Introduction to Mathematical General Relativity—lectures given by Rick Schoen

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Abstract

This series of lecture notes were taken for the topic class on mathematical General Relativity given by Professor Rick Schoen in the spring quarter of 2012 at Tsinghua University. These lectures start from basic introduction of General Relativity, and then move on to several fundamental mathematical subjects in this field. Particularly, the notes cover the conformal method of solving the vacuum constraint equations, Positive Mass Theorems and the Penrose Inequality. The materials are very good examples for the application of methods from partial differential equations and calculus of variation.

It is likely that we have numerous typos and mistakes here and there, and would appreciate it if these are brought to our attention.

Contents

1 Introduction

Reference:

- R. Wald. General Relativity.
- S. Hawking and G. Ellis, The Large Scale Structure of Space-Time.

1.1 Mathematical model

$(\mathcal{S}^{n+1}, g,$ other fields$)$ is used to model the space-time, where $\mathcal{S}^{n+1}$ is an $n+1$ dimensional smooth oriented manifold, g a Lorentz signature $(-1, \underbrace{1, \cdots, 1}_{n})$ metric, and other fields are models by tensor fields.

Flat model: $\mathbb{R}^{n+1}$:

- $x_0, x_1, \cdots, x_n$ are coordinates;
- $g = -dx_0^2 + \sum_{i=1}^n dx_i^2$ is the Lorentzian metric;
- $\langle v, w\rangle = -v_0 w_0 + \sum_{i=1}^n v_i w_i$.
- $$\text{3 types of vectors: } \begin{cases} \text{space-like:} \quad \langle v, v\rangle > 0; \\ \text{time-like:} \quad \langle v, v\rangle < 0; \\ \text{null:} \quad \langle v, v\rangle = 0. \end{cases}$$
- Time-like curves are used to model the World line;

Let $H^n \subset \mathbb{R}^{n+1}$ be a plane, then there exists a unique $v \neq 0$ up to scale, such that $H = \{w : \langle v, w\rangle = 0\}$.

- H is space-like if v is time-like, then $g|_H$ has positive signature;
- H is time-like if v is space-like, then $g|_H$ has Lorentz signature;
- H is null if $v \in H$ and $\langle v, v\rangle = 0$, then $g|_H$ is degenerate.

Let $M^n \subset \mathbb{R}^{n+1}$ be a hyper-surface, then it is space-like if T_pM is space like for all $p \in M$.

Let D be the Levi-Civita connection on $(\mathcal{S}^{n+1}, g)$, and $\{x^a : a = 0, \cdots, n\}$ local coordinates, with $\partial_a = \frac{\partial}{\partial x^a}$. Then $D_{\partial_a}\partial_b = \sum_{c=0}^n \Gamma_{ab}^c \partial_c$, where the Christoffel symbol $\Gamma_{ab}^c = \frac{1}{2} g^{cd}(g_{ad,b} + g_{db,a} - g_{ab,d})$. Given $X, Y, W, Z \in T_p\mathcal{S}$, the Riemannian curvature tensor $R(X, Y, Z, W)$ is defined in local coordinates by:

$$R(\partial_a, \partial_b, \partial_c, \partial_d) = \sum_e g_{ae} R_{bcd}^e,$$

where $R_{bcd}^e = \Gamma_{db,c}^a - \Gamma_{cb,d}^a + \Gamma \cdot \Gamma - \Gamma \cdot \Gamma$. The Ricci curvature is $R_{ab} = \sum_{c,d} g^{cd} R_{acbd}$, and Scalar curvature is $R = \sum_{a,b} g^{ab} R_{ab}$.

1.2 Einstein equation

The **Einstein equation** is given by

$$R_{ab} - \frac{1}{2} R g_{ab} = T_{ab}, \tag{1.1}$$

where T_{ab} is the stress-energy tensor of matter fields. The questions related to matter fields are:

- What does T mean?
- How to compute T?

Given v a time-like unit vector $\langle v, v\rangle = -1$, then

- $T(v,v)$ is the observed energy density of observer;
- $T(v,\cdot)^\sharp$ is energy-momentum density vector.

Standing Assumption on T—Dominant Energy Condition(DEC):

$\forall v$ unit time-like vector $\Longrightarrow T(v,\cdot)^\sharp$ is forward pointing time-like or null. Given $v = e_0, e_1, \cdots, e_n$ an o.n. basis, and let $T_{ab} = T(e_a, e_b)$, then (DEC) requires $T_{00} \geq \sqrt{\sum_{i=1}^n T_{0i}^2}$.

Special Case: $T = 0$, in dimension $n \geq 2$, then (1.1) is reduced to

$$\text{(VEE)} \quad R_{ab} = 0. \tag{1.2}$$

Proof: $g^{ab}(R_{ab} - \frac{1}{2}Rg_{ab}) = R - \frac{n+1}{2}R = 0 \Longrightarrow R = 0$.

Lagrangian formulation: The associate Lagrangian is:

$$\mathcal{L}_{HE}(g) = \int_{\mathcal{S}} R dv = \int_{\mathcal{S}} g^{ab} R_{ab} \sqrt{-\det(g)} dx. \tag{1.3}$$

Claim: consider a variation $g + th$ where h is compactly supported, then

$$\frac{d}{dt}|_{t=0} \mathcal{L}_{HE}(g + th) = \int_{\mathcal{S}} (-R_{ab} + \frac{1}{2} R g_{ab}) h^{ab} dv.$$

Proof: By [2], the first variation of scalar curvature is $\delta_g R(h) = \delta\delta h - Ric \cdot h$, so the first variation of $\mathcal{L}_{HE}$ is

$$\begin{aligned}
\delta_g \mathcal{L}_{HE}(h) &= \int_{\mathcal{S}} (\delta_g R(h)) \sqrt{-\det(g)} dx + R(\delta_g \sqrt{-\det(g)}(h)) dx \\
&= \int_{\mathcal{S}} [(\delta\delta h - Ric \cdot h) \sqrt{-\det(g)} + R \frac{1}{2} tr_g h \sqrt{-\det(g)}] dx \\
&= \int_{\mathcal{S}} [-R_{ab} + \frac{1}{2} R g_{ab}] h^{ab} dv,
\end{aligned}$$

where in the third " $=$ " we used the divergence theorem.

Gravity coupled with scalar fields: Let u be a function on $\mathcal{S}$, the Klein-Gordon action is

$$\mathcal{L}_{KG}(u, g) = \int_{\mathcal{S}} (\langle \nabla u, \nabla u + m^2 u^2 \rangle) dv. \tag{1.4}$$

The first variation equation is:

$$\frac{d}{dt}|_{t=0} \mathcal{L}_{KG}(u + t\eta, g) = \int_{\mathcal{S}} (\langle \nabla \eta, \nabla u \rangle + m^2 u^2) dv = 0, \quad \forall \eta \in C_c^\infty(\mathcal{S}).$$

Hence the Euler-Lagrange equation for u is:

$$-\Box u + m^2 u^2 = 0.$$

Now the total action for (u, g) is

$$\mathcal{L}_{total}(u,g) = \mathcal{L}_{HE}(g) + c\mathcal{L}_{KG}(u,g).$$

And the first variation is given by:

$$\delta_g \mathcal{L}_{total}(u,g) = -\int_{\mathcal{S}} (R_{ab} - \frac{1}{2} R g_{ab}) h^{ab} dv + c \int_{\mathcal{S}} T_{ab} h^{ab} dv,$$

where the Energy-Momentum tensor corresponding to u is given by

$$\int_{\mathcal{S}} T_{ab} h^{ab} dv = \delta_g \mathcal{L}_{kG}(u,g) = \delta_g \int_{\mathcal{S}} (g^{ab} u_a u_b + m^2 u^2) \sqrt{-det(g)} dx$$

$$= \int_{\mathcal{S}} [-h^{ab} u_a u_b + (|\nabla u|^2 + m^2 u^2) \frac{1}{2} tr_g h] \sqrt{-det(g)} dx.$$

So the total Euler-Lagrangian equation for g is

$$R_{ab} - \frac{1}{2} R g_{ab} = T_{ab} = -u_a u_b + \frac{1}{2}(|\nabla u|^2 + m^2 u^2) g_{ab}.$$

1.3 Initial value problem

The **initial data** is modeled by a triple (M^n, g, h), where g is a Riemmanian metric, and h a symmetric $(0, 2)$ tensor.

Problem: Given initial data, find a local evolution $(\mathcal{S}^{n+1}, g^{\mathcal{S}})$ of VEE(1.2), with

$$M \subset \mathcal{S}, \quad g = g^{\mathcal{S}}|_M, \quad h = 2nd\ f.f.$$

- The problem is solvable only if (g, h) satisfies $(n + 1)$ constraint equations;
- The constraint equations are given by $R_{ab} - \frac{1}{2} R g_{ab} = T_{ab}$ when $a = 0$.

The **constraint equations** are given by:

$$(CE) \quad \begin{cases} T_{00} = \mu = \frac{1}{2}(R + (tr_g h)^2 - |h|^2), \\ T_{0i} = J = div_g(h - (tr_g h) g). \end{cases} \tag{1.5}$$

The **vacuum constraint equations** are given by:

$$(VCE) \quad \begin{cases} 0 = \frac{1}{2}(R + (tr_g h)^2 - |h|^2), \\ 0 = div_g(h - (tr_g h) g). \end{cases} \tag{1.6}$$

2 Constraint equations

2.1 Derivation of (CE)

Given $(\mathcal{S}^{n+1}, g^{\mathcal{S}})$ the space-time, let (M^n, g, h) be the initial data set, such that (g, k) are the restriction and 2nd f.f. of $M \subset (\mathcal{S}^{n+1}, g^{\mathcal{S}})$. Take $\{e_0, e_1, \cdots, e_n\}$ an o.n. frame of $\mathcal{S}$ at $p \in M$, with $e_0 \perp M$ and $e_i \in T_pM$. Consider the Einstein equation

$$R_{ab} - \frac{1}{2} R g^{\mathcal{S}} = T_{ab}, \quad 0 \leq a, b \leq n.$$

- $a = 0, b = 0 \Longrightarrow$ (1) $\mu = T_{00} = \frac{1}{2}(R_g + (tr_g h)^2 - \|h\|^2)$;
- $a = 0, b = 1, \cdots, n \Longrightarrow$ (2) $J = T_{0i} = div_g(h - (tr_g h)g)$.

