_\Sigma |A|^2 \leq \frac{1}{16}$. If $x \in \Sigma^2$ and $dist_\Sigma(s, \partial\Sigma) \geq 2s$, then*

(i) $B_{2s}^\Sigma(x)$ is graphical over $T_x\Sigma$ of some function u, where $B_{2s}^\Sigma(x)$ is the geodesic ball of Σ, and $|\nabla u| \leq 1$ and $|Hessu| \leq 1/\sqrt{2}$;

(ii) Let Σ' be a connected component of $B_s(x) \cap \Sigma$ containing x, then $\Sigma' \subset B_{2s}^\Sigma(x)$.

Proof: See [3, §3. Chap 2].

□

Lemma 9.2 *Let Σ^2 be a simply-connect minimal surface. Fix $x \in \Sigma$, then $r(y) = dist_\Sigma(y, x)$ is a smooth function when $y \neq x$. Let K be the Gauss curvature of Σ, then*

- $$|\partial B_{r_0}^\Sigma(x)| - 2\pi r_0 = -\int_0^{r_0} \int_{B_\rho^\Sigma} K \underbrace{=}_{H=0} \frac{1}{2} \int_0^{r_0} \int_{B_\rho^\Sigma} |A|^2;$$

- $$|B_{r_0}^\Sigma(x)| - \pi r_0^2 = -\int_0^{r_0} \int_0^\tau \int_{B_\rho^\Sigma} K \underbrace{=}_{H=0} \frac{1}{2} \int_0^{r_0} \int_0^\tau \int_{B_\rho^\Sigma} |A|^2;$$

- *When $t < r_0/2$*

$$t^2 \int_{B_{r_0-2t}^\Sigma} |A|^2 \leq \int_{B_{r_0}^\Sigma} |A|^2 \frac{(r_0 - r)^2}{2} = \int_0^{r_0} \int_0^\tau \int_{B_\rho^\Sigma} |A|^2$$

$$= 2(|B_{r_0}^\Sigma| - \pi r_0^2) \leq r_0(|\partial B_{r_0}^\Sigma| - 2\pi r_0).$$

Proof: Using co-area formula and Gauss-Bonnet formula on the simply-connected domain B_r^Σ,

$$\frac{d}{dr}|B_r^\Sigma| = |\partial B_r^\Sigma|, \quad \frac{d}{dr}|\partial B_r^\Sigma| = \int_{\partial B_r^\Sigma} k_g = 2\pi - \int_{B_r^\Sigma} K,$$

where k_g is the geodesic curvature of the curve ∂B_r^Σ. Integrate the 2nd of the above, i.e. $\int_0^{r_0} \Longrightarrow$

$$|\partial B_{r_0}^\Sigma| = 2\pi r_0 - \int_0^{r_0} \int_{B_\rho^\Sigma} K.$$

Integrate again $\Longrightarrow$

$$|B_{r_0}^\Sigma| = \pi r_0^2 - \int_0^{r_0} \int_0^\tau \int_{B_\rho^\Sigma} K.$$

For the last one, if $t < \frac{1}{2}r_0$,

$$t^2 \int_{B^\Sigma_{r_0-2t}} |A|^2 \leq \int_{B^\Sigma_{r_0}} |A|^2 \frac{(r_0-r)^2}{2} = \int_0^{r_0} \frac{(r_0-r)^2}{2} \frac{d}{dr} \int_{B^\Sigma_r} |A|^2 dr$$

$$= \int_0^{r_0} (r_0-r) \underbrace{\int_{B^\Sigma_r} |A|^2}_{=\frac{d}{dr}\int_0^r \int_{B^\Sigma_r} |A|^2} dr = \int_0^{r_0} \int_0^r \int_{B^\Sigma_\rho} |A|^2.$$

Here we used the integration by part twice in the second and third $=$. Using the second equation, we get the first part. The inequality $2|B^\Sigma_{r_0}| \leq r_0|\partial B^\Sigma_{r_0}| \Longleftrightarrow$ non-positive curvature and simply-connected(This can be proved by using the integration inequality $\int_{B^\Sigma_{r_0}} (\triangle r^2 \geq 4)$, where $\triangle r^2 \geq 4$ comes from the fact that $K \leq 0$.).

□

Theorem 9.1 *Assume that Σ^2 is stable and 2-sided in $\mathbb{R}^3$. If $x \in \Sigma$ and $dist(x, \partial\Sigma) \geq r_0$, then*

$$|B^\Sigma_{r_0}| \leq \frac{4\pi}{3} r_0^2.$$

Proof: It suffices to assume $\pi_1(\Sigma) = \{1\}$, or we can pass to the universal cover $\tilde{\Sigma}$ of Σ, which is also stable and 2-sided (see Lecture 5). Since $|B^{\tilde{\Sigma}}_r| \geq |B^\Sigma_r|$, we can get the result. Using the last equality in the above lemma,

$$4(|B^\Sigma_{r_0}| - \pi r_0^2) = \int_{B^\Sigma_{r_0}} |A|^2 (r_0-r)^2 \leq \int_{B^\Sigma_{r_0}} |\nabla(r_0-r)|^2 = |B^\Sigma_{r_0}|,$$

where we used the stability inequality in $\leq$. Hence we finished by moving $|B^\Sigma_{r_0}|$ to the right hand side.

□

Corollary 9.1 *A complete stable 2-sided minimal Σ^2 in $\mathbb{R}^3$ is a plane.*

Proof: By the above lemma, Σ has no more than quadratic area growth, hence the stability inequality and logarithmic cutoff technique imply that Σ^2 is a hyperplane.

□

Corollary 9.2 *(originally due to Schoen [6]) If Σ is stable and 2-sided, $x \in \Sigma$ and $dist_\Sigma(x, \partial\Sigma) \geq r_0$, then*

$$\sup_{B^\Sigma_{r_0-\sigma}(x)} |A|^2 \leq c\sigma^{-2}.$$

Proof: By Theorem 8.2, we can reduce to prove the small total curvature condition. Using the logarithmic cutoff trick, the stability inequality and area bound,

$$\int_{B^{\Sigma}_{e^{-n}r_0}} |A|^2 \leq n^{-2} \int_{B^{\Sigma}_{r_0} \setminus B^{\Sigma}_{e^{-n}r_0}} r^{-2} \leq cn^{-1}.$$

So we can get the small total curvature condition by shrinking down the radius.

□

Theorem 9.2 *Let $\Sigma^2 \subset \mathbb{R}^3$ be simply connected and embedded. If $x \in \Sigma$, $\partial\Sigma \subset \partial B_r(x)$ and $|B_{r_0}(x) \cap \Sigma| \leq A_0 r_0^2$, then*

$$\sup_{\Sigma \cap B_{r_0/2}(x)} |A|^2 \leq c(A_0) r_0^{-2}.$$

Remark 9.1 *Non-embedded counter examples are Helicoid type singularities; Non-simply-connected counter examples are Catenoid type singularities.*

Proof: Using the simply-connectedness and quadratic area growth, we can apply the third inequality in Lemma 9.2 to get

$$\int_{\Sigma \cap B_{\theta r_0}} |A|^2 \leq c(\theta, A_0),$$

where $\theta \in (0,1)$ and $c(\theta, A_0)$ is a constant depending only on θ and A_0. Then we can start from $r_1 = \frac{3}{4} r_0$, and devide $[0, r_1]$ into N sub-intervals $[9^{-n-1} r_1, 9^{-n} r_1]$, for $n = 0, \cdots, N$. Then $\exists n \leq N$, such that

$$\int_{\Sigma \cap (B_{9(-n+1)_{r_1}} \setminus B_{9^{-n} r_1})} |A|^2 < \frac{c}{N}.$$

By rescaling, we can assume that $\int_{\Sigma \cap (B_1 \setminus B_{1/9})} |A|^2 < \frac{c}{N}$. $\Longrightarrow$ curvature estimates on the annuli region $\Sigma \cap (B_1 \setminus B_{1/9})$, $\Longrightarrow \Sigma \cap (B_1 \setminus B_{1/9})$ is locally graphical, by Maximum Principle $\Longrightarrow$ graphical on $\Sigma \cap (B_1 \setminus B_{1/9})$, hence graphical on D by embeddedness.

□

Curvature estimates under small excess assumption

Given $x \in \Sigma^k \subset \mathbb{R}^n$ and $r > 0$, the following quantity

$$\Theta_x(r) = \frac{|\Sigma \cap B_r(x)|}{\omega_k r^k}$$

is monotonously non-decreasing w.r.t. r.

Definition 9.1 *The excess of Σ in $B_r(x)$ is $\Theta_x(r) - 1 \geq 0$.*

Theorem 9.3 *Let $\Sigma^k \subset \mathbb{R}^n$ be minimal. $\exists \epsilon = \epsilon(n,k)$, if $x \in \Sigma$, $\partial\Sigma \subset \partial B_{r_0}(x)$, and $\Theta_x(r_0) - 1 < \epsilon$, then*

$$\sup_{\Sigma \cap B_{r_0/2}(x)} |A|^2 \leq r_0^{-2}.$$

Proof:

- It suffices to assume that $\Theta_y(r_1) - 1 < \epsilon$ for all $y \in B_{r_1}(x) \cap \Sigma$ by the monotonicity formula 3.1.
- By rescaling the function $(r_1 - |y|)^2 |A|^2(y)$ near the maximum point as in the proof of Theorem 8.2, we can get another minimal surface, denoted still as Σ, such that $0 \in \Sigma$, $\partial\Sigma \subset \partial B_1(0)$, $|A|^2 \leq 1$ on Σ, and $|A|^2(0) = \frac{1}{4}$. Furthermore, by the small excess condition, $|\Sigma \cap B_1(0)| \leq (1+\epsilon)\omega_k$.
- This is not possible if $\epsilon \leq \epsilon_0$, for some $\epsilon_0 > 0$ small enough, by the following argument.
- Compactness argument: consider the class

  $$\mathcal{C}_\epsilon = \{\Sigma :\ 0 \in \Sigma,\ \partial\Sigma \subset \partial B_1(0),\ |A|^2 \leq 1,\ |A|^2(0) = \frac{1}{4},\ |\Sigma \cap B_1(0)| \leq (1+\epsilon)\omega_k\}.$$

  If the curvature estimates is not true, then we can find a sequence $\{\Sigma_i\}$, with $\Sigma \cap B_1(0) \leq (1 + 2^{-i})\omega_k$. A subsequence $\Sigma_i \to \Sigma$ in C^k norm to some minimal Σ_∞, such that $\Sigma_\infty \in \mathcal{C}_0$, i.e. $|\Sigma_\infty \cap B_1(0)| = \omega_k$, $\Longrightarrow$ Σ is a disk, hence contradiction to the curvature assumption $|A|^2(0) = \frac{1}{4}$.

□

10 Schoen-Simon-Yau curvature estimates and minimal cone (2/9/2012)

10.1 Curvature estimates by Schoen-Simon-Yau when $n \leq 6$

Theorem 10.1 *(Schoen-Simon-Yau [7]) Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be a stable 2-sided minimal surface. Assume $x_0 \in \Sigma$, $\partial\Sigma \subset \partial B_{r_0}(x_0)$, $|\Sigma \cap B_{r_0}(x_0)| \leq V r_0^{n-1}$ and $n \leq 6$. Then*

$$\sup_{\Sigma \cap B_{r_0/2}(x_0)} |A|^2 \leq c(n,V) r_0^{-2}.$$

Corollary 10.1 *A complete 2-sided stable $\Sigma^{n-1} \in \mathbb{R}^n$ with $\mathbb{R}^{n-1}$ volume growth and $n \leq 6$ is a hyperplane.*

Remark 10.1 *Counterexample for $n = 7$: $\exists$ complete volume minimizing $\Sigma^7 \subset \mathbb{R}^8$, not a hyperplane*

Proof:

$$\textbf{Claim:}\quad \frac{2}{n-1}\int_\Sigma \big|\nabla|A|\big|^2 \varphi^2 \leq \int_\Sigma |\nabla\varphi|^2 |A|^2, \quad \forall \varphi \in C_c^1(\Sigma).$$

Now let us firstly prove this claim. By plug in $\varphi|A|$ to the stability inequality $-\int_\Sigma(\varphi|A|)L(\varphi|A|) \geq 0$, and using the tricks in Theorem 6.1, we have

$$\int_\Sigma \varphi^2|A|L(|A|) \leq \int_\Sigma |\nabla\varphi|^2|A|^2.$$

Using Proposition 7.2, we can get the conclusion.

