Course Plan:

Chap 1: Introduction (1.1, 1.3)
Chap 2: First order Differential Equations (2.1 - 2.6)
Chap 3: Second order Linear Differential Equations (3.1 - 3.6)
Chap 4: Higher order Linear Differential Equations (4.1 - 4.3)
Chap 7: Systems of First order Linear Differential Equations (7.1, 7.5 - 7.9)

Chap 1: Introduction

If we try to describe many phenomena in our world, we usually get equations involving derivatives, as many quantities evolve over time, and the derivatives are just rate of change.

Def: Equations containing derivatives of functions are differential equations.

A differential equation that describes some physical process is often called a mathematical model of the process.

Example: (Newton's law) \( F = ma \).

\( x \) = position of the object

\( v \) = velocity \( \frac{dx}{dt} \), the rate of change of \( x \)

\( a \) = acceleration \( \frac{dv}{dt} \), rate of change of \( v \)
Newton's Law can be described by a differential equation.

Examples:
1. \[ \frac{dy}{dt} = t^2 + 1 \]
2. \[ \frac{d^2y}{dx^2} - \frac{dy}{dx} + 1 = 0 \]
3. \[ \frac{dy}{dx} = \ln y \]
4. \[ \frac{\partial^2f(x,y)}{\partial x^2} + \frac{\partial^2f(x,y)}{\partial y^2} = 0 \] \( \rightarrow \text{PDE} \)
5. \[ y'(+t) = \cos(y(+t)) \]
6. \[ y'(+t) = \frac{1}{y(+t)} \]
7. \[ \begin{align*} x \frac{dx}{dt} &= x - xy \\ y \frac{dy}{dt} &= y + xy \end{align*} \]

Def.: An ordinary differential equation is an equation involving an unknown function of a single independent variable and its derivatives up to some fixed order.

Def.: A partial differential equation has several independent variables and involves partial derivatives.

Rmk.: We will only study ODE in this class.

Def.: The order of an ODE is the order of the highest derivative that appears in the equation.

Examples:
1. \[ y''(t) = -4y(t) \] 2nd order
2. \[ \frac{dy}{dt} = y + t \] 1st order
3. \[ y \cdot y''(t) + y'(t) + t = 0 \] 3rd order
In general, the equation $F(t, y, y', \ldots, y^{(n)}) = 0$ is an ODE of the $n$-th order.

We always assume that in our class, every ODE can be written as $y^{(n)}(t) = f(t, y, y', y'', \ldots, y^{(n-1)}).$

Sometimes, if there are two or more unknown functions, we need a system of differential equations.

**Def.** A collection of several ODEs involving more than one unknown functions is called a system of ODEs.

$$\begin{align*}
\frac{dx}{dt} &= x(t) - x(t)y(t) \\
\frac{dy}{dt} &= -y(t) + x(t)y(t).
\end{align*}$$

**Def.** (Linear / nonlinear ODE) The ODE $F(t, y, y', \ldots, y^{(n)}) = 0$ is said to be linear if $F$ is a linear function of the variables $y, y', \ldots, y^{(n)}$. Thus the general linear ODE of order $n$ is:

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t). \quad (\star)$$

An ODE that is not of the above form is nonlinear.

**Examples.**

1. $y'' + ty' + y = 0 \quad Y.$
2. $y' = y + t \quad Y.$
3. $y'' + t^2 = 0 \quad N.$
4. $\frac{d^2y}{dt^2} + \sin(y) = 0 \quad N.$
Def. A linear ODE \((x)\) is called \underline{homogenous} if \(g(t)=0\).

Def. (Solutions) A solution of the \(n\)-th order ODE on the interval \(a<t<b\) is a function \(\phi\), such that

\[
\phi', \phi'', \ldots, \phi^{(n)} \text{ exist and satisfy:}
\]

\[
\phi^{(n)}(t) = f(t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t))
\]

Example: \(\frac{dy}{dx} = x\). \(y = y(x)\)

\(y\) is a function whose derivative is equal to \(x\).

The unknown function is \(y\), and \(y\) is a function of the independent variable \(x\).