(1) Gauss Equation: $M \subset \mathcal{S}$, $X, Y, Z, W \in TM$,

$$R^M(X, Y, Z, W) = R^{\mathcal{S}}(X, Y, Z, W) + \langle \Pi(X, Z), \Pi(Y, W) \rangle - \langle \Pi(X, W), \Pi(Y, Z) \rangle,$$

where $\Pi(X, Y) = (D_X Y)^{\perp} = h(X, Y)e_0$. Plug in e_i, e_j, e_k, e_l,

$$R^M_{ijkl} = R^{\mathcal{S}}_{ijkl} - h_{ik}h_{jl} + h_{il}h_{jk}.$$

Summing over i, k and j, l respectively,

$$R^M = \sum_{i,j=1}^{n} R^M_{ijij} = \sum_{i,j=1}^{n} R^{\mathcal{S}}_{ijij} - (tr_g h)^2 + \|h\|^2.$$

Now

$$\sum_{i,j=1}^{n} R^{\mathcal{S}}_{ijij} = \underbrace{\sum_{i,j=1}^{n} R^{\mathcal{S}}_{ijij} - 2\sum_{i=1}^{n} R^{\mathcal{S}}_{0i0i}}_{=R^{\mathcal{S}}} + 2\underbrace{\sum_{i=1}^{n} R^{\mathcal{S}}_{0i0i}}_{=2R^{\mathcal{S}}_{00}} = 2(R^{\mathcal{S}}_{00} - \frac{1}{2}R^{\mathcal{S}} g^{\mathcal{S}}_{00}) = 2T_{00} = 2\mu.$$

Plug in back, we can get the first constraint equation.
(2) Codazzi Equation: $M \subset \mathcal{S}$, $X, Y, Z \in TM$,

$$R^{\mathcal{S}}(e_0, X, Y, Z) = (D_Y \Pi)(X, Z, e_0) - (D_Z \Pi)(X, Y, e_0),$$

where $\Pi(X, Y, \nu) = \langle D_X Y, \nu \rangle$, hence $\Pi(X, Y, e_0) = -h(X, Y)$. Plug in $X = e_i$, $Z = e_i$ and sum over i,

$$J(Y) = R^{\mathcal{S}}(e_0, Y) = \sum_{i=1}^{n} R^{\mathcal{S}}(e_0, e_i, Y, e_i) = \sum_{i=1}^{n} \underbrace{D_{e_i} h(e_i, Y)}_{div_g h(Y)} - D_Y(tr_g h)$$

$$= div_g(h - (tr_g h)g).$$

The (VCE) system for (M, g, h) when $n = 3$:

$$\begin{cases} \frac{1}{2}(R + (tr_g h)^2 - |h|^2) = 0, \\ div_g(h - (tr_g h)g) = 0, \end{cases}$$

has 4 equations and 12 unknowns, so it is underdetermined system. We are mainly interested in the following two cases: (1) M compact; (2) M asymptotically flat.

2.2 Conformal method

Given (M^3, g, σ, τ), where g is a given conformal class of metrics; σ is a trace-free and divergence free symmetric $(0, 2)$ tensor, i.e. $div\sigma = 0$, $tr_g\sigma = 0$; and τ is the prescribed mean curvature. Consider the conformal transformations:

$$\begin{cases} \bar{g} = \varphi^4 g, \quad \varphi > 0, \\ \bar{h} = \varphi^{-2}(\sigma + LW) + \frac{\tau}{3}\bar{g}, \end{cases} \tag{2.1}$$

where φ is a function and W a v.f.(vector field). L is defined as follows,

Definition 2.1. L is the conformal Killing operator,

$$LW = \mathcal{L}_W g - \frac{1}{3}Tr(\mathcal{L}_W g)g = D_i W_j + D_j W_i - \frac{2}{3}div(W)g. \tag{2.2}$$

2 Conformal formulae:

$$(1) \qquad R(\bar{g}) = -8\varphi^{-5}(\triangle_g \varphi - \frac{1}{8}R_g\varphi);$$

$$(2) \quad L_{\bar{g}}W = \varphi^4 L_g W,\ div_{\bar{g}}(\varphi^{-2}k) = \varphi^{-6}div_g(k),$$

where k is a trace-free symmetric $(0, 2)$ tensor.
Check (2):

$$L_{\bar{g}}W = \mathcal{L}_W\bar{g} + \text{trace term} = \varphi^4\mathcal{L}_W g + \text{trace term}.$$

So by taking trace-free parts $\Longrightarrow L_{\bar{g}}W = \varphi^4 L_g W$.

To show the second part of (2), we can use duality properties. Given k a trace-free symmetric $(0, 2)$ tensor, and W a vector field, then by integration by part

$$\int_M (div_g k)(W)d\mu_g = -\int_M k_{ij}D^j W^i d\mu = -\frac{1}{2}\int_M \langle k, L_g W\rangle d\mu_g.$$

Hence we have shown that,

Lemma 2.2. *The operators L_g and $-2div_g$ are conjugate w.r.t. $L^2(M, g)$.*

So

$$\int_M div_{\bar{g}}(\varphi^{-2}k)(W)d\mu_{\bar{g}} = -\frac{1}{2}\int_M \langle \varphi^{-2}k, L_{\bar{g}}W\rangle_{\bar{g}} d\mu_{\bar{g}} = -\frac{1}{2}\int_M \varphi^{-2-8+4+6}\langle k, L_g W\rangle_g d\mu_g$$

$$= -\frac{1}{2}\int_M \langle k, L_g W\rangle_g d\mu_g = \int_M div_g(k)(W)d\mu_g.$$

So $div_{\bar{g}}(\varphi^{-2}k)(W) = \varphi^{-6}div_g(k)(W)$.

We want $(\bar{g}, \bar{h})$ in (2.1) to satisfy the (VCE). By plugging $(\bar{g}, \bar{h})$ into (1.6), we get

$$\text{(Conformal VCE)} \begin{cases} \triangle\varphi - \frac{1}{8}R_g\varphi = \frac{1}{12}\tau^2\varphi^5 - \frac{1}{8}|\sigma + LW|^2\varphi^{-7}, \\ div_g LW = \frac{2}{3}\varphi^6 d\tau. \end{cases} \tag{2.3}$$

- By plugging in the conformal formulae to the first one in (1.6), we can get

$$-8\varphi^{-5}(\triangle_g\varphi - \frac{1}{8}R_g\varphi) + \tau^2 - \|\varphi^{-2}(\sigma + LW)\|_{\bar{g}}^2 - \frac{\tau^2}{3} = 0,$$

hence the first formula above.

- By plugging in the second one in (1.6), we can get

$$div_{\bar{g}}(\varphi^{-2}(\sigma + LW) + \frac{\tau}{3}\bar{g} - \tau\bar{g}) = \varphi^{-6}div_g(LW) - \frac{2}{3}d\tau = 0.$$

$(*)$ Find solution of (VCE) reduces to finding solution of (2.3).

2.3 CMC case

By CMC (constant mean curvature), we means

$$\tau = \text{const}, \quad \text{hence } W = 0,$$

which is a solution to the second one in (2.3). Then (2.3) is reduced to the Lichnerowicz equation:

$$\text{(CMC VCE):} \quad \triangle\varphi - \frac{1}{8}R_g\varphi = \frac{1}{12}\tau^2\varphi^5 - \frac{1}{8}|\sigma|^2\varphi^{-7}. \tag{2.4}$$

Definition 2.3. M^3 is compact, and g is a metric. We say g is

- $Y_g > 0$(Yamabe positive) $\Longleftrightarrow \exists\bar{g} \in [g]$, $R_{\bar{g}} > 0 \Longleftrightarrow L_1 = \triangle - \frac{1}{8}R_g$ has $\lambda_1 > 0$;
- $Y_g = 0$(Yamabe zero) $\Longleftrightarrow \exists\bar{g} \in [g]$, $R_{\bar{g}} = 0 \Longleftrightarrow L_1 = \triangle - \frac{1}{8}R_g$ has $\lambda_1 = 0$;
- $Y_g < 0$(Yamabe negative) $\Longleftrightarrow \exists\bar{g} \in [g]$, $R_{\bar{g}} < 0 \Longleftrightarrow L_1 = \triangle - \frac{1}{8}R_g$ has $\lambda_1 < 0$.

Here λ_1 is the first eigenvalue of the conformal laplacian operator L_1.

In the following table, we list all the existence results in the CMC case:

	$\sigma = 0, \tau = 0$	$\sigma = 0, \tau \neq 0$	$\sigma \neq 0, \tau = 0$	$\sigma \neq 0, \tau \neq 0$
$Y > 0$	No	No	Yes	Yes
$Y = 0$	Yes	No	No	Yes
$Y < 0$	No	Yes	No	Yes

Regularity issue: we assume $g \in W^{2,p}(M)$, $h \in W^{1,p}(M)$ for $p > 3$. The first one in VCE(1.6) $\Longrightarrow$ $R_g \in W^{1,p}(M)$.
Sub-super-solutions: On (M, g) compact, consider the equation

$$(*) \quad T(u) = \triangle_g u + f\big(x, u(x)\big) = 0.$$

Want: solve with $u \in W^{3,p}(M)$. Assume

$$\exists\, u_+ \in W^{3,p}(M), u_+ > 0, \quad T(u_+) \leq 0;$$

$$\exists\, u_- \in W^{3,p}(M), 0 \leq u_- \leq u_+, \quad T(u_-) \geq 0.$$

Denote $m_- = \inf_M u_1$, $m_+ = \sup_M u_+$. Assume further that

$$f : M \times [m_-, m_+] \to \mathbb{R}^1,\ f, \frac{\partial f}{\partial u} \in C^0(M \times [m_-, m_+]),\ f(x, u(x)) \in W^{1,p}(M), \text{if } u \in W^{1,p}(M).$$

The u_+ and u_- are called **sub-solution** and **super-solution** respectively.

Theorem 2.4. Under the above assumption, $\exists\, u \in W^{3,p}(M)$ with $T(u) = 0$, and $u_- \leq u \leq u_+$.

The proof will be given in §2.6.

2.4 Find sub-super-solutions for (2.4) when $\sigma \neq 0$ and $\tau \neq 0$

Let

$$T(u) = \triangle u - \frac{1}{8}Ru - \frac{1}{12}\tau^2 u^5 + \frac{1}{8}|\sigma|^2 u^{-7}, \quad u > 0.$$

$Y_g > 0$: we can assume $R > 0$ since $Y_g > 0$.

- Super-solutions: let $u_+ = C$ for $C >$ large enough, then

$$T(u_+) = -\frac{1}{8}RC - \underbrace{\frac{1}{12}\tau^2 C^5}_{\text{Dominant part}} + \frac{1}{8}|\sigma|^2 C^{-7} < 0.$$

- Sub-solutions: Firstly solve $\triangle v - \frac{1}{8}Rv = \delta_0\tau^2 - \frac{1}{8}|\sigma|^2$, since $R > 0$, $\triangle - \frac{1}{8}R$ is invertible hence the equation is always solvable. When $\delta_0 = 0$, It has a positive solution by the maximum principle. By the continuity, there exists $\delta_0 > 0$ small enough $\Longrightarrow v > 0$.

  Set $u_- = \epsilon v$, then when ϵ is small enough depending only on the positive lower bound of v,

$$
\begin{aligned}
T(u_-) &= \delta_0\tau^2\epsilon - \frac{1}{12}\tau^2\epsilon^5 v^5 - \frac{1}{8}|\sigma|^2\epsilon + \frac{1}{8}|\sigma|^2\epsilon^{-7}v^{-7} \\
&= \tau^2\underbrace{(\delta_0\epsilon - \frac{1}{12}\epsilon^5 v^5)}_{\geq 0} - \frac{1}{8}|\sigma|^2(\underbrace{\epsilon - \epsilon^{-7}v^{-7}}_{\leq 0}) \geq 0.
\end{aligned}
$$

$Y_g \leq 0$:

- Super-solutions: $u_+ =$ large constant works similarly as above;
- Sub-solutions: Take $\Lambda_0 > 0$ such that $\Lambda_0 + \frac{1}{8}R > 0$, then solve

$$\triangle v - \frac{1}{8}Rv - \Lambda_0 v = \delta_0\tau^2 - \frac{1}{8}|\sigma|^2,$$

  for small enough δ_0. Similarly as above, $u_- = \epsilon v$ works as a sub-solution when $\epsilon > 0$ small enough.

2.5 Find transversal and trace-less symmetric (0, 2) tensors

Given (M^3, g) compact Riemannian manifold, we will talk about how to find trace-less and divergence free symmetric (0, 2) tensors.