Now change $\varphi \to \varphi|A|^q$, for some $q > 0$, then we get

$$\frac{2}{n-1}\int_\Sigma \big|\nabla|A|\big|^2\varphi^2|A|^{2q} \leq \int_\Sigma |A|^2\big|\nabla(\varphi|A|^q)\big|^2 = \int_\Sigma |A|^2\big|(\nabla\varphi)|A|^q + q|A|^{q-1}\varphi(\nabla|A|)\big|^2$$

$$\leq (q^2+\epsilon)\int_\Sigma |A|^{2q}\varphi^2\big|\nabla|A|\big|^2 + (1+\frac{1}{\epsilon})\int_\Sigma |\nabla\varphi|^2|A|^{2q+2}.$$

Hence if $\boxed{q < \sqrt{\frac{2}{n-1}}}$, by moving the first term on the right hand side to the left,

$$\Longrightarrow \int_\Sigma \big|\nabla|A|\big|^2\varphi^2|A|^{2q} \leq C(q)\int_\Sigma |A|^{2q+2}|\nabla\varphi|^2,$$

$$\Longrightarrow \int_\Sigma \big|\nabla|A|^{q+1}\big|\varphi^2 \leq C(q)\int_\Sigma (|A|^{q+1})^2|\nabla\varphi|^2.$$

Set $p = q+2$,

$$\Longrightarrow \int_\Sigma \big|\nabla|A|^{p-1}\big|\varphi^2 \leq C(p)\int_\Sigma (|A|^{p-1})^2|\nabla\varphi|^2.$$

Now replace φ by $\varphi|A|^{p-1}$ in the stability inequality $\int_\Sigma |A|^2\varphi^2 \leq \int_\Sigma |\nabla\varphi|^2$,

$$\Longrightarrow \int_\Sigma |A|^{2p}\varphi^2 \leq \int_\Sigma \big|\nabla(\varphi|A|^{p-1})\big|^2 \leq 2\int_\Sigma \varphi^2\big|\nabla(|A|^{p-1})\big|^2 + |\nabla\varphi|^2|A|^{2p-2}.$$

Using the above inequality,

$$\Longrightarrow \quad \int_\Sigma |A|^{2p}\varphi^2 \leq C(p)\int_\Sigma |A|^{2p-2}|\nabla\varphi|^2.$$

Replace φ by φ^p, then

$$\int_\Sigma |A|^{2p}\varphi^2 \leq C(p)\int_\Sigma |A|^{2p-2}|\nabla\varphi^p|^2 = Cp^2\int_\Sigma (\varphi|A|)^{2p-2}|\nabla\varphi|^2$$

$$\leq C(p)\{\int_\Sigma (\varphi|A|)^{2p}\}^{(p-1)/p}\{\int_\Sigma |\nabla\varphi|^{2p}\}^{1/p}.$$

$$\Longrightarrow \quad \underline{\int_\Sigma (\varphi|A|)^{2p} \leq C(p)\int_\Sigma |\nabla\varphi|^{2p}}, \quad \forall p < 2+\sqrt{\frac{2}{n-1}}. \tag{12}$$

- **Remark:** if $\Sigma^k \subset \mathbb{R}^n$ is minimal, $\int_\Sigma |A|^{2p} \leq C$, for $2p > k \Longrightarrow$ Curvature Estimates.
- **Want:** $2p > n-1$, hence $4 + 2\sqrt{\frac{2}{n-1}} > n-1$, i.e. $\sqrt{\frac{2}{n-1}} > \frac{n-5}{2} \Longrightarrow n \leq 6$.

- When $n \leq 6$, take $2p = n-1$, and use the logarithmic cut off trick and the volume growth $\Longrightarrow$ $\int_\Sigma |A|^{n-1}$ is small in small ball, and hence the curvature estimates.
- Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be complete, stable, 2-sided, $n \leq 6$ and $\Sigma \cap B_R \leq CR^{n-1} \Longrightarrow \Sigma$ is hyperplane. Take $2p > n-1$, $\Longrightarrow \int_\Sigma (\varphi|A|)^{2p} \leq C\int_\Sigma |\nabla\varphi|^{2p} \leq \frac{C}{R^{2p}}|\Sigma \cap B_{2R}| \to 0$.

□

10.2 Minimal cone

Given $\Sigma^{k-1} \subset S^{n-1}$, the **cone** based on Σ is defined as

$$C(\Sigma) = \{\lambda x : \ x \in \Sigma, \ \lambda \geq 0\}. \tag{13}$$

Propostion 10.1 *$C(\Sigma)$ is minimal $\Longleftrightarrow$ Σ is minimal in S^{n-1} $\Longleftrightarrow$ $\triangle_\Sigma x^i + (k-1)x^i = 0$, $i = 1, \cdots, n$, where $\{x^1, \cdots, x^n\}$ is coordinates of $\mathbb{R}^n$.*

Proof: Given $\vec{X} = (x^1, \cdots, x^n)$, then $C(\Sigma)$ is minimal $\Longleftrightarrow \triangle_{C(\Sigma)}\vec{X} = 0$. Take a o.n. basis $\{e_1, \cdots, e_{k-1}\}$ for $T\Sigma$, and $e_k = \vec{X}/|\vec{X}|$, then $\{e_1, \cdots, e_k\}$ is an o.n. basis for $C(\Sigma)$. Then

$$\triangle_{C(\Sigma)}\vec{X} = \sum_{i=1}^{k-1} e_i e_i \vec{X} + e_k e_k \vec{X} - \sum_{i=1}^{k} (\nabla_{e_i} e_i)^{TC(\Sigma)} \vec{X}.$$

Here $\nabla_{e_k} e_k = \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0$, with $r = |\vec{X}|$ the radial function. $\nabla_{e_i} e_i \cdot \vec{X} = -e_i \cdot (\nabla_{e_i}\vec{X}) = -e_i \cdot e_i = -1$, hence $(\nabla_{e_i} e_i)^{TC(\Sigma)} = (\nabla_{e_i} e_i)^{T\Sigma} + (\nabla_{e_i} e_i \cdot \vec{X})\vec{X} = (\nabla_{e_i} e_i)^{T\Sigma} - \vec{X}$. So

$$\triangle_{C(\Sigma)}\vec{X} = \triangle_\Sigma \vec{X} + (k-1)\vec{X}.$$

So we prove the equivalence of the first and the third conclusion.

Now Σ is minimal in S^{n-1} if and only if $\vec{H}_\Sigma = \sum_{i=1}^{k-1} (\nabla_{e_i} e_i)^{TS^{n-1}}$, hence $(\sum_{i=1}^{k-1} (\nabla_{e_i} e_i))$ lies in the normal direction of S^{n-1}. So $\vec{H}_{C(\Sigma)} = (\sum_{i=1}^{k-1} (\nabla_{e_i} e_i) + \nabla_{e_k} e_k)^{\perp(\Sigma)} = \sum_{i=1}^{k-1} (\nabla_{e_i} e_i)^{\perp(\Sigma)} = 0$.

□

Clifford Hypersurfaces: Given $S^p(r_1) \subset \mathbb{R}^{p+1}$ and $S^q(r_2) \subset \mathbb{R}^{q+1}$, take

$$\Sigma = S^p(r_1) \times S^q(r_2) \subset \mathbb{R}^{p+q+1}.$$

Then $\Sigma \subset S^{p+q+1} \Longleftrightarrow r_1^2 + r_2^2 = 1$. Given $(\vec{x}, \vec{y}) \in \Sigma$, where $\vec{x} \in S^p$, and $\vec{y} \in S^q$, hence $\triangle_\Sigma \vec{x} = -\frac{p}{r_1^2}\vec{x}$, and $\triangle_\Sigma \vec{y} = -\frac{q}{r_2^2}\vec{y}$. Hence

$$\Sigma^{p+q} \subset S^{p+q+1} \text{ is minimal, } \Longleftrightarrow \frac{p}{r_1^2} = \frac{q}{r_2^2} = p+q.$$

Hence such class of Σ form lots of examples of minimal suffices in S^n and hence minimal cones in $\mathbb{R}^{n+1}$.

Tangent Cone at ∞: Given $\Sigma^{n-1} \subset \mathbb{R}^n$ complete, volume minimizing, which is not a hyperplane, then

$$\lambda_i^{-1}\Sigma \to C, \quad \lambda_i \to \infty,$$

where C is a non-flat, volume minimizing cone.

- So existence of such $\Sigma \Longrightarrow \exists$ nonflat volume minimizing cone C_1^m, $m \leq n-q$ with an isolated singularity at 0.
- J. Simons: when $m \leq 7$, no such cones exists $\Longrightarrow \Sigma$ such does not exists if $n \leq 7$.
- $p = q = 3 : C\big(S^3(1/\sqrt{2}) \times S^3(1/\sqrt{2})\big)$ is stable, and area minimizing;
- $p = 1, q = 5 : C\big(S^1(1/\sqrt{6}) \times S^5(\sqrt{5/6})\big)$ is stable, but not area minimizing.

11 Classical Plateau Problem (2/14/2012)

Plateau Problem: Given $\Gamma^{k-1} \subset \mathbb{R}^n$, with $\partial\Gamma = 0$, find $\Sigma^k \subset \mathbb{R}^n$, such that $\partial\Sigma = \Gamma$, and $|\Sigma| = \min\{|\Sigma_1| : \Sigma_1 = \Gamma\}$.

- What are the competetors?
- 1930 J. Douglas, T. Rado proved the Classical Plateau Problem:
- Γ a piecewise C^1 Jordan curve in $\mathbb{R}^n$, consider all $u : D \to \mathbb{R}^n$, with D the unit disk on $\mathbb{C}$, satisfying $u : \partial D \to \Gamma$ is a homeomorphism.

Theorem 11.1 *(Classical Plateau Problem) $\exists\, u : D \to \mathbb{R}^n$ of least area among:*

$$X_\Gamma = \{v \in W^{1,2}(D, \mathbb{R}^n) \cap C^0(\overline{D}, \mathbb{R}^n) : \ v : \partial D \to \Gamma \textit{ is monotone \& onto}\}.$$

The map u is harmonic, almost conformal i.e. $|u_x| = |u_y|$ and $u_x \cdot u_y = 0$, a.e. on D, and $u : \partial D \to \Gamma$ is a homeomorphism.

Given $u \in W^{1,2}(D)$ and (x_1, x_2) or (x, y) coordinates on D, the pull back metric is $(g_{ij}) = (u^*\delta)_{ij} = u_{x_i} \cdot u_{x_j} \in L^1(D)$.

Definition 11.1 *The **area** of u is:*

$$A(u) = \int_D \sqrt{det\big(u^*(\delta_{ij})\big)}dx_1dx_2 = \int_D \sqrt{|u_x|^2|u_y|^2 - (u_x \cdot u_y)^2}dxdy.$$

Note: given $\varphi : D \to D$ diffeomorphism, then $A(u \circ \varphi) = A(u)$.

The $W^{1,2}$ norm of u is:

$$\|u\|^2_{W^{1,2}(D)} = \int_D (|u|^2 + |\nabla u|^2)dxdy,$$

where $|\nabla u|^2 = |u_x|^2 + |u_y|^2$.

Definition 11.2 *The **energy functional** of u is*

$$E(u) = \frac{1}{2}\int_D |\nabla u|^2 dxdy.$$

Lemma 11.1 *$A(u) \le E(u)$ with* equality $\Longleftrightarrow$ *u is almost conformal.*

Proof: By Cauchy-Schwartz

$$A(u) = \int_D |u_x \wedge u_y| dxdy \le \frac{1}{2}\int_D |u_x|^2 + |u_y|^2.$$

When equality holds, Cauchy-Schwartz implies the almost conformal property.