We can also write \(y'(t) = t\). (by change \(x\) to \(t\)).

Solutions of this ODE is \(y(x) = \frac{1}{2}x^2 + C\) \(\text{undetermined constant}\).

Sometimes, we will have additional information for the ODE, called \underline{initial conditions} which allow us to fix the arbitrary constant in our solutions.

Example: \(\frac{dy}{dx} = x\), \(y(1) = 2\). \(\Rightarrow\) \(\frac{1}{2} (1)^2 + C = 2 \Rightarrow C = \frac{3}{2}\)

\(\Rightarrow\) \(y(x) = \frac{1}{2} x^2 + \frac{3}{2}\).
**Ex:** Suppose $f(x)$ is any continuous function. Consider the ODE:

$$\frac{dy}{dx} = f(x) \implies y(x) = \int f(x) \, dx + C.$$

The **Fundamental Theorem of Calculus** tells us that every continuous function $f(x)$ has an antiderivative $\int f(x) \, dx$.

**Ex:** $\frac{dy}{dx} = y$. Find a function whose derivative is equal to itself.

An: $y(x) = C \cdot e^x$. (Note: $e^x + C$ is not a solution).

**Ex:** 2nd order ODE: $y''(x) = x$

$$\implies y'(x) = \int x \, dx + C_1 = \frac{1}{2} x^2 + C_1,$$

$$\implies y(x) = \int \left( \frac{1}{2} x^2 + C_1 \right) \, dx + C_2 = \frac{1}{6} x^3 + C_1 x + C_2$$

2 undetermined constants.

**Ex:** The most general first order ODE has the form:

$$\frac{dy}{dx} = F(x, y),$$

where $F(x, y)$ is any “reasonably nice” function of both the independent variable $x$ and the unknown function $y$.

**Remk:** 1. Not every ODE has a solution. (Existence)

   2. If an ODE has a solution, we may need to consider if it has other solutions. (Uniqueness)
§. Direction Fields

**Def.** (Direction Fields). We know that the slope of the tangent line to the graph of a function \( y = y(x) \) at a point \((x_0, y(x_0))\) is equal to \( y'(x_0) \).

\[ \Rightarrow \text{The slope of a solution to the ODE } y' = f(x, y) \text{ which passes through a point } (x_0, y_0) \text{ is equal to } f(x_0, y_0). \]

Hence, the function \( f(x, y) \) defines a slope field or direction field which can be used to sketch solution curves.

\[ \frac{dy}{dx} = -y + 1. \]

**Ex.** \( f(x, y) = -y + 1 \)

Solu.: \( y = 1 + ce^{-t} \).

• if \( c > 0 \), \( y \) decreases to 1
• if \( c < 0 \), \( y \) increases to 1
• if \( c = 0 \), \( y = 1 \).

The solution "\( y = 1 \)" is special, and it is called an "equilibrium solution".

**Def.** Equilibrium solutions for the ODE \( y' = f(x, y) \) are solutions \( y(x) \) such that \( y'(x) = 0 \), i.e., \( y(x) \) is a constant.
They can be easily found by $F(x, y) = 0$.

1. $y' = (y+1)(y-2)$. Let $(y+1)(y-2) = 0 \Rightarrow y = -1 \lor y = 2$.
2. $y' = y(y-x)$. Let $y(y-x) = 0 \Rightarrow y = 0$.
3. $y' = \ln(y)$. Let $\ln(y) = 0 \Rightarrow y = 1$.

Let us draw the direction fields for $y' = 0$.

- $y = -1 \lor y = 2 \Rightarrow y' = 0$
- $y > 2 \Rightarrow$
  \[ y' = (y+1)(y-2) > 0 \]
- $-1 < y < 2 \Rightarrow$
  \[ y' = (y+1)(y-2) < 0 \]
- $y < -1 \Rightarrow$
  \[ y' = (y+1)(y-2) > 0 \]

Remark: 0. To plot the direction fields, we do not have to solve the ODE.

1. It gives a good picture of the overall behavior of solutions of an ODE.