Definition 2.5. Given $p > 1$, denote

$$\mathcal{X}_{2,p} = \text{the set of all } W^{2,p} \text{ vector fields on } M;$$

$$\mathcal{T}_{1,p} = \text{the set of all } W^{1,p} \text{ trace-free symmetric (0, 2) tensors on } M.$$

The conformal Killing operator L defined in Definition 2.1 is then a bounded operator

$$L : \mathcal{X}_{2,p} \to \mathcal{T}_{1,p}.$$

- $L^* = -2div_g^{\#}$ by Lemma 2.2, where L^* is the $L^2(M, g)$ adjoint operator for L and $\omega^{\#}$ is the dual vector field for any ω 1-form.
- $L^* \circ L : \mathcal{X}_{2,p} \to \mathcal{X}_{0,p}$ is a self-adjoint(w.r.t. $L^2(M, g)$) and elliptic operator(which will be shown later).

- $$ker(L^* \circ L) = kerL = \{\text{conformal Killing v.f.}\}.$$

  This comes from:
  $$0 = \int_M \langle L^* \circ LW, W\rangle d\mu = \int_M \|LW\|^2 d\mu \implies LW = 0.$$

- If No conformal Killing fields, then
  $$L^* \circ L : \mathcal{X}_{2,p} \to \mathcal{X}_{0,p} \text{ is an isomorphism.}$$

  Generally,
  $$L^* \circ L : ker(L)^\perp \to \mathcal{X}_{0,p} \text{ is an isomorphism.}$$

- $\forall\ k \in \mathcal{T}_{1,p}$, $L^*k \in \mathcal{X}_{0,p} \cap (kerL)^\perp (L^*k \in (kerL)^\perp$ comes from the duality property). Then $\exists\ !\ W \in \mathcal{X}_{2,p} \cap (kerL)^\perp$, with $L^* \circ LW = L^*k$. Then
  $$k = LW + \sigma,$$
  where $\sigma = k - LW$ is transversal, i.e. $div_g \sigma = -\frac{1}{2}L^*\sigma = -\frac{1}{2}L^*(k - LW) = 0$, and trace-less since both k and LW are trace-less. Clearly LW and σ are orthogonal w.r.t. $L^2(M, g)$, since
  $$\int_M \langle LW, \sigma\rangle d\mu = \int_M \langle W, L^*\sigma\rangle d\mu = 0.$$

Hence we have proved the following decomposition proposition,

Propostion 2.6.
$$\mathcal{T}_{1,p} = \mathcal{T}\mathcal{T}_{1,p} \oplus_\perp L(\mathcal{X}_{2,p}),$$
where $\mathcal{T}\mathcal{T}_{1,p}$ is the set of transversal(divergence free) and trace-less $W^{1,p}$ symmetric $(0, 2)$ tensors, and the decomposition is $L^2(M, g)$ orthogonal.

Now we are left to check that $L^* \circ L$ is elliptic.

- $$L^* \circ L = -\frac{1}{2} div^\# \circ L.$$

- Take normal coordinates $\{x^1, x^2, x^3\}$ at $p \in M$, and $W = W^1 \frac{\partial}{\partial x^i} \in \mathcal{X}^\infty(M)$, then
  $$(LW)_{ij} = W_{i,j} + W_{j,i} - \frac{2}{3}(\sum W_{i,i})\delta_{ij},$$
  hence
  $$div(LW)_i = W_{i,jj} + W_{j,ij} - \frac{2}{3}W_{k,ki} = W_{i,jj} + \frac{1}{3}W_{j,ji} + \text{lower order terms(l.o.t.)}$$

- Now let us calculate the symbol of $div(LW)$. Take $\xi \in \mathbb{R}^3$, $\xi \neq 0$, then then symbol is

$$T_\xi(W)_i = |\xi|^2 W_i + \frac{1}{3}\xi_i \xi_j W_j.$$

To check that $T_\xi : W \to T_\xi(W)$ is nontrivial for $\xi \neq 0$, we assume $T_\xi W = 0$, then

$$\sum_i T_\xi(W)_i \xi_i = |\xi|^2 W \cdot \xi + \frac{1}{3}|\xi|^2 W \cdot \xi = 0,$$

hence $\Longrightarrow W \cdot \xi = 0$, hence $T_\xi W = |\xi|^2 W = 0, \Longrightarrow W = 0$. So T_ξ is nontrivial, and $\div \circ L$ is elliptic.

2.6 Proof of Sub-super-solution method

Proof. (of Theorem 2.4) Take $\Lambda \gg 1$, and rewrite T as

$$T(u) = \underbrace{(\triangle u - \Lambda u)}_{=L_1 u} + F(x, u(x)),$$

where $F(x, u(x)) = \Lambda u + f(x, u(x))$. We can chose Λ such that $\frac{\partial F}{\partial u} = \Lambda + \frac{\partial f}{\partial u} > 0$ on $M \times [m_-, m_+]$, and clearly L_1 is invertible.

Take $u_0 = u_+ \in W^{3,p}(M)$, and inductively $u_{i+1} = -L_1^{-1}\big(F(x, u_i(x))\big)$. Then we claim that we can repeat this induction for all i, $u_i \in W^{3,q}(M)$ for all $q > 1$ and

$$(*)_i : \quad u_0 \geq u_1 \geq \cdots \geq u_i \geq u_{i+1} \geq u_-.$$

Proof: To repeat this induction, we only need $m_- \leq u_i \leq m_+$. This follows from $(*)_i$. When $i = 0$, $u_0 = u_+$ satisfies all the property. Assuming $u_i \in W^{3,q}(M)$ and $(*)_i$, i.e. $u_{i-1} \geq u_i \geq u_-$, we show $u_i \geq u_{i+1} \geq u_-$ and $u_{i+1} \in W^{3,q}(M)$.

- Since $u_i \in W^{3,q}(M)$ and $m_- \leq u_i \leq m_+$, $f(x, u(x)) \in W^{1,q}(M)$ for all $q > 1$ since $f, \partial f \in C^0(M \times [m_-, m_+])$, hence $F(x, u(x)) = \Lambda u + f(x, u(x)) \in W^{1,q}(M)$. The elliptic regularity tells us that $u_{i+1} = -L_1^{-1} F(x, u(x))$ is well-defined and lies in $W^{3,q}(M)$.
- Using $\partial_u F \geq 0$,

$$L_1 u_i = -F(x, u_{i-1}) \leq -F(x, u_i) = L_1 u_{i+1},$$

$\Longrightarrow L_1(u_i - u_{i+1}) \leq 0 \Longrightarrow u_i - u_{i+1} \geq 0$ by Maximum Principle.

-

$$L_1 u_{i+1} = -F(x, u_i) \leq -F(x, u_-) \leq L_1 u_-,$$

$\Longrightarrow L_1(u_{i+1} - u_-) \leq 0 \Longrightarrow u_{i+1} - u_- \geq 0.$

Using property $(*)_i$ and elliptic regularity for $L_1 u_{i+1} + F(x, u_i) = 0$, we know u_i have uniform $W^{3,q} - norms$ for any $q > 0$. So the compactness tells us that $u_i \to u \in W^{3,p}(M)$ for the given $p > 1$, and

$$L_1 u_{i+1} + F(x, u_i) = 0 \longrightarrow L_1 u + F(x, u(x)) = 0.$$

□

2.7 Non-CMC cases 1

Given free data (M, g, σ, τ) with M compact, we want to solve the Conformal VCE (2.3). In terms of u, the conformal VCE can be rewritten as:

$$\text{(VCE)} \quad \begin{cases} \triangle u - \frac{1}{8} R_g u = \frac{1}{12}\tau^2 u^5 - \frac{1}{8}\|\sigma + LW\|^2 u^{-7}, \\ div_g LW = \frac{2}{3} u^6 d\tau. \end{cases}$$

- Extension of Sub-super solutions:

  Assume: (M, g) has no conformal-Killing v.f. then $L^* \circ L = -2 div_g \circ L$ is an isomorphism $\Longrightarrow$

  $$\text{Given } u, \text{ there exists a v.f. } W_u \text{ such that } div_g LW = \frac{2}{3} u^6 d\tau.$$

- Now consider the operator:

$$T(u, W) = \triangle u - \frac{1}{8} R_g u - \frac{1}{12}\tau^2 u^5 + \frac{1}{8}\|\sigma + LW\|^2 u^{-7}.$$

Definition 2.7. $u_+ > 0$ is a global super-solution, if

$$T(u_+, W_u) \leq 0, \quad \forall\, 0 \leq u \leq u_+.$$

$u_- > 0$ is a global sub-solution, if

$$T(u_-, W_u) \geq 0, \quad \forall\, u_+ \geq u \geq u_-.$$

Theorem 2.8. (M. Holst, G. Nagy, G Tsogtgerel: Arxiv:[gr-qc]0712.0798) If $\exists\, u_- \leq u_+$ global sub-super-solutions, then $\exists\, u$ and W_u solutions of the conformal VCE (2.3).

Theorem 2.9. (D. Maxwell: Arxiv:[gr-qc]0804.0874) Assume that $g \in W^{2,p}(M)$ and $\sigma, \tau \in W^{1,p}(M)$ for $p > 3$ satisfy one of the following:

1. $Y_g > 0$, $\sigma \neq 0$;
2. $Y_g = 0$, $\sigma \neq 0$, $\tau \neq 0$;
3. $Y_g < 0$, $\exists \hat{g} \in [g]$, such that $R_{\hat{g}} = -\frac{2}{3}\tau^2$.

If $\exists\, u_+ \in W^{3,p}(M)$ global super-solution, then $\exists$ a solution of the conformal VCE (2.3).

Proof. (of Condition 3). Since $Y_g < 0$, and $R_{\hat{g}} = -\frac{2}{3}\tau^2$, if we write $\hat{g} = v^4 g$ with $v > 0$, then

$$\triangle v - \frac{1}{8}R_g v = \frac{1}{12}\tau^2 v^5.$$

Hence $\forall\, W \in \mathcal{X}(M)$,

$$T(v, W) = \frac{1}{8}\|\sigma + LW\|^2 v^{-7} \geq 0.$$

Take $\epsilon > 0$ small enough, then

$$T(\epsilon v, W) = (\epsilon - \epsilon^5)\frac{\tau^2}{12}v^5 + \frac{1}{8}\|\sigma + LW\|^2 v^{-7} \geq 0.$$

So ϵv works as a global sub-solution when $\epsilon \ll 1$ such that $\epsilon v \leq u_+$. □

Propostion 2.10. (D. Maxwell) If $Y_g > 0$ and $\|\sigma\|_\infty$ small enough, then $\exists$ a solution of the conformal VCE (2.3).