□

Corollary 11.1 *If u is a critical point of $E(\cdot)$, and u is conformal, then u is a critical point of $A(\cdot)$.*

Definition 11.3

$$A_\Gamma = \inf\{A(v):\ v \in X_\Gamma\}, \quad E_\Gamma = \inf\{E(v):\ v \in X_\Gamma\},$$

where X_Γ is defined in Theorem 11.1.

Propostion 11.1

$$A_\Gamma = E_\Gamma.$$

Proof: By the above lemma, $A_\Gamma \le E_\Gamma$ clearly. Choose $u \in X_\Gamma$, with $A(u) < A_\Gamma + \epsilon$(may assume that u is smooth in D).

- Suppose u is an immersion. Then $(D, g = u^*\delta_{\mathbb{R}^n})$ is a Riemman surface, where $g_{ij} = u_{x_i} \cdot u_{x_j}$. By the uniformization theorem, $\exists\ \varphi : (D, \delta) \to (D, g)$ conformal diffeomorphism, i.e. $\varphi^* g = \lambda^2 \delta$, hence
$$(u \circ \varphi)^* \delta_{\mathbb{R}^n} = \varphi^*(u^*\delta) = \varphi^* g = \lambda^2 \delta_D.$$
Then $u \circ \varphi : D \to \mathbb{R}^n$ is conformal, hence $E(u \circ \varphi) = A(u \circ \varphi) = A(u)$,
$$\Longrightarrow \quad E_\Gamma \le A_\Gamma + \epsilon.$$
(Remark: we allow the boundary value $u|_{\partial D}$ to change.)
- In general, define $u^s : D \to \mathbb{R}^{n+2}$ by $u^s(x, y) = \big(u(x, y), sx, xy\big)$, for $s > 0$ small enough. Now the pullback metric $\tilde{g}_{ij} = ((u^s)^*\delta_{\mathbb{R}^{n+2}})_{ij} = g_{ij} + s^2\delta_{ij}$. So $\exists \varphi$ diffeomorphism of D, such that $u^s \circ \varphi$ is conformal. Then
$$\begin{aligned} E_\Gamma \le E(u \circ \varphi) = E\big(\varphi : (D, \delta) \to (D, g)\big) = \frac{1}{2}\int_D |D\varphi|_g^2 dxdy \\ \le \frac{1}{2}\int_D |D\varphi|_{\tilde{g}}^2 dxdy = E\big(\varphi : (D, \delta) \to (D, \tilde{g})\big) \le E(u^s \circ \varphi) \\ = A(u^s \circ \varphi) = A(u^s) = A(D, \tilde{g}) \le A(D, g) + \epsilon(s) \\ = A(u) + \epsilon(s) < A_\Gamma + 2\epsilon. \end{aligned}$$
Here $\epsilon(s) \to 0$ as $s \to 0$. Hence we finished the proof.

□

Proof: (of Theorem 11.1) By the above lemma, we want to achieve E_Γ.

Step 1: Dirichlet problem: given $v \in X_\Gamma$, $v : D \to \mathbb{R}^n$, $v \in W^{1,2}(D) \cap C^0(\overline{D})$, then $\exists$ u of least energy in $\mathcal{C}_v = \{w \in W^{1,2}(D) : \ w - v \in W_0^{1,2}(D)\}$. This is equivalent to find a $u \in W^{1,2}(D) \cap C^0(\overline{D}) \cap C^\infty(D)$ which is a solution of

$$\begin{cases} \triangle u = 0 & \text{in } D \\ u = v & \text{on } \partial D. \end{cases}$$

There are several methods: $i)$ Perron method; $ii)$ Poison kernel; $iii)$ Variational method. Now let us use the variational method.

- Take $\{u_i\}$ in $\mathcal{C}_v$, with $E(u_i) \to \inf_{w \in \mathcal{C}_v} E(w)$.
- $\{u_i\}$ are bounded in $W^{1,2}(D)$. $\int_D |\nabla u_i|^2 \leq C$ comes from $E(u_i) \leq C$. Now by Poincaré inequality,

$$\int_D (u_i - v)^2 \leq C \int_D |D(u_i - v)|^2 \leq 2C \int_D |Du_i|^2 + |Dv|^2 \leq C'.$$

  So $\int_D u_i^2 \leq c(\int_D v^2 + C')$.
- Rellich Lemma $\Longrightarrow u_{i'} \rightharpoonup u$ weakly in $W^{1,2}(D)$, and $u_{i'} \to u$ in $L^2(D)$. Furthermore, $u - v \in W_0^{1,2}(D)$ since $W_0^{1,2}(D)$ is weakly closed in $W^{1,2}(D)$.
- Lower semi-continuity of E:

$$\Longrightarrow \quad E(u) \leq \liminf E(u_{i'}) = \inf_{w \in \mathcal{C}_v} E(w).$$

- $u \in \mathcal{C}_v \Longrightarrow u$ is weakly harmonic, $\Longrightarrow u \in C^\infty(D)$.
- $u \in C^0(\overline{D})$. (barrier argument)

  Take $v_i \to v$ uniformly on $\overline{D}$, with $v_i \in C^\infty(\overline{D})$. Solve $\triangle u_i = 0$ in D, $u_i = v_i$ on ∂D, $\Longrightarrow$ $u_i \in C^\infty(\overline{D})$. Maximum Principle implies:

$$\max_{\overline{D}} |u_i - u_j| = \max_{\partial D} |v_i - v_j| \to 0, \quad i, j \to \infty,$$

  $\Longrightarrow u_i \to u$ uniformly in $\overline{D}$, hence $u \in C^0(\overline{D})$.

Step 2: Minimize over boundary parametrization.

Propostion 11.2 *(Courant-Lebesgue Lemma) Given $u : D \to \mathbb{R}^n$, $u \in W^{1,2}(D) \cap C^0(\overline{D})$ and $E(u) \leq K$, then $\forall \delta \ll 1$ and $x \in \partial D$, $\exists \rho \in [\delta, \delta^{1/2}]$ and an arc $C_\rho = (\partial B_\rho(x)) \cap D$, such that*

$$|u(C_\rho)|^2 \leq \frac{2\pi K}{|\log \rho|}.$$

Proof:

$$|u(C_\rho)|^2 \le (\int_{C_\rho} |Du|ds)^2 \le 2\pi\rho \int_{C_\rho} |Du|^2 ds.$$

Now integrate over $\rho \in [\delta, \delta^{1/2}]$,

$$\int_\delta^{\delta^{1/2}} \frac{u(C_\rho)^2}{\rho} d\rho \le \int_\delta^{\delta^{1/2}} 2\pi(\int_{C_\rho} |Du|^2 ds) d\rho \le 2\pi E(u) \le 2\pi K.$$

Hence $\inf_{[\delta,\delta^{1/2}]} |u(C_\rho)|^2(-\log\rho) \le \int_\delta^{\delta^{1/2}} \frac{u(C_\rho)^2}{\rho} d\rho \le 2\pi K$.

□

Key Problem: if φ is a conformal diffeomorphism of D, i.e. $\varphi \in PSL(2,\mathbb{R})$, $\varphi = \frac{\alpha z+\beta}{\gamma z+\delta}$, then

$$E(u \circ \varphi) = E(u).$$

The less of compactness when we minimize over all the possible boundary parametrizations comes from the un-compactness of $PSL(2,\mathbb{R})$.

□

12 Continuity of Plateau Problem and Harmonic maps (2/16/2012)

12.1 Continuity of the Proof of Theorem 11.1

- 3 Point Condition: Fix some orientation on both ∂D and Γ. Given $\{p_i : i = 1,2,3\} \subset \partial D$ and $\{q_i : i = 1,2,3\} \subset \Gamma$ monotone, i.e. $p_1 < p_2 < p_3\}$ w.r.t. the fixed orientation, introduce

$$X_\Gamma^* = \{u \in X_\Gamma : u(p_i) = q_i\}.$$

Lemma 12.1 *Given $v \in X_\Gamma$, $\exists \varphi : D \to D$ Mobiüs transform, such that $u = v \circ \varphi \in X_\Gamma^*$, and $E(u) = E(v)$.*

Proof: Since $v \in X_\Gamma$, $\exists\, r_1, r_2, r_3 \in \partial D$, with $v(r_i) = p_i$. The monotonicity of $v : \partial D \to \Gamma$ implies that $r_1 < r_2 < r_3$. Then $\exists \varphi : D \to D$ Mobiüs, with $\varphi(p_i) = r_i$, $i = 1,2,3$. Hence $u = v \circ \varphi$ satisfies the requirement.

□

Corollary 12.1

$$E_\Gamma = \inf\{E(u) : u \in X_\Gamma^*\}.$$

Lemma 12.2 *Given $K > 0$, and $X_{\Gamma,K}^* = \{u \in X_\Gamma^* : E(u) \le K\}$, then $\{u|_{\partial D} : u \in X_{\Gamma,K}^*\}$ is uniformly equi-continous.*

Proof: Given $\epsilon > 0$, using the Courant-Lebesgue Lemma, $\exists\sqrt{\delta} < \min_{i\neq j=1,2,3}\frac{d(p_i,p_j)}{2}$, and $\exists C_\rho$(= circle centered at $x \in \partial D$, $\rho \in [\delta, \delta^{1/2}]$), such that $|u(C_\rho)| \leq \frac{c\sqrt{K}}{|\log\rho|^{1/2}} < \epsilon$. By our choice of δ, there can be at most one p_i inside C_ρ. Using the monotonicity, the sub-arc of ∂D inside C_ρ must be mapped to the short arc inside $u(C_\rho)$.(or there would be two $q_i s$ inside $u(C_\rho)$, contradiction to the monotonicity.) Hence we have the equip-continuity.

□

Theorem 12.1 $\exists\, u \in X_\Gamma$, *with* $E(u) = E_\Gamma$.

Proof: Take a minimizing sequence $\{u_i\} \subset X_\Gamma$, with $E(u_i) \to E_\Gamma$. By firstly applying Lemma 12.1 and then the Dirichlet Problem in the above section, we can assume that

$$\begin{cases} u_i \in X_\Gamma^* \\ \triangle u_i = 0. \end{cases}$$

By weak compactness of bounded set in $W^{1,2}(D)$ and Lemma 12.2, there exists a subsequence $\{u_{i'}\}$

$$u_{i'} \rightharpoonup u, \text{weakly in } W^{1,2}(D), \quad u_{i'} \Longrightarrow u, \text{ uniformly on } \partial D.$$

The uniform convergence on ∂D implies that $u|_{\partial D}$ is monotone, onto and $u(p_i) = q_i$, $i = 1, 2, 3$. By Maximum Principle

$$u_{i'} \Longrightarrow u, \text{ uniformly in } D, \text{ hence } \triangle u = 0.$$

Now $u \in X_\Gamma^*$ clearly. Hence $E_\Gamma \leq E(u) \leq \liminf E(u_{i'}) = E_\Gamma$, so $E(u) = E_\Gamma$.

□

The following corollary finished the proof of Theorem 11.1.

Corollary 12.2 u *is harmonic, almost conformal, and* $A(u) = A_\Gamma$.

Proof: The harmonicity of u is trivial. Since $u \in X_\Gamma$,

$$A_\Gamma \leq A(u) \leq E(u) = E_\Gamma.$$

So by Lemma 11.1, we have $A(u) = E(u) = A_\Gamma = E_\Gamma$. Hence Lemma 11.1 implies that u is almost conformal.

□

Propostion 12.1 *The set of* ***branch points*** $\{x \in D :\ |\nabla u| = 0\}$ *is discrete.*

Proof: $\{x :\ |\nabla u| = 0\} \subset \{x :\ \frac{\partial u}{\partial z} = 0\}$ is discrete, since $\frac{\partial u}{\partial z}$ is holomorphic(i.e. $\frac{\partial}{\partial \bar{z}}(\frac{\partial u}{\partial z}) = \triangle u = 0$).