Proof. Since $Y_g > 0$, we can choose g with $R_g > 0$. Fix $\epsilon > 0$ small, and assume $\|\sigma\|_{L^\infty} < \epsilon_1(\epsilon)$ with $\epsilon_1(\epsilon) > 0$ another small number depending on ϵ. Want to show that $u_+ = \epsilon$ is a global super-solution, then for any $0 < u \leq u_+$

$$\begin{aligned} T(\epsilon, W_u) &= -\frac{1}{8}R_g\epsilon - \frac{1}{12}\tau^2\epsilon^5 + \frac{1}{8}\|\sigma + LW\|^2\epsilon^{-7} \\ &\leq -\frac{1}{8}R_g\epsilon + \frac{1}{4}(\|\sigma\|^2 + \|LW\|^2)\epsilon^{-7}. \end{aligned}$$

Since $div_g LW_u = \frac{2}{3}u^6 d\tau$, the elliptic regularity tells us

$$\|W_u\|_{2,p} \leq C\|u^6\|_{0,p} \leq C\epsilon^6.$$

Since $p > 3$, Soblev embedding implies that $\|\nabla W_u\|_{L^\infty} \leq C\|W\|_{2,p} \leq C\epsilon^6$. Plugging the estimates back, $T(\epsilon, W_u) \leq -\frac{1}{8}R_g\epsilon + \frac{1}{4}(\epsilon_1(\epsilon) + C\epsilon^{12})\epsilon^{-7}$. So by taking $\epsilon_1(\epsilon)$ small enough, we have $T(\epsilon, W_u) \leq 0$, hence $u_+ = \epsilon$ is a super-solution. □

Propostion 2.11. (D. Maxwell) $\frac{\max_M |d\tau|}{\min_M |\tau|} < \epsilon \Longrightarrow$(2.3) is solvable.

2.8 Non-CMC cases 2

Proof. (Theorem 2.9) Take a cutoff function $\xi(t) \in C^\infty(\mathbb{R}^1)$, such that $\xi(t) \geq 0$, $\xi(t) = 0$ for $|t| \geq \min_M u_+$, and $\xi(0) = 1$. Let

$$\chi_\epsilon(t) = t + \epsilon\xi(t),$$

then $\chi(0) = \epsilon$. Define the regularized system:

$$(*)_\epsilon \quad \begin{cases} \triangle u - \frac{1}{8}R_g u = \frac{1}{12}\tau^2 u^5 - \frac{1}{8}\|\sigma + LW\|^2\chi_\epsilon(u)^{-7}, \\ div_g LW = \frac{2}{3}u^6 d\tau. \end{cases} \tag{2.5}$$

- $u = 0$ is a global sub-solution for $(*)_\epsilon$. Let

$$T_\epsilon(u) = \triangle u - \frac{1}{8}R_g u - \frac{\tau^2}{12}u^5 + \frac{1}{8}\|\sigma + LW\|^2\chi_\epsilon(u)^{-7}.$$

  Then $T_\epsilon(0) = \frac{1}{8}\|\sigma + LW\|^2\epsilon^{-7} \geq 0$ for any W.
- u_+ is a global super-solution of $(*)_\epsilon$, since

$$T_\epsilon(u_+) = T(u_+) \leq 0,\ \forall\, W \text{ solution of } divLW = \frac{2}{3}u^6 d\tau, u \leq u_+.$$

- HNT Theorem 2.8 $\Longrightarrow \exists\, u_\epsilon > 0$, solutions of $(*)_\epsilon$, such that $u_\epsilon \leq u_+$.

Main Estimates:(ME) If $\sigma \neq 0$, then $\exists\, \delta > 0$, such that $\min_M u_\epsilon \geq \delta, \forall \epsilon > 0$.

- (ME) $\Longrightarrow u_\epsilon$ have uniformly bounded $W^{2,p}$ norms $\Longrightarrow \exists\, \epsilon_i \to 0$, such that $u_{\epsilon_i} \to u$ weakly in $W^{2,p}$ hence strongly in $C^{1,\alpha}$ since $p > 3$. Similarly, W_{ϵ_i} have uniformly bounded $W^{2,p}$ norms, so $W_{\epsilon_i} \to W$ weakly in $W^{2,p}$ hence strongly in $C^{1,\alpha}$. So u and W are solutions of (VCE).

Proof of Main Estimates:

- $\chi_\epsilon(u) \leq u + \epsilon$.
- u_ϵ is a super-solution:

$$\triangle u_\epsilon + Q_\epsilon u_\epsilon \leq 0,$$

  where $Q_\epsilon = -\frac{1}{8}R_g - \frac{1}{12}\tau^2 u_\epsilon^4$. Since $|R_g|$ and $|u_\epsilon| \leq |u_+| + \epsilon$ are uniformly bounded, $\max_M |Q_\epsilon|$ is uniformly bounded. By De Giorgi-Morser iteration or estimates of integral kernel for $-\triangle - Q_\epsilon$(Proposition 8 and Proposition 9 in D. Maxwell Arxiv:[gr-qc]0804.0874), we have

$$\min_M u_\epsilon \geq C\int_M u_\epsilon d\mu.$$

- Suppose $\exists\, \epsilon_i \to 0$, such that $\min_M u_i \to 0$ ($u_i = u_{\epsilon_i}$).
- Since u_i are uniformly bounden, W_i have uniformly $W^{2,p}$ norm. By extracting a subsequence, we may assume

$$LW_i \to LW \text{ in } C^0 \text{ norm.}$$

  Hence $\sigma + LW_i \Longrightarrow \sigma + LW$ in C^0. Since

$$\int_M \|\sigma + LW\|^2 d\mu = \int_M \|\sigma\|^2 + \|LW\|^2 \geq \int_M \|\sigma\|^2 > 0,$$

  $\exists\, \Omega \neq \emptyset$, Ω open and $\alpha > 0$, such that $\|\sigma + LW_i\|^2 \geq \alpha > 0$ in Ω for $i \gg 1$.

- By integrating the equation,

$$\int_M \|\sigma + LW_i\|^2 \chi_\epsilon(u_i)^{-7} d\mu = \int_M (-\triangle u_i + \frac{1}{8} R u_i + \frac{\tau^2}{12} u_i^5) \leq C.$$

So $C \geq \int_M \|\sigma + LW_i\|^2 \chi_\epsilon(u_i)^{-7} d\mu \geq \alpha \int_\Omega \chi_\epsilon(u)^{-7} d\mu$. So

$$C \leq |\Omega|^2 \leq (\int_\Omega \chi_i(u_i)^{1/2} \chi_i(u_i)^{-1/2})^2 \leq (\int_M \chi_i(u_i))(\int_\Omega \chi_i(u_i)^{-1})$$

$$\leq (\int_M (u_i + \epsilon_i) d\mu)(\int_\Omega \chi_i(u_i)^{-7})^{1/7} |\Omega|^{6/7}.$$

Combing all the above,

$$C_1 \leq \int_M u_i d\mu + \epsilon_i |M| \leq C_2 \min_M u_i + \epsilon_i |M|.$$

Hence $C_1 - \epsilon_i |M| \leq C_2 \min_M |u_i|$, a contradiction to our assumption.

□

2.9 Non-CMC non-smallness

Dahl, Gicquaud, Humbert (Arxiv:[gr-qc]1012.2188) talked about the case $\tau \in C^1(M)$ and $\tau > 0$:

Either (VCE) has a solution or $\exists$ nonzero W satisfying:

$$(B.E.) \qquad divLW = \sqrt{\frac{2}{3}} \|LW\|_g \frac{d\tau}{\tau}.$$

Here we introduce another regularization of the Conformal VCE:

$$(*)_\Lambda \quad \begin{cases} \triangle u - \frac{1}{8} R_g u = \frac{1}{12} \tau^2 u^5 - \frac{1}{8} \|\sigma + LW\|^2 u^{-7}, \\ div_g LW = \frac{2}{3} \Psi_\Lambda(u)^6 d\tau. \end{cases} \tag{2.6}$$

where $\Phi_\Lambda(t)$ is a cutoff function, such that $\Phi_\Lambda(t) = 0$ for $t \leq 0$, $\Phi_\Lambda(t) = t$ for $0 \leq t \leq \Lambda$ and $\Phi_\Lambda(t) = \Lambda$ for $t \geq \Lambda$.

Now since $\tau > 0$, $u_+ = C_\Lambda$ is a global super-solution for some$C_\Lambda \gg 1$ depending on Λ as shown in the following inequality. Given $u \leq u_+ = C$, W_u is always uniformly bounded in $W^{2,p}$. Then

$$T(u_+, W_u) = -\frac{1}{8} R_g C - \frac{1}{12} \tau^2 C^5 - \frac{1}{8} \|\sigma + LW\|^2 C^{-7} < 0.$$

By using the regularization as in the above section, there always exists a solution u_Λ, with $\delta_\Lambda \leq u_\Lambda \leq C_\Lambda$ for some $\delta_\Lambda > 0$ (δ_Λ depending on Λ.)

Question: What is $\lim_{\Lambda\to\infty} u_\Lambda = ?$

Blow up analysis: Let u be a large solution of $(*)_\Lambda$, and $\bar{g} = u^4 g$. Now consider a blow up of (M, g) on the scale of $\epsilon > 0$. Our aim is to blow up the metric g at the maximum point of u, and then keep track of the blow-up or blow-down of other free data (σ, τ) and conformal data (u, W) such that the initial data $(\bar{g}, \bar{h})$ are unchanged.

Let $g^{(\epsilon)} = \epsilon^{-2} g$. Then the scaling for the data are:

- $R_{g^{(\epsilon)}} = \epsilon^2 R_g$;
- $u_\epsilon = \epsilon^{1/2} u$;
- $\sigma^{(\epsilon)} = \epsilon\sigma$;
- $L_\epsilon W = \epsilon^{-2} LW$;
- $W^{(\epsilon)} = \epsilon^3 W$;
- $\tau^{(\epsilon)} = \tau$.

Now let us take a look of the scaling:

$$\bar{g} = u^4 g = \epsilon^2 u^4 g^{(\epsilon)} = u_\epsilon^4 g^{(\epsilon)} \Longrightarrow u_\epsilon = \epsilon^{1/2} u;$$

$$\bar{h} = u^{-2}(\sigma + LW) + \frac{1}{3}\tau\bar{g} = u_\epsilon^{-2}(\sigma^{(\epsilon)} + L_\epsilon W^{(\epsilon)}) + \frac{1}{3}\tau^{(\epsilon)}\bar{g} = u^{-2}(\epsilon^{-1}\sigma^{(\epsilon)} + \epsilon^{-1} L_\epsilon W^{(\epsilon)}) + \frac{1}{3}\tau\bar{g}$$

$$\Longrightarrow \sigma^{(\epsilon)} = \epsilon\sigma,\ \tau^{(\epsilon)} = \tau \text{ and } L_\epsilon W^{(\epsilon)} = \epsilon LW;$$

$$(LW)_{ij} = \nabla_i(g_{jp}W^p) + \nabla_j(g_{ip}W^p) + \text{trace term} \Longrightarrow$$

$$(L_\epsilon W)_{ij} = \nabla_i^\epsilon(g_{ip}^{(\epsilon)} W^p) + \cdots = \epsilon^{-2}(LW)_{ij};$$

$$L_\epsilon W^{(\epsilon)} = \epsilon LW = \epsilon^{-2} LW^{(\epsilon)} \Longrightarrow W^{(\epsilon)} = \epsilon^3 W.$$

Then the system is blowed up to

$$(*)_{\epsilon,\Lambda} \quad \begin{cases} \triangle_\epsilon u_\epsilon - \frac{1}{8} R_\epsilon u_\epsilon = \frac{1}{12}\tau^2 u_\epsilon^5 - \frac{1}{8}\|\sigma_\epsilon + L_\epsilon W^{(\epsilon)}\|^2 u_\epsilon^{-7}, \\ div_{g_\epsilon} L_\epsilon W^{(\epsilon)} = \frac{2}{3}\Psi_{\epsilon^{1/2}\Lambda}(u_\epsilon)^6 d\tau. \end{cases} \tag{2.7}$$

Now take $p \in M$ such that $u(p) = \max_M u \gg 1$, and let

$$\epsilon^{1/2} = u(p)^{-1},$$

hence $\max_M u_\epsilon = u_\epsilon(p) = 1$. Now take $\{x^1, x^2, x^3\}$ as normal coordinates for g centered at p, and let $x^i = \epsilon y^i$, then

$$g^{(\epsilon)} = \epsilon^{-2} g_{ij}(x) dx^i dx^j = \underbrace{g_{ij}(\epsilon y)}_{=\delta_{ij}+o(\epsilon^2) \text{ for bounded } y} dy^i dy^j.$$

We claim that as $\Lambda_i \to \infty$ and $\epsilon_i \to 0$:

$$u_i, W_i \to u, W \quad \text{uniformly in } W^{2,p}_{loc}.$$

- Bounds on $W^{(\epsilon)}$:

$$\|div_g LW\|_g = \|\frac{2}{3}\Psi_\Lambda(u)^6 d\tau\|_g \leq \frac{2}{3}u^6\|d\tau\|_g \leq C\epsilon^{-3},$$

$$\Longrightarrow \|W\|_g + \|LW\|_g \leq C\|W\|_{2,p} \leq C\epsilon^{-3}.$$

  Then $\|L_\epsilon W^\epsilon\|^2_{g^{(\epsilon)}} = \epsilon^4\|\epsilon LW\|^2_g \leq C\epsilon^6\epsilon^{-6} = C$. So if we normalize $W^\epsilon(p) = 0$, then W^ϵ is uniformly bounded on any compact subset under the y-coordinates.

- Boundes on $\sigma^{(\epsilon)}$:

$$\|\sigma^{(\epsilon)}\|^2_{g^{(\epsilon)}} = \epsilon^4\|\epsilon\sigma\|^2_g = \epsilon^6\|\sigma\|^2_g.$$

- Claim: On $B^{(\epsilon)}_R(0)$ with fixed $R > 0$,

$$\inf_{B^\epsilon_R(0)} u_i \geq \delta(R), \text{ for some } \delta(R) > 0 \text{ depending only on } R.$$

  This comes from the same argument as in the above section by using the estimates for super-solutions. The only ingredient we need to address is $\int_{B^\epsilon_R(0)} \|\sigma^{(\epsilon)} + L_\epsilon W^{(\epsilon)}\|^2_{g^\epsilon} d\mu \geq \delta(R) > 0$. Using Maximum Principle at $y = 0$, since $u_\epsilon(0) = \max u$, $\triangle_\epsilon u_\epsilon(0) \leq 0 \Longrightarrow$

$$-\frac{1}{8}R_\epsilon \geq \frac{1}{12}\tau^2 - \frac{1}{8}\|\sigma^{(\epsilon)} + L_\epsilon W^{(\epsilon)}\|^2 \Longrightarrow$$

$$\frac{1}{8}\|\sigma^{(\epsilon)} + L_\epsilon W^{(\epsilon)}\|^2(0) \geq \frac{1}{12}\tau_0^2 + \frac{1}{8}\epsilon^2 R(p) \geq C > 0, \quad \text{for } \epsilon \text{ small enough.}$$

  So we get the desired esteems by using the uniform Holder norm bound of $L_\epsilon W^{(\epsilon)}$, which comes from the elliptic regularity.

- The estimates on $W^{(\epsilon)}$ and $\sigma^{(\epsilon)}$, together with the fact that $0 < \delta(R) \leq u_\epsilon \leq 1$ give uniform bounded on u_i and W_i , hence the convergence.

Blow up equations on $\mathbb{R}^3$:

$$\begin{cases} \triangle u = \frac{1}{12}\tau_0^2 u^5 - \frac{1}{8}\|LW\|^2 u^{-7}, \\ divLW = \frac{2}{3}\Psi_\lambda(u)^6\tau_1, \end{cases} \tag{2.8}$$

where $\tau_0 = \tau(p_\infty)$ and $\tau_1 = \sum_{i=1}^3 a_i dy^i = d\tau(p_\infty) = \frac{\partial\tau}{\partial x^i}dx^i = \frac{\partial\tau}{\partial y^i}dy^i$. Moreover, u and ∇W are bounded .

3 Asymptotically flat manifold

3.1 Introduction and motivation for asymptotical flatness

(M^n, g, h) is an Initial data set satisfying the (CE). Here we will briefly discuss the notion of asymptotically flatness. Roughly speaking (M^n, g, h) is asymptotically flat if outside a compact set

$K \subset M$, $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B$, i.e. $M \setminus K \sim \mathbb{R}^n \setminus B^n$. Moreover denote $\{x^1, \cdots, x^n\}$ to be the coordinates on $\mathbb{R}^n \setminus B$, then we assume roughly

$$g_{ij}(x) = \delta_{ij} + \text{decaying terms};$$

$$h_{ij}(x) = \text{decays as } x \to \infty.$$

Compared to Newtonian gravity, we will discuss our main interests in the asymptotical flat theory.

- Newtonian Gravity: in $\mathbb{R}^3$. Let $\rho > 0$ be the mass density on a region Ω, where $\rho \equiv 0$ outside Ω, then the gravitational potential φ satisfies:

$$\triangle\varphi = -4\pi\rho, \quad \varphi \to 0, \text{at } \infty.$$

  The gravitational force is $F = \nabla\varphi$, while the total mass is $m = \int_\Omega \rho$, and the center of mass is $\vec{C} = \frac{1}{m}\int_\Omega \vec{x}\rho dx$.
- We can also define mass m, center of mass $\vec{C}$, linear momentum $\vec{P}$ and angular-momentum $\vec{J}$(static quantities of gravity fields) for the initial data set.
- Newtonian Case: The gravitational potential has expansion:

$$\varphi(x) = \frac{m}{|x|} + \frac{\vec{a}\cdot\vec{x}}{|x|^3} + O(|x|^{-3}).$$

  Here m is the mass. This is because:

$$-4\pi m = -4\pi\int_{\mathbb{R}^3}\rho = \int_{\mathbb{R}^3}\triangle\varphi = \int_{\partial B_{r\to\infty}}\frac{\partial\varphi}{\partial r} = -\int_{\partial B_{r\to\infty}}\frac{m}{r^2} = -4\pi m.$$

  In fact, $\exists\,!\,\vec{C} \in \mathbb{R}^3$, such that

$$\varphi(x) = \frac{m}{|\vec{x} - \vec{C}|} + O(|x|^{-3}),$$

  where $\vec{C}$ is the center of mass(can be checked similarly by integration by parts). In fact, the first term is the potential for point mass m centered at $\vec{C}$.
- Schwartzchild Solution: $(\mathbb{R}^n \setminus \{0\}, g, h = 0)$, where

$$g_{ij}(x) = \underbrace{(1 + \frac{m}{2|x|^{n-2}})}_{=u}{}^{\frac{4}{n-2}}\delta_{ij}.$$

  and u is a harmonic function on $\mathbb{R}^n$ when $m > 0$.

  $R_g = -\frac{4(n-1)}{n-2}u^{-\frac{n+2}{n-2}}(\triangle u) = 0$, so it is vacuum.

  Furthermore, it is a Static Black hole.

Let $r = |x|$, $\xi = \frac{x}{|x|} \in S^{n-1}$, then

$$dr^2 + r^2 d\xi^2 = e^{2t}(dt^2 + d\xi^2),$$

where $d\xi^2$ is the standard metric on S^2, and $r = e^t$, with $t \in (-\infty, \infty)$. So we can rewrite g as

$$\begin{aligned} g &= (1 + \frac{m}{2r^{n-2}})^{\frac{4}{n-2}}(dr^2 + r^2 d\xi^2) \\ &= (1 + \frac{m}{2r^{n-2}})^{\frac{4}{n-2}}(e^{\frac{n-2}{2}t})^{\frac{4}{n-2}}(dt^2 + d\xi^2) \\ &= (e^{\frac{n-2}{2}t} + \frac{m}{2e^{\frac{n-2}{2}t}})^{\frac{4}{n-2}}(dt^2 + d\xi^2). \end{aligned}$$

The minimum of the coefficient is achieved when $r^{n-2} = e^{(n-2)t} = m/2$. Hence $r = (\frac{m}{2})^{\frac{1}{n-2}}$ corresponds a minimal surface, hence a horizon.

3.2 Mathematical definition

Definition 3.1. (M^n, g, h) is called asymptotically flat (with one end), if:

- $\exists\, K \subset M$ compact, such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B$. Let $\{x^1, \cdots, x^n\}$ be the local coordinates given by $\mathbb{R}^n \setminus B$.
- $g \in C^{2,\alpha}(M)$, $h \in C^{1,\alpha}(M)$, and

$$g_{ij} = \delta_{ij} + \gamma_{ij},\ \gamma \in W^{2,p}_{-q}(M),\ q > \frac{n-2}{2}, p > n, \text{ and } h \in W^{1,p}_{-1-q}(M).$$

- The mass density μ and momentum density J in (CE) (1.5) satisfy: $\mu, J \in C^{0,\alpha}_{-q_0}(M)$, $q_0 > n$.

Now we will give definition and show a list of properties of the **weighted Soblev and Hölder spaces** $W^{n,p}_{-q}$ and $C^{0,\alpha}_{-q}$ in the following. The main references are [1, 7].

Definition 3.2. The weighted Soblev norm and Hölder norms are defined by

$$\|f\|_{W^{2,p}_{-q}} = \|f\|_{W^{2,p}(K)} + \Big(\int_{\mathbb{R}^n \setminus B} \sum_{|\beta| \le 2} (|x|^{q+|\beta|}|\partial^\beta f|)^p |x|^{-n} dx\Big)^{1/p};$$

$$\|f\|_{C^{0,\alpha}_{-q}} = \sup_{x \in \mathbb{R}^n \setminus B} (|x|^q |f(x)|) + \big[|x|^{q+\alpha} f\big]_\alpha,$$

where $[\cdot]_\alpha$ is the Holder coefficient.

(1) Soblev embedding theorem:

$$\sup_{x\in\mathbb{R}^n\setminus B} |x|^q|f(x)| \le C\|f\|_{W^{1,p}_{-q}}, \quad p>n.$$

More precisely, we have: $\sup_{|x|\ge R}|f(x)| \le o(R^{-q})$.

(2) Let $(g-\delta)\in W^{2,p}_{-q}$, $p>n$, $q>\frac{n-2}{2}$, and $Lu=\triangle_g u+Q(x)u$, where $Q(x)\in C^0$ and $|x|^{2+\delta}|Q(x)|\le C$ with $\delta>0$, then:

- $L: W^{2,p}_{-q_1}\to W^{0,p}_{-2-q_1}$ is a bounded operator;
- If $q_1\in(0,n-2)$, then L is a Fredholm operator of index=0;
- If $q_1\in(n-2,n-1)$, then L is a Fredholm operator of index=-1.

  Remark: 1°: Consider the Laplacian of g: $\triangle_g: W^{2,p_1}_{-q_1}\to W^{0,p_1}_{-2-q_1}$. When $q_1>0\Longrightarrow\triangle_g$ injective, and $Kernel(\triangle)$ =harmonic functions.

  2°: $\exists\, v(x)\in W^{2,p_1}_{-q_2}$ with $q_2<n-2$ such that $v(x)=|x|^{2-n}$ near ∞; moreover $f_0=\triangle_\delta v(x)\in W^{0,p_1}_{-2-q_3}$, $\forall q_3>n-2$ since $\triangle_\delta v(x)=0$ near ∞.