□

Propostion 12.2 *If Γ is a $C^{k,\alpha}$-curve, $k \geq 2$, $0 < \alpha < 1$, then $u \in C^{k,\alpha}(\overline{D})$, with finite number of branch points, and boundary branch points are isolated.*

Theorem 12.2 *(Osserman) When $n = 3$, the solution u has no interior branch points.*

Open Question: When $n = 3$, can there be a boundary branch points. (True when boundary is analytic.)

- In $\mathbb{R}^n$, $n \geq 4$, the minimizing solution has branch points, e.g. $z \to (z^2, z^3) \in \mathbb{C}^2 \simeq \mathbb{R}^4$.

Propostion 12.3 *The solution u is a homeomorhpism on ∂D to Γ.*

- Since u is monotone an onto, if u is not homeomorphism, then u must map a sub-interval of ∂D to a point on Γ. By a reflection argument, we can reflect conformally near the interval where u is a constant, and extend u to a conformal harmonic map in a neighborhood of the interval, then u maps an interval to a point. In fact, u maps the direction tangent to ∂D to 0, but the normal direction not to 0(if 0, then u has branch points along an interval, contradiction to the fact that branch points are discrete), contradiction to the conformal property.

12.2 Harmonic maps

Motivation: find minimal Σ^2 in an arbitrary Riemannian manifold (M^n, g).

Definition 12.1 *Given $u : (\Omega^k, \gamma) \to (M^n, g)$, where (Ω^k, γ) is a k-dimensional Riemannian manifold possibly with boundary, the **harmonic energy** is:*

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 d\mu_\gamma,$$

*where in local coordinates $\{x^1, \cdots, x^k\}$ of Ω, $\{u^1, \cdots, u^n\}$ of M, the **harmonic energy density** is*

$$|\nabla u|^2 = tr_\gamma(u^* g) = \sum_{i,j=1}^{k} \gamma^{ij} \langle du(\frac{\partial}{\partial x^i}), du(\frac{\partial}{\partial x^j}) \rangle_g = \sum_{i,j,\alpha,\beta} \gamma^{ij} g_{\alpha\beta} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}.$$

*Critical point of E is called **harmonic maps**.*

Lemma 12.3 *The Euler-Lagrange equation of E is call the **harmonic map equation(HE)**:*

$$\triangle_\gamma u^\alpha + \sum_{i,j,\beta,\delta} \gamma^{ij} \Gamma^\alpha_{\beta\delta}\big(u(x)\big) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\delta}{\partial x^j} = 0, \quad \alpha = 1, \cdots, n. \tag{14}$$

Another form of (HE): Consider an isometric embedding $M^n \subset \mathbb{R}^N$, then

$$u : \Omega \to \mathbb{R}^N, \quad u(\Omega) \subset M,$$

and

$$E(u) = \frac{1}{2} \sum_{\alpha=1}^{N} \int_\Omega |\nabla_\gamma u^\alpha|^2 d\mu_\gamma.$$

So the variational problem can be viewed as a constraint problem.

- Take a variation vector fields $\frac{d}{dt}|_{t=0}u_t = \eta$, where η is tangent to M;
- $$\frac{d}{dt}|_{t=0}E(u_t) = \int_\Omega \langle \nabla u, \nabla \eta \rangle d\mu_\gamma.$$
- The Euler-Lagrange equation is: $(\triangle_\gamma u)^T = 0$, or

$$\triangle_\gamma u^\alpha = [(\triangle_\gamma u)^\perp]^\alpha = \sum_{i,j=1}^{k} \gamma^{ij} A^\alpha_{u(x)}(\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j}), \tag{15}$$

where A is the second fundamental form of $M \subset \mathbb{R}^N$.
- For example, $M = S^n \subset \mathbb{R}^{n+1}$, then $\triangle u^\alpha = -|\nabla u|^2 u^\alpha$.

Bochner Formula: Given $u : (\Omega^k, \gamma) \to (M^n, g)$ harmonic and smooth, then

$$\begin{aligned} \frac{1}{2}\triangle|\nabla u|^2 = |\nabla\nabla u|^2 + g_{\alpha\beta}\gamma^{ik}\gamma^{jl}R^\Omega_{ij}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j} \\ - \sum_{i,j}\langle R^M\big(du(e_i), du(e_j)\big)du(e_i), du(e_j)\rangle, \end{aligned} \tag{16}$$

where R^Ω_{ij} is the Ricci curvature of (Ω, γ), R^M the sectional curvature of (M, g), and $\{e_1, \cdots e_k\}$ an o.n. basis on (Ω, γ).

- When Ω and M are all compact,

$$\triangle|\nabla u|^2 \geq -c_1|\nabla u|^2 - c_2|\nabla u|^4,$$

where c_1, c_2 are two positive constants.
- If $R^M \leq 0, \Longrightarrow \triangle|\nabla u|^2 \geq -c_1|\nabla u|^2$, then

$$\sup_{B_{R/2}} |\nabla u|^2 \leq C\int_{B_R} |\nabla u|^2.$$

- If does not assume $K_M \leq 0$, we have counterexamples to the gradient estimates.

Example: $\exists\ u_i : S^2 \to S^2$, holomorphic, but with strong dilation which maps a small neighborhood of the south pole to almost all of S^2, satisfying: $|\nabla u_i|^2(0) \to \infty$, $\int_{S^2} |\nabla u_i|^2 = 8\pi$, and each u_i is energy minimizing.

Lemma 12.4 *Given $u : (\Omega^2, \gamma) \to (M^n, g)$, the area is $A(u) = \int_\Omega \sqrt{det(u^*g)}dx$.*

$$A(u) \leq E(u),$$

with " $=$ " if and only if u is almost conformal.

13 Sacks-Uhlenbeck's theorem (2/21/2012)

13.1 Hopf differential

In the case (Σ^2, γ) has dimension 2, let $z = x + iy$ be the local complex coordinates w.r.t. the conformal structure determined by γ.

Definition 13.1 *Given $u : (\Sigma^2, \gamma) \to (M^n, g)$, the Hopf differential Φ is defined by:*

$$\Phi = \varphi(z)dz^2,$$

where

$$\varphi(z) = |\frac{\partial u}{\partial x}|^2 - |\frac{\partial u}{\partial y}|^2 - 2i\langle\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\rangle.$$

Propostion 13.1 *Viewing $E(u, \gamma)$ as a functional of both u and γ, then:*

1. *$\Phi \equiv 0 \Longleftrightarrow E(u, \gamma)$ is critical w.r.t. compactly supported variation of γ;*
2. *Φ is holomorphic $\Longleftrightarrow E(u, \gamma)$ is critical for domain variations, i.e. $u_t = u \circ f_t$, where f_t are $1-$parameter family of diffeomorphisms, with $f_t = id$ near $\partial\Sigma$.*

Proof: Consider $\gamma_t : -\epsilon < t\epsilon$, and $\dot{\gamma}_0 = h$. Since

$$E(u, \gamma) = \frac{1}{2}\int_\Sigma \sum_{i,j=1}^{2} \gamma^{ij}\langle\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j}\rangle\sqrt{det\gamma}dx^1dx^2,$$

and

$$\begin{cases} \frac{d}{dt}|_{t=0}\gamma_t^{ij} = -h^{ij} = -\gamma^{ip}\gamma^{jq}h_{pq}, \\ \frac{d}{dt}\sqrt{det\gamma} = \frac{1}{2}tr_\gamma(h)\sqrt{det\gamma}, \end{cases}$$

we have

$$\delta_\gamma E(u, \gamma) = -\int_\Sigma h^{ij}\underbrace{(\langle\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j}\rangle - \frac{1}{2}|\nabla u|^2\gamma_{ij})}_{T_{ij}-\text{stree-energy tensor}}d\mu_\gamma.$$

Proof of 1: locally $\gamma_{ij} = \lambda^2\delta_{ij}$, so

$$T = \begin{bmatrix} \frac{1}{2}(-|u_x|^2 + |u_y|^2), -\langle u_x, u_y\rangle \\ -\langle u_x, u_y\rangle, \frac{1}{2}(|u_x|^2 - |u_y|^2) \end{bmatrix} = 0 \Longleftrightarrow \Phi = 0.$$

Proof of 2: **Claim**: Φ is holomorphic $\Longleftrightarrow div_\Sigma T = 0$.
(On (Σ, z), trace-free, divergence free (0, 2)-tensors are $1 : 1$ correspondence to holomorphic quadratic differentials. **Pf**: in local conformal coordinates $z = x^1 + ix^2$, let $T = T_{ij}$, hence $T_{11} = -T_{22}$(trace free), $T_{11,1} + T_{12,2} = 0$, $T_{21,1} + T_{22,2} = 0$(divergence free). If we let $\Phi = \varphi(z)dz^2 = (T_{11} - T_{22} - 2iT_{12})dz^2$, then it is easy to check $\frac{\partial}{\partial\bar{z}}\varphi = 0$.)

- Using the composition $f_t : (\Sigma, \gamma) \to (\Sigma, (f_t^{-1})^*\gamma)$ and $u : (\Sigma, (f_t^{-1})^*\gamma) \to M$

$$E(u \circ f_t, \gamma) \equiv E(u, \underbrace{(f_t^{-1})^*\gamma}_{\gamma_t}).$$

- $\frac{d}{dt}|_{t=0} RHS = \int_\Sigma \langle h, T \rangle d\mu_{ga}$. Let $\vec{X} = \frac{d}{dt}|_{t=0} f_t$, we have $h = \frac{d}{dt}|_t \gamma_t = -\mathcal{L}_X \gamma = -(X_{i,j} + X_{j,i})$. So

$$\frac{d}{dt}|_{t=0} E(u \circ f_t, \gamma) = \delta_\gamma E = \int_\Sigma (X_{i,j} + X_{j,i}) T^{ij} d\mu_\gamma = 2 \int_\Sigma X_{i,j} T^{ij} \mu_\gamma$$

$$= - \int \langle X, divT \rangle d\mu_\gamma = 0.$$

Hence $\delta_\gamma E = 0 \iff divT = 0$, for $\delta\gamma = \mathcal{L}_X \gamma$.

□

Remark 13.1 • *When u is smoothly harmonic $\Longrightarrow \frac{d}{dt} E(u \circ f_t, \gamma) = 0$ for f_t 1-parameter family of diffeomorhpisms $\Longrightarrow \Phi$ is holomorphic;*

- *When u is weakly harmonic, $u \circ f_t$ may not be C^1 variations of u.*

Corollary 13.1 1. *When $u : S^2 \to M$ is smoothly harmonic map, then u is almost conformal;*

2. *When $u : \mathbb{R}^2 \to M$ is smoothly harmonic and $\int_{\mathbb{R}^2} |\nabla u|^2 dx < \infty$, then u is almost conformal.*

Proof: Let $\Phi(z) = \varphi(z) dz^2$ on $S^2 \setminus \{+\infty\}$, with $\varphi(z)$ an entire holomorphic function.

- <u>Pf of 2</u>: $\int_{\mathbb{R}^2} |\nabla u|^2 < \infty \Longrightarrow \int_{\mathbb{C}} |\varphi| < \infty \Longrightarrow \varphi \equiv 0$.
- <u>Pf of 1</u>: Φ is regular at ∞, hence $\varphi(z) dz^2$ is regular at ∞. Let $\zeta = 1/z$, then $\Phi = \varphi(1/\zeta)(-\frac{d\zeta}{\zeta^2})^2 = (\varphi(1/\zeta)/\zeta^4)(d\zeta)^2$. Since $\varphi(1/\zeta)/\zeta^4$ is regular near $\zeta = 0$, $\Longrightarrow |\varphi(z)| \leq \frac{C}{|z|^4} \Longrightarrow \varphi \equiv 0$.

□

Remark 13.2 • *Only true when S^2(u is critical & u is conformal invariant & S^2 has only one conformal structure).*

- *Not true for Σ_g with $g \geq 1$.*

13.2 Sacks-Uhlenbeck's method

<u>**Sacks-Uhlenbeck**</u>**:** If (M^n, g) is a compact Riemannian manifold and $\pi_k(M) \neq 0$ for some $k \geq 2$, then $\exists$ a nontrivial $u : S^2 \to M$, which is harmonic(hence almost conformal).