  3°: $\triangle_g u=f$ for $f\in W^{0,p}_{-2-q_1}$ is not always solvable, since f_0 lies in $W^{0,p}_{-2-q_1}$ but $\triangle_g u=f_0$ is not solvable in $W^{2,p}_{-q_1}$. However $\triangle u=f+cf_0$ is solvable for $u\in W^{2,p}_{-q_1}$ for some constant c.

(3) Suppose L is as in (2), & $Lu=f$ for $f\in W^{0,p}_{-q_1}$ with $q_1\in(n-2,n-1)$, & $u\in W^{2,p}_{-q_2}$, $q_2>0$. Then $u(x)=a|x|^{2-n}+v_1(x)$ where $v_1\in W^{2,p}_{-q_1}$.

Proof:(Sketch) The equation can be rewritten as: $\triangle_g u=f-Q(x)u$. In fact, $u\in W^{2,p}_{-q_2}$ and $|x|^{2+\delta}|Q(x)|\le C$ implies that $Q(x)u\in W^{2,p}_{-q_2-2-\delta}$. Assuming $q_2<n-2$(or the statement is trivially true). Using the isomorphism $\triangle_g: W^{2,p}_{-q}\to W^{0,p}_{-2-q}$ for $q\in(0,n-2)$, we know that $u\in W^{2,p}_{-q_2-\delta}$. By iteration, $u\in W^{2,p}_{-q_2-n\delta}$, such that $q_2+n\delta<n-2$ and $q_2+(n+1)\delta>n-2$. Now $\triangle u=f-Q(x)u$ is solvable in the sense that $\exists!\ v_1\in W^{2,p}_{-q_2-(n+1)\delta}$ such that $\triangle v_1=f(x)-Q(x)u-cf_0$, where $f_0=\triangle v$ as in (2) with $v(x)=|x|^{2-n}$ near ∞. Hence $\triangle(v_1+cv)=f-Q(x)u$, and the injectivity of $\triangle$ implies that $u(x)=v_1+cv(x)$. Now we can finish the proof by using iteration to show that $v_1\in W^{2,p}_{-q_1}$ as above.

(4) $Lu=\triangle u+Q(x)u$ with $|Q(x)|\le C|x|^{-2-\alpha}$, $\alpha>0$. If $\int Q_-^{n/2}d\mu<\epsilon_0$ for some small $\epsilon_0>0$, where $Q_-(x)=-Q(x)$ if $Q(x)\le 0$ and $Q_-(x)=0$ if $Q(x)\ge 0$, then L is injective on $W^{2,p}_{-q}$ if $p>n$ and $q>\frac{n-2}{2}$.

Proof: Suppose $Lu=0$ and $u\in W^{2,p}_{-q}\Longrightarrow\nabla u\in L^2$ since

$$\int|\nabla u|^2=\int(|\nabla u|^2|x|^{2(q+1)}|x|^{-2n/p})(|x|^{-2(q+1)}|x|^{2n/p})$$

$$\leq \Big(\int (|\nabla u||x|^{q+1})^p|x|^{-n}dx\Big)^{2/p}\Big(\int |x|^{(\frac{2n}{p}-2(q+1))\frac{p}{p-2}}\Big)^{(p-2)/p},$$

where the first term is bounded by the $W^{2,p}_{-q}$−norm of u, and the exponent of the second term $(\frac{2n}{p} - 2(q+1))\frac{p}{p-2} < -n$(can be checked by multiplying both side by $p-2$) which is also bounded. Now $\int u(\triangle u + Q(x)u) = 0, \Longrightarrow$

$$\int |\nabla u|^2 = -\int Q(x)u^2 \leq \int Q_- u^2 \leq (\int Q_-^{n/2})^{\frac{2}{n}}(\int u^{\frac{2n}{n-2}})^{\frac{n-2}{n}}.$$

Using the Soblev inequality $C(\int u^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq \int |\nabla u|^2, \Longrightarrow u \equiv 0$ when ϵ_0 small enough.

Definition 3.3. The ADM energy is:

$$E = \frac{1}{2(n-1)\omega_{n-1}} \lim_{R\to\infty} \int_{\partial B_R} \sum_{i,j} (g_{ij,i} - g_{ii,j})\nu^j d\xi;$$

the ADM linear-momentum is:

$$P_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{R\to\infty} \int_{\partial B_R} \sum_j \pi_{ij}\nu^j d\xi,$$

where $\omega_{n-1} = vol(S^{n-1})$ and $i = 1, 2, \cdots, n$.

Theorem 3.4. If (M, g, h) is as above, then the ADM energy and linear-momentum E and P exist and are continuous in the following sense:

- $(g_i, h_i) \to (g, h)$ in $W^{2,p}_{loc} \times W^{1,p}_{loc}$, <u>&</u> if $\|g_i - \delta\|_{W^{2,p}_{-q}} + \|h_i\|_{W^{1,p}_{-q}} \leq C$, <u>&</u> $\|\mu\|_{C^{0,\alpha}_{-q_0}} + \|J_i\|_{C^{0,\alpha}_{-q_0}} \leq C, \Longrightarrow E_i \to E$ and $P_i \to P$.

Proof. The scalar curvature has local expansion:

$$\underline{R = g_{ij,ij} - g_{ii,jj} + O(\gamma, \partial^2\gamma) + O\big((\partial\gamma)^2\big)},$$

where $\gamma = g - \delta$.

- $R = \mu + O(h^2)$, where $\mu \sim |x|^{-q_0}$, and $|h(x)| \leq C|x|^{-1-q}$, hence $|h^2(x)| \leq C|x|^{-2-2q} \leq C|x|^{-n}$, since $q > \frac{n-2}{2} \Longrightarrow 2 + 2q > n$. Hence $R \sim |x|^{-n-\delta}$, $\delta > 0$ for $|x|$ large, hence integrable;
- $\partial\gamma \sim |x|^{-1-q}$ for $|x|$ large, hence $(\partial\gamma)^2 \sim |x|^{-n-\delta}$, $\delta > 0$, hence integrable;
- Since $\gamma \sim |x|^{-q}$,

$$\int O(\gamma, \partial^2\gamma)dx \sim \int |x|^{-q}|\partial^2\gamma|dx = \int |x|^{-2q-2+\frac{n}{p}}|x|^{q+2-\frac{n}{p}}|\partial^2\gamma|dx$$

$$\leq \Big(\int (|x|^{q+2}|\partial^2\gamma|)^p \frac{dx}{|x|^n}\Big)^{1/p}\Big(\int |x|^{(-2q-2+\frac{n}{p})\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}.$$

The first term is bounded by the $W^{2,p}_{-q}$ norm of γ, and the second one is bounded since $(-2q-2+\frac{n}{p})\frac{p}{p-1} < -n$(can be checked by multiplying both sides by $(p-1)$). Moreover, the integration of the last term on $\mathbb{R}^3 \setminus B_R$ decays as $R^{-\delta}$, $\delta > 0$.

Using the decay estimates above, we can get:

$$|\int_{M\setminus B_{R_0}} (g_{ij,ij} - g_{ii,jj})|dx \leq CR_0^{-\delta}.$$

Furthermore, $\forall\epsilon > 0$, given R_1, R_2 large enough, and using the divergence theorem,

$$\int_{\partial B_{R_2}} (g_{ij,i} - g_{ii,j})\nu^j d\xi - \int_{\partial B_{R_1}} (g_{ij,i} - g_{ii,j})\nu^j d\xi = \int_{B_{R_2}\setminus B_{R_1}} (g_{ij,ij} - g_{ii,jj})dx \leq \epsilon.$$

Now the continuity of the ADM energy E comes as a corollary as follows: since (g_i, h_i, μ_i, J_i) have uniform bounded norms, we can always choose a large enough $R \gg 1$, such that $\int_{\partial B_R}(g_{ij,i} - g_{ii,j})\nu^j d\xi$ approximate E by $\epsilon > 0$ for all i. Now local $W^{2,p}_{loc} \times W^{1,p}_{loc}$ implies that the surface integral $\int_{\partial B_R}(g_{ij,i} - g_{ii,j})\nu^j d\xi$ converge as $i \to \infty$. Hence we get the convergence of E.

The divergence constraint can be expended locally as:

$$\underline{J = div_g(\pi) = g^{ik}(\pi_{ij,k} + \pi\cdot\Gamma) = \pi_{ij,j} + \gamma\cdot(\partial\pi) + \pi\cdot(\partial\gamma)}.$$

- $J \sim |x|^{-q_0}$, hence $J \sim |x|^{-n-\delta}$, $\delta > 0$;
- $\pi \sim |x|^{-1-q}$ and $\partial\gamma \sim |x|^{-1-q}$, hence $\pi\cdot(\partial\gamma) \sim |x|^{-2-2q} \sim |x|^{-n-\delta}$, $\delta > 0$;
- Since $\gamma \sim |x|^{-q}$, and $\partial\pi \in L^p_{-2-q}$, $\gamma\cdot(\partial\pi)$ works similarly as $O(\gamma, \partial^2\gamma)$ term above.

Combining them together, we can get the uniform integrability of $\int_{\mathbb{R}^3\setminus B_R} \pi_{ij}\nu^j d\xi$. Hence the existence and continuity of P follows similarly as E. □

We will eventually give the proof of the following famous results:

Theorem 3.5. (Positive Energy Theorem [9, 10, 11]) If $\mu \geq |J|$, and satisfy the decay conditions, then $E \geq 0$. Moreover, $E = 0$ only if the data is trivial, i.e. $(M^n, g, h) \hookrightarrow \mathbb{R}^{n,1}$.

Theorem 3.6. (Positive Mass Theorem [5]) If $\mu \geq |J|$, and satisfy the decay conditions, then $E \geq |P|$. Moreover, $E = |P|$ only if the data is trivial, i.e. $(M^n, g, h) \hookrightarrow \mathbb{R}^{n,1}$.

Remark 3.7. The ADM mass m is defined by $m^2 = E^2 - |P|^2$.

4 Density Theorems

We will talk about the Density Theorems, especially the (VCE) case, i.e. $\mu = 0$, $J = 0$.

4.1 Case $\pi = 0$

In this case, initial data set reduces to (M^n, g), and (VCE) reduces to $\underline{R_g \equiv 0}$.

Definition 4.1. An asymptotically flat initial data (M^n, g) has *Conformally flat Asymptotics*, if $g = u^{\frac{4}{n-2}}\delta$ outside some compact set, with

$$\triangle u = 0 \text{ near } \infty, \text{ and } u(x) \to 1 \text{ at } \infty.$$

Then $u(x) = (1 + \frac{E}{2}|x|^{2-n}) + O(|x|^{1-n})$ near ∞.

Definition 4.2.

$$\mathcal{S} = \{g : \ g - \delta \in W^{2,p}_{-q}, \ R_g \equiv 0\}.$$

Theorem 4.3. There exists a dense subset of $\mathcal{S}$ with conformally flat asymptotics. In particular, given $g \in \mathcal{S}$, $\epsilon > 0$, $\exists\, \bar{g}$ with conformally flat asymptotics, with $\|g - \bar{g}\|_{W^{2,p}_{-q}} < \epsilon$, and $|E - \bar{E}| < \epsilon$.

Proof. Given $\sigma \gg 1$, take a cutoff function $\zeta(r)$, such that $\zeta(r) = 1$ for $0 \leq r \leq \sigma$, $\zeta(r) = 0$ for $r \geq \sigma$, and $|\zeta'(r)| \leq \frac{C}{\sigma} \sim \frac{C}{r}$ for $\sigma \leq r \leq 2\sigma$. Let

$$\hat{g}_{ij} = \zeta(|x|)g_{ij} + (1 - \zeta(|x|))\delta_{ij},$$

in A.F. coordinates $\{x^1\}$. Hence $\hat{g} \equiv g$ in B_σ and $\hat{g} \equiv \delta$ outside $B_{2\sigma}$.