<u>**2 Improvement:**</u>

1. Meeks-Yau: $\pi_2(M)$ is generated by minimal 2-spheres.

   Question: when $k = 2$, given $v : S^2 \to M$ not homotopic to point, $\exists\, u$ harmonic and homotopic to v?

   Result: given $[v] \in \pi_2(M)$, then $[v] = [u_0] + \cdots + [u_k]$, where $\{u_0, \cdots, u_k\}$ are minimal stable spheres.
2. Micallef-Moore: If $\pi_k(M) \neq 0$ for $k \geq 3$, then $\exists\, u$ minimal with Morse Index(for the energy functional E) $\leq k - 2$.

Key Idea:

Definition 13.2 *Given $\alpha > 1$ and $u \in W^{1,2\alpha}(S^2, M) \subset C_0(S^2, M)$, the α-**energy** is defined by*

$$E_\alpha(u) = \frac{1}{2} \int_{S^2} (1 + |\nabla u|^2)^\alpha d\mu.$$

Remark 13.3 *When $\alpha = 1$, $E_1(u) = 2\pi + E(u)$.*

$\alpha > 1$, Finding critical point u_α for E_α is much easier.
$\alpha_i \searrow 1$: expect $u_{\alpha_i} \to u$, with u harmonic. In fact, we have good convergence if $|\nabla u_{\alpha_i}|^2$ is uniformly bounded.

Lemma 13.1 *Euler-Lagrangian equation for E_α:*

1. *If $u \in W^{1,2\alpha}(S^2)$, and u is critical for E_α, then u is a weak solution of:*

$$div\big((1 + |\nabla u|^2)^{\alpha-1} \nabla u^i\big)^T = 0, \quad M \subset \mathbb{R}^N \& i = 1, \cdots, N,$$

$$\Longleftrightarrow div\big((1 + |\nabla u|^2)^{\alpha-1} \nabla u^i\big) = (1 + |\nabla u|^2)^{\alpha-1} \gamma^{pq} A^i(\frac{\partial u}{\partial x^p}, \frac{\partial u}{\partial x^q}).$$

2. *If u is smooth,*

$$\triangle u^i + (\alpha - 1)\frac{\langle \nabla |\nabla u|^2, \nabla u^i \rangle}{1 + |\nabla u|^2} = \sum_{p,q=1}^{2} \gamma^{pq} A^i(\frac{\partial u}{\partial x^p}, \frac{\partial u}{\partial x^q}), \quad i = 1, \cdots, N.$$

14 Sacks-Uhlenbeck's theorem continued (2/23/2012)

Theorem 14.1 *If (M^n, g) is compact and $\pi_2(M) \neq 0$, then $\exists$ nontrivial $u : S^2 \to M$ harmonic and almost conformal.*

Propostion 14.1 *If $\alpha > 1$, given $v : S^2 \to M$, then $\exists\, u \in C^\infty(S^2)$ with $E_\alpha(u) = min\{E_\alpha(w) : w \in W^{1,2\alpha}(S^2)$ & u homotopic to $v\}$.*

Proof: Minimal u exists by direct method(minimization).

- $u \in W^{1,2\alpha}(S^2) \implies u \in W^{2,2}(S^2)$ follows from Morrey Thm 1.11.1;
- $u \in W^{2,2}(S^2) \implies u \in C^\infty(S^2)$ follows from the standard elliptic estimates.

□

Lemma 14.1 *For $(\alpha - 1)$ small enough, if u is critical for E_α, then for $p \geq 3$,*

$$\|u\|_{2,p} \leq C\big(1 + (\sup_{S^2} |\nabla u|^2)^{\frac{p-1}{p}} (\int_{S^2} |\nabla u|^2)^{1/p}\big).$$

Proof: Since u is smooth by the above lemma, by the Euler-Lagrangian equation:

$$|\triangle u| \leq (\alpha - 1)|\nabla\nabla u| + C|\nabla u|^2.$$

Hence standard L^p estimates imply that:

$$\|u\|_{2,p} \leq C(\|u\|_{0,p} + \|\triangle u\|_{0,p}) \leq C\big(1 + (\alpha - 1)\|u\|_{2,p} + \big\||\nabla u|^2\big\|_{0,p}\big).$$

When $(\alpha - 1)$ is small enough,

$$\|u\|_{2,p} \leq C\big(1 + (\int_{S^2} |\nabla u|^{2p})^{1/p}\big) \leq C\big(1 + (\sup_{S^2} |\nabla u|^2)^{\frac{p-1}{p}} (\int_{S^2} |\nabla u|^2)^{1/p}\big).$$

□

Proof: (of **Theorem 14.1**) Take a sequence $\alpha_1 \searrow 1$ and corresponding critical mapping $u_i = u_{\alpha_i}$.
Case 1: $\exists$ subsequence $u_{i'}$ with $|\nabla u_{i'}| \leq C$, then $u_{i'}$ satisfy uniform $C^{2,\alpha}$ estimates by Lemma 14.1 and standard elliptic theory, $\implies u_{i'} \to u$ in C^2 and u is our solution;
Case 2: $\lambda_i = \max_{S^2} |\nabla u_i|^2 \to +\infty$. $\exists\, p_i \in S^2$, with $|\nabla u_i(p_i)|^2 = \lambda_i$. Assume a subsequence $p_i \to p \in S^2$.

- Let $\gamma_i = \lambda_i \gamma$, where γ is the standard metric on S^2, then γ_i becomes flatter.
- Then we have the following scaling equality:

$$|\nabla u_i|^2_{\gamma_i} = \lambda_i^{-1} |\nabla u_i|^2_\gamma \implies \int_{S^2} |\nabla u_i|^2_{\gamma_i} d\mu_{\gamma_i} = \int_{S^2} |\nabla u_i|^2_\gamma d\mu_\gamma;$$

$$|\nabla\nabla u_i|^2_{\gamma_i} = \lambda_i^{-2} |\nabla\nabla u_i|^2_\gamma \implies \int_{S^2} |\nabla\nabla u_i|^q_{\gamma_i} d\mu_{\gamma_i} = \lambda_i^{1-q} \int_{S^2} |\nabla\nabla u_i|^q_\gamma d\mu_\gamma.$$

- Using the above scaling equality and the estimates in Lemma 14.1,

$$\implies \lambda_i^{p-1} \int_{S^2} |\nabla\nabla u_i|^p_{\gamma_i} \leq C\big(1 + (\lambda_i)^{p-1} \int_{S^2} |\nabla u_i|^2_{\gamma_i}\big),$$

$$\implies \int_{S^2} |\nabla\nabla u_i|^p_{\gamma_i} \leq C\big(\lambda_i^{-(p-1)} + \int_{S^2} |\nabla u_i|^2_{\gamma_i}\big).$$

- So u_i are uniformly in $W^{2,p}_{loc}(S^2,\gamma_i)$. Since (S^2,γ_i,p_i) locally converge to $(\mathbb{R}^2,\delta,0)$, using the Sobolve embedding,
$$u_{i'} \to u \text{ in } C^{1,1/2}\text{-loc on } \mathbb{R}^2,$$
and $|\nabla u(0)| = 1 = \max_{\mathbb{R}^2}|\nabla u|$.
- The Euler-Lagrangian equation is scaled to:
$$\lambda_i \triangle_{\gamma_i} u_i^j + (\alpha_i - 1)\frac{\lambda_i\langle\nabla(\lambda_i|\nabla u_i|^2_{\gamma_i}), \nabla u_i^j\rangle_{\gamma_i}}{1+\lambda_i|\nabla u_i|^2_{\gamma_i}} = \sum_{p,q=1}^{2}\lambda_i\gamma_i^{pq}A^j(\frac{\partial u_i}{\partial x^p}, \frac{\partial u_i}{\partial x^q}),$$
$$\Longrightarrow \triangle_{\gamma_i} u_i^j + (\alpha_i - 1)\frac{\langle\nabla|\nabla u_i|^2_{\gamma_i}, \nabla u_i^j\rangle_{\gamma_i}}{\lambda_i^{-1}+|\nabla u_i|^2_{\gamma_i}} = \sum_{p,q=1}^{2}\gamma_i^{pq}A^i(\frac{\partial u_i}{\partial x^p}, \frac{\partial u_i}{\partial x^q}).$$
Here the second term $(\alpha_i - 1)\frac{\langle\nabla|\nabla u_i|^2_{\gamma_i}, \nabla u_i^j\rangle_{\gamma_i}}{\lambda_i^{-1}+|\nabla u_i|^2_{\gamma_i}} \le (\alpha_i-1)C|\nabla\nabla u_i|_{\gamma_i}$, where $|\nabla\nabla u_i|_{\gamma_i}$ is uniformly bounded in $L^p(S^2,\gamma_i)$. So the second term converges to 0 in $L^p_{loc}(\mathbb{R}^2,\delta)$.
- $\Longrightarrow u$ is a $C^{1,1/2}$ weakly harmonic map. Elliptic estimates imply that $u:\mathbb{R}^2\to M$ is a C^∞ harmonic map. Furthermore, u is nontrivial, since $|\nabla u(0)| = 1$ and $E(u) < \infty$.
- Claim: u can be extended to a smooth harmonic map on $S^2 = \mathbb{R}^2\cup\{\infty\}$.

  In $(\mathbb{R}^2, z)$, $z\to\zeta = 1/z$ is conformal, which takes ∞ to 0. Then
$$v(\zeta) = u(1/\zeta) \quad \text{is harmonic on } \mathbb{C}\setminus\{0\} \;\&\; E(v)<\infty.$$
  Since $u:\mathbb{C}\to M$ is harmonic $\&$ $E(u)<\infty$, so u is almost conformal by Corollary 13.1, hence v is also harmonic.

  Key step: v is continuous at $\zeta = 0(v\in W^{1,2}\cap C^0(\Sigma^k)$ $\&$ weakly harmonic $\Longrightarrow v$ smooth).

  1°. By Courant-Lebesgue Lemma 11.2, $\exists\, r_i\to 0$ such that $\max_{\partial D_{r_i}} d_M\big(v(s_1),v(s_2)\big)\to 0$;

  2°. Claim: The oscillation of v on $D_{r/2}$ is bounded by a multiply of the energy $E_{D_r}(v)$ and boundary oscillation on ∂D_r:
$$\max_{D_{r/2}} d_M\big(v(s_1),v(s_2)\big) \le C(E_{D_r}(v)) + \max_{\partial D_r} d_M\big(v(s_1),v(s_2)\big).$$
  If $\exists\, s_1,s_2\in\partial D_r$, such that $d_M\big(v(s_1),v(s_2)\big)$ is too large, we can cover the image of $v(D_r)$ by several unit balls in M. Then by the monotonicity formula of minimal surfaces(v is harmonic and almost conformal, hence minimal), each portion of $v(D_r)$ inside the unit ball has a fixed mount of area, which then makes the total area of $v(D_r)$ too large than $E(D_r)$, a contradiction.
- Hence v is smooth on $\mathbb{C}$, which means that u can be extended to a nontrivial harmonic map on S^2. Finished.

□

15 Colding-Minicozzi's min-max sphere (3/1/2012) (by Xin Zhou)

Motivation: Given a Riemannian manifold (M, g), want to find unstable minimal spheres in $C^0 \cap W^{1,2}(S^2, M)$ by direct variational method.

In fact, any $\pi_3(M)$ representative $u : S^3 \to M$ can be viewed as a 1-parameter family of maps $S^2 \to M$, i.e. $S^3 = S^2 \times [0,1]$, with $S^2 \times \{0\}$ and $S^2 \times \{1\}$ mapped to points.

Definition 15.1 *The variational space is define as:*

$$\Omega = \big\{u(t,x) \in C^0\big([0,1], C^0 \cap W^{1,2}(S^2, M)\big) : \ u(0), u(1) = \textit{point map}\big\}.$$

Given $\beta \in \Omega$, denote $[\beta]$ to be the homotopy class of β in Ω. The area width of $[\beta]$ is

$$W_A = \inf_{u\in[\beta]} \max_{t\in[0,1]} Area\big(u(t)\big);$$

the energy width of $[\beta]$ is

$$W_E = \inf_{u\in[\beta]} \max_{t\in[0,1]} E\big(u(t)\big).$$

Propostion 15.1

$$W_A = W_E.$$

Hence we denote the width by $W = W_A = W_E$.

Theorem 15.1 *(Colding-Minicozzi) Given (M, g) and $\rho \in \Omega$, such that $\rho \in \pi_3(M)$ is nontrivial, then*

1. *$\exists\, \gamma^j \subset \Omega$, $j = 1, \cdots, \infty$, such that $\max_{t\in[0,1]} E\big(\gamma^j(t)\big) \searrow W$;*
2. *$\forall\, \epsilon > 0$, $\exists\, J \gg 1$ and $\delta > 0$, such that if $j > J$, for any t with*

$$E\big(\gamma^j(t)\big) - W > -\delta,$$

   $\exists$ a collection of harmonic spheres(hence almost conformal) $u_i : S^2 \to M$, $i = 0, \cdots, l$, such that

$$d_V\big(\gamma^t, \sqcup_{i=0}^{l} u_i\big) < \epsilon,$$

   where d_V is the varifold distance;

2′. *or $\forall\, t_i$ such that $E\big(\gamma^j(t_j)\big) \to W$, a subsequence $\gamma^j(t_j)$ converge to a collection of harmonic spheres $\{u_0, \cdots, u_l\}$ in the sense of bubble-tree convergence.*

Remark 15.1 *In fact, $2' \Longrightarrow 2$, or the bubble-tree convergence $\Longrightarrow$ varifold convergence.*

Bubble-tree convergence: (Definition 3.6 in [3])

Roughly a sequence of $u^j \in W^{1,2}(S^2, M)$ is said to bubble-tree converge to a collection of harmonic spheres $\{u_i\}_{i=0}^{l}$, if

1. $u^j \rightharpoonup u_0$ weakly(up to a subsequence) in $W^{1,2}(S^2)$;

2. $\exists\, \mathcal{S}_0 = \{x_0^1, \cdots, x_0^{k_1}\} \subset S^2$, such that $u^j \to u_0$ strongly in $W^{1,2}(K)$ for any K compact subset of $S^2 \setminus \mathcal{S}_0$;
3. Near each $x_0^i \in \mathcal{S}_0$, $\exists\, D_{ij} : S^2 \to S^2$ conformal dilation, which takes a small ball centered at x_0^i to the lower-hemisphere, and $u^j \circ D_{ij}$ converges to u_i in the sense of step 1 and 2;
4. We have the energy identity:

$$\lim_{j\to\infty} E(u^j) = \sum_{i=1}^{l} E(u_i).$$

Key ideas: Variational Method. Given $\beta \in \Omega$, with $[\beta] \in \pi_3(M)$ nontrivial,

0). **Mollify the minimizing sequence:** Find minimizing sequence $\tilde{\gamma}^j(t) \in [\beta]$ such that $\tilde{\gamma}^j \in C^0\big([0,1], C^2(S^2, M)\big)$;

1). **Almost conformal reparametrization:** Reparametrize $\tilde{\gamma}^j(t) \to \gamma^j(t) = \tilde{\gamma}^j\big(h^j(t)(\cdot), t\big)$, where $h^j(t) : (S^2, g_0) \to (S^2, \tilde{\gamma}^j(t)^* g + \delta_j^2 g_0)$ is continuous 1-parameter family of conformal isotopies, hence $\gamma^j \in [\beta]$, $Area\big(\gamma^j(t)\big) = Area\big(\tilde{\gamma}^j(t)\big)$, and

$$\max_{t\in[0,1]} E\big(\gamma^j(t)\big) - Area\big(\gamma^j(t)\big) \to 0;$$

2). **Tightening:** $\gamma^j(t) \to \rho^j(t)$, by local harmonic replacement(Perron method), hence $\rho^j \in [\beta]$, $E\big(\rho^j(t)\big) \le E\big(\gamma^j(t)\big)$, and $\rho^j(t)$ is almost harmonic if $|E\big(\rho^j(t)\big) - W| \ll 1$.

Step 0: Using a mollification method, we have

Lemma 15.1 *Given $\beta \in \Omega$, $\exists \tilde{\gamma}^j \in [\beta]$, with $\max_{t\in[0,1]} Area\big(\tilde{\gamma}^j(t)\big) \searrow W_A$, and $\tilde{\gamma}^j(t) \in C^0\big([0,1], C^2(S^2, M)\big)$.*

Step 1: Reparemetrization.

Propostion 15.2 *$\exists\, \gamma^j \in [\beta]$, $Area\big(\gamma^j(t)\big) = Area\big(\tilde{\gamma}^j(t)\big)$, and*

$$\max_{t\in[0,1]} E\big(\gamma^j(t)\big) - Area\big(\gamma^j(t)\big) \to 0.$$

In fact, Proposition 15.1 is a direct corollary.
Proof: (of Proposition 15.1) $W_A \le W_E$ is clearly true since $Area(\cdot) \le E(\cdot)$. Then

$$W_E \le \lim_{j\to\infty}\big[\max_{t\in[0,1]} E\big(\gamma^j(t)\big)\big] = \lim_{j\to\infty}\big[\max_{t\in[0,1]} Area\big(\gamma^j(t)\big)\big] = W_A.$$

□

Lemma 15.2 *(Uniformization)*

- *$\forall\, C^1$ metric g on S^2, $\exists!\, C^{1,1/2}$ isotopy $h : (S^2, g_0) \to (S^2, g)$, fixing three points, and conformal diffeomorphism;*

- *If g_1, g_2 are two C^1 metrics, and $g_i \geq \epsilon g_0$, let h_1, h_2 be the unique conformal isotopic diffeomorphism, then*

$$\|h_1 - h_2\|_{C^2 \cap W^{1,2}(S^2,S^2)} \leq C(\epsilon, \|g_i\|_{C^1})\|g_1 - g_2\|_{C^0}.$$

Sketch of proof: Pull h back to $h : (\mathbb{C}, dwd\bar{w}) \to (\mathbb{C}, g = \lambda^2|dz + \mu(z)d\bar{z}|^2)$.

- h satisfy $\overline{(h_w)} = \mu\big(h(w)\big)h_w$ a quasi-linear elliptic system;
- Apriori estimates $\Longrightarrow$ results.

Proof: (of Proposition 15.2)

□

Step 2: Tightening.

Propostion 15.3 *$\exists\, \epsilon_0 > 0$, and continuous $\Psi : [0, \infty) \to [0, \infty)$, $\Psi(0) = 0$ depending on M. $\forall \gamma \in \Omega$ with no non-constant harmonic slice, i.e. $\gamma(t)$ is not harmonic unless $\gamma(t) = pt$, then $\exists\ \gamma \to \rho$ deformation, such that $\rho \in [\gamma]$, $E\big(\rho(t)\big) \leq E\big(\gamma(t)\big)$, and if $E\big(\gamma(t)\big) \geq \frac{W}{2}$, then*

(B) *$\forall\ \mathcal{B}$ finite collection of balls on S^2, with $\int_{\mathcal{B}} |\nabla\rho(t)|^2 < \epsilon_0$, let $v : \frac{1}{8}\mathcal{B} \to M$ be the energy-minimizing harmonic map, with $v|_{\frac{1}{8}\partial\mathcal{B}} = \rho(t)|_{\frac{1}{8}\partial\mathcal{B}}$, then*

$$\int_{\frac{1}{8}\mathcal{B}} |\nabla\rho(t) - \nabla v|^2 \leq \Psi\big[E\big(\gamma(t)\big) - E\big(\rho(t)\big)\big].$$

Harmonic replacement.

16 Introduction to the Willmore conjecture (3/6/2012)

Willmore Conjecture in R^3: $\Sigma^2 \subset \mathbb{R}^3$ compact, embedded surface, the **Willmore energy** is defined by

$$W(\Sigma) = \int_\Sigma H^2 d\Sigma,$$

where $H = \frac{1}{2}(k_1 + k_2)$ is the normalized mean curvature.

- $W(S^2) = 4\pi$.

Conjecture: If Σ is a torus, then $W(\Sigma) \geq 2\pi^2$, "$=$" only if Σ is the Clifford torus.

Conformal invariance of W: $\Sigma^k \subset (M^{k+1}, g)$, A second fundamental form, $\mathring{A}$ trace-free part of A, then $\hat{W}$ is conformal invariant:

$$\hat{W}(\Sigma, g) = \int_\Sigma |\mathring{A}|^k d\mu_g = \hat{W}(\Sigma, e^u g).$$

Proof: Given local coordinates $\{x^1, \cdots, x^k\}$ on $\Sigma \subset (M, g)$, then the 2nd f.f. is h_{ij}. Let $\hat{g} = e^u g$, then

$$\hat{h}_{ij} = e^{u/2}(h_{ij} + \frac{\partial u}{\partial \nu})g_{ij}.$$

Hence $\mathring{\hat{h}}_{ij} = e^{u/2}\mathring{h}_{ij}$, and $|\mathring{A}|_g = e^{-u/2}|\mathring{A}|_{\hat{g}}$.

□

When $\underline{k=2}$, k_1, k_2 principal curvatures,

$$|\mathring{A}|^2 = \frac{1}{2}(k_1 - k_2)^2 = 2H^2 - 2k_1k_2.$$

Since Gauss curvature $K = k_1k_2$ in $\mathbb{R}^3$,

$$\int_\Sigma H^2 = \frac{1}{2}\int_\Sigma |\mathring{A}|^2 + 2\pi\chi(\Sigma).$$

<u>Consider S^3:</u> View $\mathbb{R}^3$ as stereographic projection to S^3, which is a conformal transformation.

$$\begin{aligned} W(\Sigma) &= \frac{1}{4}\int_\Sigma (k_1 - k_2)^2 d\Sigma + 2\pi\chi(\Sigma) \\ &= \frac{1}{4}\int_\Sigma (4H^2 - 4k_1k_2)d\Sigma + 2\pi\chi(\Sigma) \\ &= \int_\Sigma \big(H^2 - 4(K_\Sigma - 1)\big) + 2\pi\chi(\Sigma) \\ &= \int_\Sigma (1 + H^2)d\Sigma, \end{aligned}$$

where K_Σ is the Gauss curvature of $\Sigma \subset S^3$, and $K_\Sigma = 1 + k_1k_2$ by Gauss formula. Hence when project $\Sigma \subset \mathbb{R}^3$ to $\Sigma \subset S^3$, the **Willmore energy** is

$$\underline{W(\Sigma) = \int_\Sigma (1 + H^2)d\Sigma.} \tag{17}$$

- If $H = 0$, then $W(\Sigma) = |\Sigma|$;
- <u>Clifford torus</u>: let $(x, y) \in \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, Clifford torus is

  $$T_c = S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) = \big\{(x, y) \in \mathbb{R}^4 : \ |x|^2 = |y|^2 = 1/2\big\} \subset S^3.$$

  $W(T_c) = A(T_c) = 2\pi^2$.

17 Outline of Marques-Neves's proof of Willmore conjecture [2] (3/8/2012)

- $\Sigma^2 \subset S^3$, with H the mean curvature, the **Willmore energy** is defined as

$$W(\Sigma) = \int_\Sigma (1 + H^2) d\Sigma.$$

  $W(equator) = 4\pi$; if Σ is not embedded $\Longrightarrow W(\Sigma) \geq 8\pi > 2\pi^2$.
- $T_c^2 = S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \subset S^3$ is the Clifford torus, $K = H = 0$ flat,

$$W(T_c^2) = 2\pi^2.$$

Theorem 17.1 *If Σ has genus ≥ 1, then $W(\Sigma) \geq 2\pi^2$, with equality only if Σ is a Clifford torus.*

Theorem 17.2 *If $H = 0$, and genus ≥ 1, then $|\Sigma| \geq 2\pi^2$, with equality only for Clifford torus.*

Theorem 17.3 *If Σ is embedded, $g(\Sigma) \geq 1$, then $\exists$ minimal surface $\tilde{\Sigma}$ with $4\pi < |\tilde{\Sigma}| \leq W(\Sigma)$.*

Remark 17.1 *(Hopf-Almgren) $\Sigma \subset S^3$, $H = 0$, and $\pi_1(\Sigma) = \{1\}$, then Σ is an equator.*
Pf: *Let $f : \Sigma \to S^3$ be the immersion, and $\Pi(\cdot, \cdot)$ the 2nd f.f. of Σ, then the Hopf-differential*

$$\varphi(z) = \Pi(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial z}) dz^2$$

is holomorphic quadratic(In local coordinates, the 2nd f.f. h_{ij} is trace-free, and divergence-free $\sum_j \nabla^j h_{ij} = \sum_j \nabla^i h_{jj} = \nabla H = 0$ since S^3 is constant curvature, hence φ is holomorphic by section 13.1). Hence $\Pi \equiv 0$.