- Denote $\gamma = g - \delta$, then near ∞ of M^n,

  $$\hat{g} - g = (\zeta - 1)\gamma;$$

  $$\partial(\hat{g} - g) = (\zeta - 1)\partial\gamma + (\partial\zeta)\gamma;$$

  $$\partial^2(\hat{g} - g) = (\zeta - 1)\partial^2\gamma + 2(\partial\zeta)(\partial\gamma) + (\partial^2\zeta)\gamma.$$

  Since $|\zeta'| \leq \frac{C}{r}$, $|\zeta''| \leq \frac{C}{r^2}$, we have

  $$\|\hat{g} - g\|_{W^{2,p}_{-q}} \leq C\|\gamma\|_{W^{2,p}_{-q}(M \setminus B_\sigma)} \to 0, \sigma \to \infty.$$

- <u>Problem</u>: $\hat{R} \neq 0$ in $B_{2\sigma} \setminus B_\sigma$.

  <u>Claim</u>: $\int_M |\hat{R}|^{n/2} \leq C\sigma^{-\delta}$ and $\|\hat{R}\|_{W^{0,p}_{-2-q}} \leq C\sigma^{-\delta}$ for some $\delta > 0$.

  (Coming from direct integration estimates.)

- <u>Want to Solve</u>:

  $$(*) \quad \hat{L}u \equiv \hat{\triangle}u - c(n)\hat{R}u = 0; \quad u \to 1 \text{ at } \infty,$$

  where $c(n) = \frac{n-2}{4(n-1)}$.

  Let $v = u - 1 \Longrightarrow \hat{L}v = \hat{L}(-1) = c(n)\hat{R}$, where $v \in W^{2,p}_{-q}$.

  Assume the following section, we have

  $$\|v\|_{W^{2,p}_{-q}} \leq C\|\hat{R}\|_{W^{0,p}_{-2-1}} \to 0, \text{ as } \sigma \to \infty.$$

- Need: $\hat{L}: W^{2,p}_{-q} \to W^{0,p}_{-2-q}$ isomorphism, where $q \in (\frac{n-2}{2}, n-2)$.

  It follows from $\int_M |\hat{R}_-|^{n/2} < \epsilon_0$.
- So v exists. Set $u = 1 + v$, then u solves $(*)$.
- Claim: $u \geq 0 (u > 0$ then follows from the Maximum Principal).

  Let $\Omega = \{u < 0\}$, then $\bar{\Omega}$ is compact. Solve

$$\hat{L}u = 0, \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega.$$

$$0 = -\int_\Omega u\hat{L}u = \int_\Omega |\hat{\nabla}u|^2 + c(n)\hat{R}u^2 \geq C(\int_\Omega u^{\frac{2n}{n-2}})^{\frac{n-2}{n}} - c(n)(\int_\Omega u^{\frac{2n}{n-2}})^{\frac{n-2}{n}}(\int_\Omega \hat{R}^{\frac{n}{2}})^{\frac{2}{n}}$$
$$\geq (C - \epsilon)(\int_\Omega u^{\frac{2n}{n-2}})^{\frac{n}{n-2}},$$

  $\Longrightarrow u = 0$.
- Set $\bar{g} = u^{\frac{4}{n-2}}\hat{g}$, then $\bar{R} = 0$, and $\bar{g} \in \mathcal{S}$ since $\bar{g} - \delta = (u^{\frac{4}{n-2}} - 1)\delta \in W^{2,p}_{-q}$ near ∞.

$$\|\bar{g} - g\| \leq \|\bar{g} - \hat{g}\| + \|\underbrace{\hat{g} - g}_{\text{small}}\|,$$

  where $\|\bar{g} - \hat{g}\| = \|(u^{\frac{4}{n-2}} - 1)\hat{g}\| \leq C\|u - 1\|_{W^{2,p}_{-q}}$ is small. Hence $\bar{g}$ is an approximation of g when σ is large enough. The approximation of the AMD energy E by $\bar{E}$ follows from Theorem 3.4.

□

Remark 4.4. J. Corvino showed that $\{g \in \mathcal{S}: \; g \equiv (1 + \frac{E}{2|x|^{n-2}})^{\frac{4}{n-2}}\delta \text{ near } \infty\}$ is also dense in $\mathcal{S}$.

4.2 General cases $\pi \neq 0$

Definition 4.5.

$$\mathcal{S} = \{(g, \pi) : W^{2,p}_{-q} \times W^{1,p}_{-1-q} : \; \mu = 0, J = 0\}.$$

Definition 4.6. (g, π) satisfy *harmonic asymptotics*, if outside a compact set we have

$$g = u^{\frac{4}{n-2}}\delta, \quad \pi = u^{\frac{2}{n-2}}(L_Y\delta - (div_\delta Y)\delta),$$

for a function $u > 0$ and a vector field Y, where $L_Y\delta = Y_{i,j} + Y_{j,i}$.

Denote $\mathcal{L}Y = L_Y\delta - (div_\delta Y)\delta$ here. Then

$$(div\mathcal{L}Y)_i = (Y_{i,j} + Y_{j,i} - (\sum Y_{k,k})\delta_{ij})_{,j} = \triangle Y_i.$$

Remark 4.7. Since $\pi = h - (tr_g h)g$, $tr_g\pi = (1-n)tr_g h$, so $\|\pi\|^2 = \|h\|^2 + n(tr_g h)^2 - 2(tr_g h)^2$. Hence

$$R + (tr_g h)^2 - \|h\|^2 = 0, \iff R + \frac{1}{n-1}(Tr\pi)^2 - \|\pi\|^2 = 0.$$

So the (VCE) is change to

$$(VCE): \quad \begin{cases} R + \frac{1}{n-1}(Tr\pi)^2 - \|\pi\|^2 = 0, \\ div_g \pi = 0. \end{cases}$$

Under harmonic asymptotics, the (VCE) is changed to

$$(VCE): \quad \begin{cases} c(n)^{-1}\triangle u + \big(|\mathcal{L}Y|^2 - \frac{1}{n-1}(Tr\mathcal{L}Y)^2\big)u = 0, \\ \triangle Y^i + \frac{2(n-1)}{n-2}u^{-1}u_k(\mathcal{L}Y)_i^k - \frac{2}{n-2}u^{-1}u_i Tr(\mathcal{L}Y) = 0. \end{cases} \tag{4.1}$$

Claim: using spherical harmonic expansion at infinity:

$$u(x) = 1 + a|x|^{2-n} + O(|x|^{2-n-\delta}), \quad a = \frac{E}{2};$$

$$Y^i(x) = b_i|x|^{2-n} + O(|x|^{2-n-\delta}), \quad b_i = -\frac{n-1}{n-2}P_i,$$

where E and P_i are the ADM energy and linear-momentum. Let us show the second one,

$$P_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{\sigma\to\infty} \int_{\partial B_\sigma} u^{\frac{2}{n-1}}(Y_{i,j} + Y_{j,i} - (\sum Y_{k,k})\delta_{ij})\frac{x^j}{|x|}d\xi$$

$$= c \lim_{\sigma\to\infty} \int_{\partial B_\sigma} [b_i(|x|^{2-n})_j + b_j(|x|^{2-n})_i - b_k(|x|^{2-n})_k\delta_{ij}]\frac{x^j}{|x|}d\xi$$

$$= c \lim_{\sigma\to\infty} \int_{\partial B_\sigma} [b_i(2-n)|x|^{1-n} + (b\cdot x)(2-n)\frac{x^i}{|x|^{n+1}} - (b\cdot x)(2-n)\frac{x^i}{|x|^{n+1}}]d\xi$$

$$= \frac{2-n}{n-1}b_i.$$

Theorem 4.8. (Corvino-Schoen) There is a dense subset of $\mathcal{S}$, consisting of (g,π) which have harmonic asymptotics.

Proof. Take the cutoff function $\zeta(r)$ as in Theorem 4.3. Let

$$\hat{g} = \zeta(|x|)g + (1-\zeta(|x|))\delta, \quad \hat{\pi} = \zeta(|x|)\pi.$$

Similarly as in Theorem 4.3, we have $\|\hat{g} - g\| \le \epsilon$, $\|\hat{\pi} - \pi\| \le \epsilon$ for σ large enough. $(\hat{g}, \hat{\pi})$ do not satisfy the (VCE) only in $B_{2\sigma} \setminus B_\sigma$.
Look for solutions:

$$\bar{g} = u^{\frac{4}{n-2}}\hat{g}, \quad \bar{\pi} = u^{\frac{2}{n-2}}(\hat{\pi} + \hat{\mathcal{L}}Y),$$

where u is a positive function, Y a vector field, and $\mathcal{L}Y = L_Y\hat{g} - (div_{\hat{g}}Y)\hat{g}$.

Constraint Map:

$$\Phi : W^{2,p}_{-q} \times W^{1,p}_{-1-q} \to W^{0,p}_{-2-q} \times W^{0,p}_{-2-q},$$

$$\Phi(g,\pi) = (R_g + \frac{1}{n-1}(Tr\pi)^2 - \|\pi\|^2, div_g\pi).$$

Clearly, $\Phi^{-1}(0,0) = \mathcal{S}$.

Given $(g,\pi) \in W^{2,p}_{-q} \times W^{1,p}_{-1-q}$, $u \in W^{2,p}_{-q}$ and $Y \in W^{2,p}_{-q}$, define:

$$T(u,Y) = \Phi\big(u^{\frac{4}{n-2}}g, u^{\frac{2}{n-1}}(\pi + \mathcal{L}_gY)\big).$$

So $T(0,0) = \Phi(g,\pi)$.

Hope:

$$\mathcal{L}_1 \equiv DT_{(1,0)} : \mathcal{D} = W^{2,p}_{-q} \times W^{2,p}_{-q} \to \mathcal{R} = W^{0,p}_{-2-q} \times W^{0,p}_{-2-q}$$

is a compact perturbation of laplacian, hence is Fredholm of index 0.

- If $DT_{(1,0)}$ is an isomorphism, then $\hat{DT}_{(1,0)}$ is an isomorphism when σ large enough, and results follows from IFT (Inverse Function Theorem).
- $\mathcal{L}_{1=DT_{(1,0)}}$ has finitely dimensional kernel N, finitely dimensional cokernel K, and closed range.
- We have direct sum: $\mathcal{D} = N \oplus \mathcal{D}_1$, and $\mathcal{R} = K \oplus \mathcal{R}_1$. Hence $\mathcal{L}_1 : \mathcal{D}_1 \to \mathcal{R}_1$ is an isomorphism.