Let $I^n = [0,1]^n$, and

$$\Phi : I^n \to \mathcal{Z} = \mathcal{Z}_2(S^3),$$

where $\Phi(x)$ is a surface(current), and Φ is continuous. Let

$$\Pi = \text{relative homotopy class of } \Phi = \{\Phi' \sim \Phi\},$$

i.e. $\exists\ \Phi_t : I^n \to \mathcal{Z}$, $0 \leq t \leq 1$, $\Phi_0 = \Phi$, $\Phi_1 = \Phi'$ in I^n, and $\forall\ t \in [0,1]$, $x \in \partial I^n$, we have $\Phi_t(x) = \Phi(x)$.

Definition 17.1 *Width of Π:*

$$L(\Pi) = \inf_{\Phi' \in \Pi} \sup_{x \in I^n} |\Phi'(x)|.$$

Theorem 17.4 *(Almgren-Pitts) If $L(\Pi) > \sup_{x \in \partial I^n} |\Phi(x)|$, then $\exists$ smooth minimal embedded $\tilde{\Sigma}$(possibly with multiplicity), such that*

$$|\tilde{\Sigma}| = L(\Pi).$$

Moreover, $\tilde{\Sigma}$ is the limit of some min-max sequences, i.e. $\exists\ x_j \in I^n$, such that $\Phi^j(x_j) \to \tilde{\Sigma}$.

Propostion 17.1 *$F : S^3 \to S^3$ is conformal and Σ minimal, then $|F(\Sigma)| \le |\Sigma|$.*

Proof: By conformal invariance of $W(\Sigma)$,

$$|F(\Sigma)| \le \int_{F(\Sigma)} (1+H^2)d\Sigma = W\big(F(\Sigma)\big) = W(\Sigma) = |\Sigma|.$$

□

<u>**Standard family of conformal transformations:**</u>

- conformal transformation $v \in B^4 \to 0$:

$$F_v(x) = \frac{1-|v|^2}{|x-v|^2}(x-v) - v,$$

- $|F_v(\Sigma)| < |\Sigma|$, if $v \neq 0$, and if Σ is not S^2.(?)

<u>**Index of minimal surfaces in S^3**</u>**:**

- $\Sigma^2 \subset S^3$, $H = 0$, stability operator $L\varphi = \triangle\varphi + (2+|A|^2)\varphi$, index form:

$$Q(\varphi,\varphi) = -\int_\Sigma \varphi L\varphi = \int |\nabla\varphi|^2 - (2+|A|^2)\varphi^2.$$

$$I(\Sigma) = \text{Index of } \Sigma = \#\text{ of negative eigenvalues of } L.$$

- $I(S^2) = 1$, $L\varphi = \triangle\varphi + 2\varphi$, $\lambda_0 = -2$, $\lambda_1 = 0$.
- $I(T_c^2) = 5$, $L\varphi = \triangle\varphi + 4\varphi$, $\lambda_0 = -4$ with multiplicity 1 and eigenfunction $u_0 = const$; $\lambda_1 = -2$, with multiplicity 4 and eigenfunctions: $\cos(\sqrt{2}\theta_1)$, $\sin(\sqrt{2}\theta_1)$, $\cos(\sqrt{2}\theta_2)$ and $\sin(\sqrt{2}\theta_2)$; $\lambda_2 = 0$.
- <u>$\Sigma^2 \subset S^3$</u>, $N = (N_1, N_2, N_3, N_4)$ is unit normal of Σ, then by the translation invariance of the cone $C(\Sigma)$,

$$\triangle_\Sigma N + |A|^2 N = 0.$$

  So

$$LN = \triangle N + (2+|A|^2)N = 2N.$$

- Each N_i is an eigenfunction with eigenvalue -2. Since N_i changes sign(since $N \cdot x = 0$ for all $x \in \Sigma$), hence not the first eigenvalue, so $\lambda_0 < -2$.
- Furthermore, $N_1, \cdots, N_4$ is linearly independent, unless Σ is S^2(or N must be constant since $N_1, \cdots, N_2$ already satisfy 4-relations), so

$$I(\Sigma) \ge 5,$$

  unless Σ is S^2.

Propostion 17.2 *(F. Urbano 1990)If Σ is not a Clifford torus, then $I(\Sigma) > 5$.*

Given Σ embedded in S^3, with N unit normal, and $d(x)$ =signed distance function to Σ, $-\pi \leq d(x) \leq \pi$. Let

$$\Sigma_t = \{x : \ d(x) = t\} = \partial\{d(x) < t\}, \quad t \in [-\pi, \pi].$$

Propostion 17.3

$$|\Sigma_t| \leq W(\Sigma).$$

Proof: The smooth map $\psi_t : \Sigma \to \Sigma_t$ is given by $\psi_t(y) = \cos(t)y + \sin(t)N(y)$, where $y \in \Sigma$, and $N(y)$ the unit normal. Hence if $\{e_1, e_2\}$ is the o.n. principal basis of $T_y\Sigma$,

$$D\psi_t|_y e_i = (\cos(t) + \sin(t)k_i)e_i.$$

So

$$Area(\Sigma_t) = \int_\Sigma det(D\psi_t)d\Sigma = \int_\Sigma (\cos t + k_1 \sin t)(\cos t + k_2 \sin t)d\Sigma,$$

while

$$\begin{aligned}(\cos t + k_1 \sin t)(\cos t + k_2 \sin t) &= \cos^2 t + \underbrace{(k_1 + k_2)}_{=2H} \sin t \cos t + \underbrace{k_1 k_2}_{\leq H^2} \sin^2 \\ &\leq 1 - \sin^2 t + 2H \cos t \sin t + H^2(1 - \cos^2 t) \\ &= 1 + H^2 - (\sin t + H \cos t)^2.\end{aligned}$$

□

<u>Canonical family:</u>

Given $\Sigma \subset S^3$ embedded, N =unit normal, define:

$$\Phi : B^4 \times [-\pi, \pi] \to \mathcal{Z}_2,$$

$$\Phi(v, t) = \big(F_v(\Sigma)_t\big) = \Sigma_{(v,t)}, \quad v \in B^4, \ t \in [-\pi, \pi],$$

where $\mathcal{Z}_2$ is the set of 2-currents.

Propostion 17.4

$$\sup_{(v,t) \in B^4 \times [-\pi, \pi]} |\Phi(v, t)| \leq W(\Sigma).$$

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Let $\Sigma \subset S^3$ be embedded and $g(\Sigma) \geq 1$.

Making $\Sigma_{(v,t)}$ continuous:

Denote S^3_+ and S^3_- be the connected components of $S^3 \setminus \Sigma$, with N pointing to S^3_+. Then

$$\lim_{v \to p} \Sigma_{(v,t)} = \partial B_t(-p), \quad \text{for } p \in S^3_+;$$

$$\lim_{v \to p} \Sigma_{(v,t)} = \partial B_{\pi+t}(p), \quad \text{for } p \in S^3_-.$$

When $p \in \Sigma$ and $v \to p$ with angle θ, i.e. informally $v - p$ forms an angle θ with the position vector p,

$$\lim_{v \to p (\text{with angle } \theta)} \Sigma_{(v,t)} = \partial B_{\frac{\pi}{2}-\theta+t}(-\sin(\theta)p - \cos(\theta)N(p)).$$

(It is easy to check that the above are consistent when $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$.)

Now introduce the polar coordinates near Σ. Given $(s, \theta) \in [0, \epsilon) \times [-\delta, \delta]$ and $p \in \Sigma$, then define

$$P(s, \theta, p) = (1 - s)(\cos(\theta)p + \sin(\theta)N(p)).$$

This map has a geometric explanation. Given $x \in B^4$ near Σ, let $s(x) = d(x, S^3)$ and θ the signed distance between $\frac{x}{|x|}$ and Σ, say $\theta = dist_\Sigma(x/|x|, p)$ for some $p \in \Sigma$, then $x = P(s, \theta, p)$.

Definition 18.1 *Define the ϵ neighborhood of Σ as:*

$$N_\epsilon(\Sigma) = \{P(s, \theta, p) : \ p \in \Sigma, \ s \geq 0, \ s^2 + \theta^2 < \epsilon^2\}.$$

It is not hard to find a continuous map

$$F : \overline{B^4} \to \overline{B^4},$$

such that $F : B^4 \setminus N_\epsilon(\Sigma) \to B^4$ is a diffeomorphism; $F : S^3 \to S^3$ is identity and $F : N_\epsilon(\Sigma) \to S^3$ is given by nearest point projection, i.e. given $x = P(s, \theta, p)$, then $F(s, \theta, p) = \cos(\theta)p + \sin(\theta)N(p)$.

Canonical family:

Firstly, define

$$C : \overline{B^4} \times [-\pi, \pi] : \begin{cases} C(v,t) = \Sigma_{(F(v),t)} & \text{when } v \in B^4 \setminus N_\epsilon(\Sigma); \\ C(v,t) = \partial B_t(-v) & \text{when } v \in S^3_+ \setminus N_\epsilon(\Sigma); \\ C(v,t) = \partial B_{\pi+t}(v) & \text{when } v \in S^3_- \setminus N_\epsilon(\Sigma); \\ C(v,t) = \partial B_{\frac{\pi}{2}-\theta+t}(-\sin(\theta)p - \cos(\theta)N(p)) & \text{when } v = F(s, \varphi, p) \in N_\epsilon(\Sigma), \end{cases}$$

where $\theta = \tan^{-1}(\frac{\varphi}{\sqrt{\epsilon^2 - \varphi^2}})$.

Given $x \in (\partial B^4 \cup N_\epsilon(\Sigma))$, $\exists\,!\ \bar{Q}(x)$ and $s(x)$, so that

$$C(x, s(x)) = \partial B_{\pi/2}(\bar{Q}(x)),$$

has radius $\frac{\pi}{2}$. Here $s : (\partial B^4 \cup N_\epsilon(\Sigma)) \to [-\pi/2, \pi/2]$ and $\bar{Q} : (\partial B^4 \cup N_\epsilon(\Sigma)) \to S^3$ is given in the definition of C.

Key Property: $\bar{Q} : S^3 \to S^3$ is continuous and has degree g(will be proved in the next section).

Definition 18.2 *The canonical family $\Phi : I^5 \to \mathcal{Z}_2$ associated to $\Sigma \subset S^3$ is defined as:*

$$\Phi(v,t) = C\big(f(v), (2t-1)\pi + s(f(v))\big), \quad (v,t) \in I^4 \times I = I^5,$$

where $f : I^4 \to B^4$ is a diffeomorphism and $s : \overline{B^4} \to [-\pi/2, \pi/2]$ is an extension of $s : (\partial B^4 \cup N_\epsilon(\Sigma)) \to [-\pi/2, \pi/2]$.

- Φ is continuous on I^5;
- $\Phi(p, 1/2)$ is an equator when $p \in \partial I^4$;
- $\Phi(v,0) = \Phi(v,1) = 0$ in $\mathcal{Z}_2$.

Definition 18.3 *Let*

$$\Pi = \textit{relative homotopy class of } \Phi(\textit{fixed on } \partial I^5).$$

Theorem 18.1 *If genus $g(\Sigma) \geq 1$, the width $L(\Pi) > 4\pi$.*

Proof: Since $\max_{x\in\partial I^5} |\Phi(x)| = 4\pi$, if the statement is false, then $\exists\ \varphi_i \in \Pi$, such that

$$\max_{x\in I^5} |\varphi_i(x)| \leq 4\pi + \frac{1}{i}.$$

Now $\Phi_{\partial I^5} = \Phi_{(I^4\times\{0\})\cup(I^4\times\{1\})\cup(\partial I^4\times I)}$, where

$$|\Phi| : \partial I^4 \times \{1/2\} \to \mathbb{RP}^3 = \text{space of unoriented equators in } S^3,$$

has degree $2g$(since θ is a genus g covering). Then there could not exist any continuous extension of $|\Phi|$ to any oriented submanifold $S \subset I^5$ with $\partial S = \partial I^4 \times \{1/2\}$.

- Given $\epsilon > 0$, let

  $$A(i) = \text{connected component of } \{t=0\} \subset I^5 \text{ in } \{x \in I^5 :\ d_V\big(|\varphi_i(x)|, \mathcal{Z}_0\big) > \epsilon\},$$

  where d_V is the varifold distance and $\mathcal{Z}_0$ is the space of unoriented equators.
- Claim: For i sufficient large, $A(i) \cap \{t=1\} \subset I^5 = \emptyset$.
- If not, $\exists$ continuous path $\gamma_i(t) \subset A(i)$ from $\{t=0\}$ to $\{t=1\}$. Let

  $$\Pi_1 = \text{homotopy class of } \gamma_i,$$

  which is homotopic to any vertical path on $\partial I^4 \times I$, hence homotopically nontrivial. Then

  $$\max_{t\in[0,1]} \big||\varphi_i|(\gamma_i(t))\big| \leq 4\pi + 1/i.$$

  If we run the min-max theory on Π, we must get a nontrivial embedded minimal surface Σ, since Π_1 is nontrivial, and Σ must have area less or equal to 4π, hence an equator. Furthermore, the min-max theory tells us that there exists a min-max sequence $\gamma_i(t_i)$, such that

  $$d_V\big(\gamma_i(t_i), \Sigma\big) \to 0,$$

  under the varifold distance d_V, hence a contradiction to the construction of $A(i)$.

So $A(i)$ gives an continuous extension of $\Phi|_{\partial I^4\times I}$, hence a contradiction.

□

Proof: (of Theorem 17.2) $\exists$ connected smooth oriented embedded Σ of least area.

- Construct $\Phi : I^5 \to \mathcal{Z}_2(S^3)$, hence $\max_{x\in I^5} |\Phi(x)| = W(\Sigma) = |\Sigma|$;
- If Σ is not the Clliford torus T_c, then Σ has index greater or equal to 6, $\Longleftrightarrow$ Φ can be homotopic to small max area;
- Let Π be the homotopic class of Φ,

$$L(\Pi) = \tilde{\Sigma} < |\Sigma|,$$

  where $\tilde{\Sigma}$ is a connect smooth oriented embedded minimal surface by the discussion in the next section, a contradiction to the minimality of Σ.

□

Proof: (of Theorem 17.3) Given Σ oriented embedded surface of genus $g \geq 1$ in S^3, we can similarly construct the canonical family $\Phi : I^5 \to \mathcal{Z}_2(S^3)$, and $\max_{x\in I^5} |\Phi(x)| \leq W(\Sigma)$. Run the min/max, we get a embedded minimal surface $\tilde{\Sigma}$, such that $|\tilde{\Sigma}| \leq \max_{x\in I^5} |\Phi(x)| \leq W(\Sigma)$.

□

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Let $\Phi : I^4 \times I \to \mathcal{Z}_2$ be the canonical family. For any $p \in \partial I^4$, $\Phi(p,t)$ are spheres, with $\Phi(p, 1/2)$ an equator. Denote $\bar{Q} : S^3 \to S^3$ the map from $S^3 \sim \partial I^4$ to the center of $\Phi(p, 1/2)$ given above.

Theorem 19.1

$$deg(\bar{Q}) = g.$$

Proof: Since $S^3 \setminus \Sigma = S^3_+ \cup S^3_-$, we can introduce

$$\tilde{S^3} = S^3_+ \cup \Sigma \times [-\frac{\pi}{2}, \frac{\pi}{2}] \cup S^3_-.$$

The map $\bar{Q}$ can be views as $\bar{Q} : \tilde{S^3} \to S^3$, given by

$$\bar{Q}(p) = \begin{cases} -p, \quad \text{when } p \in S^3_+; \\ -\sin(\theta)p - \cos(\theta)N(p), \quad \text{when } (p,\theta) \in \Sigma \times [-\pi/2, \pi/2]; \\ p, \quad \text{when } p \in S^3_-. \end{cases}$$

So

$$deg(\bar{Q}) = \frac{1}{2\pi^2} \int_{\tilde{S^3}} (det(d\bar{Q}))d\mu.$$

<u>Orientation on S^3</u>: a basis $v_1, v_2, v_3 \in T_pS^3$ is positively oriented $\Longleftrightarrow$ $\{v_1, v_2, v_3, p\}$ is positively oriented in $\mathbb{R}^4$.

- Map $p \to -p$ is positively oriented on S^3, since if $\{e_1, e_2, e_3\}$ positive on T_pS^3, then $\{-e_1, -e_2, -e_3, -p\}$ positively on $\mathbb{R}^4$.
- So
$$\int_{\tilde{S^3}} det(d\bar{Q})d\mu = \underbrace{|S^3_+| + |S^3_-|}_{2\pi^2} + \int_{\Sigma\times[-\pi/2,\pi/2]} det(d\bar{Q})d\mu.$$
- Given $p \in \Sigma$, let e_1, e_2 be an positively oriented o.n. principal basis, i.e. $D_{e_i}N(p) = k_ie_i$ where k_1, k_2 are principal curvatures of Σ. Then $\{e_1, e_2, N(p), p\}$ forms a positively oriented basis of $\mathbb{R}^4$.

  $\{e_1, e_2, \frac{\partial}{\partial\theta}\}$ forms a positive basis for $\Sigma \times [-\pi/2, \pi/2]$.
- Since $Q(p, \theta) = -\sin(\theta)p - \cos(\theta)N(p)$,
$$d\bar{Q}(e_i) = -\sin(\theta)e_i - \cos(\theta)k_ie_i = (-\sin(\theta) - \cos(\theta)k_i)e_i;$$
$$d\bar{Q}(\frac{\partial}{\partial\theta}) = -\cos(\theta)p + \sin(\theta)N(p).$$
- Note $\{e_1, e_2, \underbrace{-\cos(\theta)p + \sin(\theta)N(p)}_{e_3}\}$ forms an o.n. basis at $T_{\bar{Q}(p,\theta)}S^3$. Furthermore, $\{e_1, e_2, -\cos(\theta)p+ \sin(\theta)N(p)\}$ is negatively oriented. This is because $\{e_1, e_2, -\cos(\theta)p+\sin(\theta)N(p), -\sin(\theta)p- \cos(\theta)N(p)\}$ is negatively oriented in $\mathbb{R}^4$(by taking the wedge product, we get $-e_1\wedge e_2\wedge N(p)\wedge p$).
- So
$$d\bar{Q}(e_1 \wedge e_2 \wedge \frac{\partial}{\partial\theta}) = -deg(d\bar{Q})e_1 \wedge e_2 \wedge e_3,$$
  and we derive
$$deg(d\bar{Q}) = -(\sin\theta + k_1\cos\theta)(\sin\theta + k_2\cos\theta) = -(\sin^2\theta + \sin\theta\cos\theta(k_1 + k_2) + \cos^2\theta).$$
  Hence
$$\int_{\Sigma\times[-\pi/2,\pi/2]} deg(d\bar{Q}) = -\int_\Sigma (\frac{\pi}{2} + \frac{\pi}{2}k_1k_2)d\mu_\Sigma = -\frac{\pi}{2}\int_\Sigma (\underbrace{1 + k_1k_2}_{K_\Sigma})d\mu_\Sigma$$
$$= -\frac{\pi}{2}2\pi(2 - 2g) = 2\pi^2(g - 1).$$

Adding all the above together, we finish the proof.

□

Doing the Min-max: Let $\mathcal{Z}_k(M^n)$ be the space of integral currents with **flat topology**, which roughly means that $\Sigma_i \to \Sigma$ if $\Sigma_i - \Sigma = \partial R_i$ and $|R_i| \to 0$.

Theorem 19.2 *(Almgren)*
$$\pi_{n-k}(\mathcal{Z}_k(M^n)) = H_n(M, \mathbb{Z}) = \mathbb{Z}.$$

- $\Phi : S^{n-k} \to \mathcal{Z}_k(M^n)$ continuous in the homotopical notion;
- $\Pi =$ homotopy class of Φ;
- $L(\Pi) =$ min/max of Π;
- (Almgren) $L(\Pi)$ is achieved by stationary varifold;
- $k = n-1$, $n = 3$(Pitts) or $n < 7$(Schoen-Simon), the stationary varifold can be achieved by smooth embedded hypersurface.

M-N paper: $\Phi : I^5 \to \mathcal{Z}_2(S^3)$ is continuous in flat topology.

- $t \to \Sigma_t$ is continuous in flat norm. If $t_1 < t_2$, then $\Sigma_{t_2} - \Sigma_{t_1} = \partial\{t_1 < d(x,\Sigma) < t_2\}$ has small volume when $t_2 - t_1$ is small;
- $v \to \Sigma_{v,t}$ is continuous even when $v \to S^3$;
- $2\pi^2 \geq L(\Pi) > 4\pi$ if Σ has area less or equal to $2\pi^2 \Longleftrightarrow$ the min/max is achieved by a smooth embedded Σ, then Σ must have multiplicity 1 and connected $\Longleftrightarrow \Sigma$ is orientable.

Urbano's theorem: (Proof of Proposition 17.2)

- $\Sigma^2 \subset S^3$ embedded, and $H = 0$. $I(\Sigma) \leq 5 \Longleftrightarrow \Sigma = S^2, T_c^2$.

  If $I(\Sigma) > 5$, then $\lambda_0 < -2$, and $\lambda_1 = -2$ has a 4 dimensional eigenspace generated by the normal vector $N = (N_1, N_2, N_3, N_4)$.
- If $I(\Sigma) \leq 5$, then $I(\Sigma) < 5$ only if Σ is an equator by discussion in Section 17. Let the eigenfunction of λ_0 be u_0. If $I(\Sigma) = 5$, then every function φ such that $\int_\Sigma \varphi u_0 = 0$ must satisfy:
  $$Q(\varphi,\varphi) = \int (|\nabla\varphi|^2 - (2+|A|^2)\varphi^2) \geq -2\int \varphi^2.$$
  Let $\psi : \Sigma \to S^3$, then $\exists v \in B^4$, with F_v defined in Section 17, such that $\int_\Sigma F_v \circ \psi = 0$. Denote $F_v \circ \psi = (\tilde{\psi}_1, \cdots, \tilde{\psi}_4)$, then
  $$\sum_{i=1}^4 \int_\Sigma [|\nabla\tilde{\psi}_i|^2 - (2+|A|^2)\tilde{\psi}_i^2] \geq -2\sum_{i=1}^4 \tilde{\psi}_i^2.$$
  Hence
  $$2|\Sigma| = 2E(\psi) \underbrace{\geq}_{\text{“}= \, only \; if \; F=id} E(F_v\circ\psi) = \int_\Sigma |\nabla F_v \circ \psi|^2 \geq \int_\Sigma \underbrace{|A|^2}_{2(1-K)} = 2|\Sigma| - \int_\Sigma K da.$$
  So $\int_\Sigma K da \geq 0$, hence Σ is either S^2 or a torus. When Σ is a torus, $F_v = id$, hence $\psi = (x_1, \cdots, x_4)$, which are eigenfunctions of $\triangle x_i = -2x_i$, so
  $$\triangle x = -2x = -|A|^2 x.$$
  Hence $|A|^2 = 2$, so $K = 0$.

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