Theorem 4.9. $\forall (g,\pi) \in W^{2,p}_{-q} \times W^{1,p}_{-1-q}$,

$$D\Phi_{(g,\pi)} : W^{2,p}_{-1} \times W^{1,p}_{-1-q} \to W^{0,p}_{-2-q} \times W^{0,p}_{-2-q} \text{ is surjective.}$$

- $\exists\, V^{\text{finite-dim}} \subset \mathcal{D}$, such that $D\Phi_{(g,\pi)} : V \to K$ is an isomorphism. By perturbing elements in V a little bit to get another $\tilde{V}$ containing all compactly supported elements, we get an isomorphism $D\Phi_{(g,\pi)} : \tilde{V} \to \tilde{K}$. Moreover, small perturbation still keeps the direct sum $\mathcal{R} = \mathcal{R}_1 \oplus \tilde{K}$.
- Define $\mathcal{L}_2 : \mathcal{D}_1 \oplus \tilde{V} \to \mathcal{R}_1 \oplus \tilde{K} = \mathcal{R}$ by $\mathcal{L}_2 = \mathcal{L}_1 \oplus D\Phi_{(g,\pi)}$. Then
  - $\mathcal{L}_2$ is surjective, and isomorphic from $\mathcal{D}_1$ to $\mathcal{R}_1$.
  - $\tilde{V}$ does not affect the Asymptotics, since elements in $\tilde{V}$ are all compactly supported.

Using the IFT, we get

$$\hat{T}(u,Y) + \Phi(\Delta g, \Delta\pi) = \Phi\big(u^{\frac{4}{n-2}}\hat{g} + \Delta g, u^{\frac{2}{n-1}}(\hat{\pi} + \mathcal{L}_{\hat{g}}Y) + \Delta\pi\big) = 0.$$

So let $(\bar{g},\bar{\pi}) = (u^{\frac{4}{n-2}}\hat{g} + \Delta g, u^{\frac{2}{n-1}}(\hat{\pi} + \mathcal{L}_{\hat{g}}Y) + \Delta\pi)$, we have $(\bar{g},\bar{\pi}) \in \mathcal{S}$, $(\bar{g},\bar{\pi})$ have harmonic asymptotics, and $(\bar{g},\bar{\pi})$ is a good approximation of (g,π) when σ large enough. $\square$

Now we give the proof of Theorem 4.9.

Proof. (of Theorem 4.9)

- Image of $D\Phi_{g,h}$ is closed with finite co-dimension in $W^{0,p}_{-2-q}$.
- If not surjective, $\exists(\xi, Z) \in ker(D\Phi^*_{g,h}) \subset W^{0,p'}_{2+q-n}$, where $W^{0,p'}_{2+q-n}$ is the dual of $W^{0,p}_{-2-q}$ under the paring $W^{0,p}_{-2-q} \times W^{0,p'}_{2+q-n} \to \mathbb{R}$.
- Must show: $\xi = 0$, $Z = 0$.

$$D\Phi^*_{(g,h)}(\xi, Z) = 0 \Longleftrightarrow \begin{cases} \xi_{;ij} - \xi R_{ij} + hDZ + \cdots = 0, \\ \frac{1}{2}(Z_{i,j} + Z_{j,i}) + 2\xi h_{ij} = 0. \end{cases}$$

If $\xi \sim |x|^{-p_1}$ and $|Z| \sim |x|^{-p_2}$, then:

- l.o.t.(low order terms) decays faster than $|x|^{-p_1-2}$ in the first equation;
- $\Longrightarrow \xi$ decays faster than $|x|^{-p_1}$;
- $\Longrightarrow Z$ decays faster than $|x|^{-p_2}$;
- Boost trap $\Longrightarrow |\xi| = O(|x|^{-N})$, $|Z| = O(|x|^{-N})$, $\forall N$.

So $\xi, Z \equiv 0$ by unique continuation argument. □

5 Positive Energy Theorem

5.1 Stability and Positive Energy Theorem

Given $\Sigma^{n-1} \subset M^n$ with $\Sigma = \partial\Omega$. Consider $(M, g, h) \subset \mathcal{S}^{n+1}$ where $\mathcal{S}^{n+1}$ is a space-time. Now take τ a unit future pointed normal to M; ν unite outer-normal of Σ w.r.t. Ω in M, then

$$\underline{\tau + \nu}: \text{ is the outward forward pointing vector.}$$

Define the expansion:

$$\theta = div_\Sigma(\tau + \nu) = \sum_{i=1}^{n-1} \langle D_{e_i}(\tau + \nu), e_i \rangle = \big(-H + Tr_\Sigma(h) \big).$$

Here θ measures the Rate of change of area form along $\tau + \nu$. Let the mean curvature vector be $\vec{H} = -\sum_i \langle \nu, D_{e_i} e_i \rangle \nu$.

Definition 5.1. 1) Σ is outer trapped if $\theta \le 0$ on Σ;

2) Σ is MOTS (marginal outer trapped surface) if $\theta \equiv 0$.

Remark 5.2. Initial data sets with such Σ are models for Blackhole initial data.

$\underline{h=0}$: (M^n, g) A.F. Outer-trapped $\Longleftrightarrow$ $\vec{H}$ is outward pointing.
Basic Existence Theorem: If Σ_0 is outer-trapped, then $\exists$ Σ stable minimal hypersurface surrounding Σ_0.
Idea: Since Σ_0 is outer-trapped, the mean curvature vector $\vec{H}$ points outward. Consider $\partial B_R \subset M$ in the exterior region, then ∂B_R has $\vec{H}$ pointing inward. Consider oriented surfaces homologous to Σ_0, and can find a minimizing representative Σ^{n-1} using GMT (geometric measure theory) since Σ_0 and ∂B_R are barriers.

- Σ^{n-1} is smooth if $n \leq 7$;
- May have singularities for $n \geq 8$. (Co-dimension at least 7)

What does stability mean?

Given Σ a minimal surface, i.e. $\vec{H} = 0$, with unit normal vector ν. Take $\varphi(x) \in C_c(\Sigma)$, and consider the deformation vector field $\varphi(x)\nu(x)$, and construct $\Sigma_t = exp_x\big(t(\varphi(x) + \nu(x))\big)$. Stability means

$$\frac{d^2}{dt^2}|\Sigma_t| \geq 0, \quad \forall \varphi.$$

- Σ is stable $\Longleftrightarrow \forall \varphi \in C^1_c(\Sigma)$

$$\int_\Sigma \big(Ric(\nu,\nu) + \|A\|^2\big)\varphi^2 d\mu \leq \int_\Sigma \|\nabla\varphi\|^2 d\mu.$$

$\Longleftrightarrow \lambda_0(-L) \geq 0$, where the Jacobi operator $L\varphi = \triangle\varphi + \big(Ric(\nu,\nu) + \|A\|^2\big)\varphi$.

$$\Longleftrightarrow -\int_\Sigma \varphi L \varphi \geq 0.$$

- Claim: Gauss equation $\Longrightarrow Ric(\nu,\nu) + \|A\|^2 = \frac{1}{2}(R_M - R_\Sigma + \|A\|^2)$.

  Take $e_n = \nu$, and $\{e_i\}$ tangent to Σ, $i = 1, \cdots, n-1$, then $R_M = \sum_{i,j=1}^n R^M_{ijij}$;

$$R_\Sigma = \sum_{i,j=1}^{n-1} R^\Sigma_{ijij} = \sum_{i,j=1}^{n-1} R^M_{ijij} + \underbrace{\sum_{i,j} h_{ii}h_{jj}}_{=H^2=0} - \|h\|^2;$$

$$R_M - R_\Sigma = 2\sum_{i=1}^{n-1} R^M_{inin} + \|A\|^2 = 2Ric(\nu,\nu) + \|A\|^2.$$

- Σ is stable means:

$$\int_\Sigma \frac{1}{2}(R_M - R_\Sigma + \|A\|^2)\varphi^2 \leq \int_\Sigma |\nabla\varphi|^2 d\mu, \quad \forall \varphi \in C^1_c(\Sigma).$$

5.2 Proof of the Positive Energy Theorem

Theorem 5.3. (**Positive Energy Theorem**) $3 \leq n \leq 7$. Assume (M^n, g) A.F. i.e. $g \in W^{2,p}_{-q}$, with $p > n$, $q > \frac{n-2}{2}$, and $R_g \geq 0$, with $|R_g| \leq C|x|^{-q_1}$ for $q_1 > n$. Then $E \geq 0$ and $E = 0$ only if (M, g) is isometric to $\mathbb{R}^n$.

Proof.
Step 1: Density Theorem $\Longrightarrow \exists\, \bar{g}$ conformally flat near ∞, i.e. $\bar{g} = u^{\frac{4}{n-2}}\delta$, and $\bar{E} \leq E + \epsilon$ for any $\epsilon > 0$.

- Want $\tilde{g} = u^{\frac{4}{n-2}} g$, with $\tilde{R} \equiv 0$, and $\tilde{E} \leq E$

$$\Longrightarrow \triangle u - c(n) R_g u = 0, \quad u \to 1 \text{ at } \infty,$$

  which always has solution since $R_g \geq 0$. Since $0 < u < 1$ on M, the expansion of u near ∞ is $u(x) = 1 + a|x|^{2-n} + l.o.$ with $a \leq 0$. So $\tilde{E} = E + Ca \leq E$.
- Density Theorem $\Longrightarrow$ for positive $E > 0$, may assume g is conformally flat near ∞.

Step 2: If $E < 0$, and $g = u^{\frac{4}{n-2}}\delta$ near ∞, with $u(x) = 1 + \frac{E}{2}|x|^{2-n} + l.o. \Longrightarrow \exists$ trapped slab. In fact, let $\{x^1, \cdots, x^n\}$ be A.F. coordinates near ∞, then the region S_Λ between $H_{\pm\Lambda} = \{x^n = \pm\Lambda\}$ is a trapped slab.

- Take $\{x^1, \cdots, x^{n-1}\}$ be coordinates on H_Λ, with $\partial_i = \frac{\partial}{\partial x^i}$, then ∂_n is normal to H_Λ. Now $(\nabla_{\partial_i}\partial_j)^\perp = \Gamma^n_{ij}\partial_n$, where

$$\Gamma^n_{ij} = \frac{1}{2} u^{-\frac{4}{n-2}}(g_{in,j} + g_{jn,i} - g_{ij,n}) = -\frac{1}{2} u^{-\frac{4}{n-2}} \frac{\partial}{\partial n}(u^{\frac{4}{n-2}}\delta_{ij}) = -\frac{2}{n-2} u^{-1} u_n < 0,$$

  where $\frac{\partial u}{\partial x^n} = \frac{2-n}{2} E \frac{x^n(=\Lambda)}{|x|^n} + O(|x|^{-n}) < 0$ for Λ large and $E < 0$. Hence $H_\Lambda = g^{ij}\Gamma^n_{ij} < 0$.

Step 3: $\exists$ stable Asymptotically planar minimal hypersurface.

- Cutoff the slab S_Λ by a large cylinder $\mathcal{C}_R = \{x = (\hat{x}, x^n) : |\hat{x}| = R\}$, where $\hat{x} = \{x^1, \cdots, x^{n-1}\}$, and $R \gg 1$.
- Let $\Gamma_h = \{x^n = h\} \cap \mathcal{C}_R$.
- Solve the Plateau Problem for $\Gamma_h \Longrightarrow \exists\, \Sigma^{n-1}_{h,R}$ smooth by GMT (geometric measure theory), with

$$|\Sigma_{h,R}| = \min\{|\Sigma| : \ \Sigma \text{ oriented}, \ \partial\Sigma = \Gamma_h\}.$$

- Claim: If $|h| \leq \Lambda$, then $\Sigma_{h,R} \subset S_\Lambda$.

  Proof: If $\Sigma_{h,R}$ does not lie in S_Λ, it must be tangent to some H_{Λ_1} (or $H_{-\Lambda_1}$ which is the similar) with $\Lambda_1 > \Lambda$, and lies totally below H_{Λ_1}. As H_{Λ_1} has mean curvature vector pointing upward, it violates the Maximum Principle.

- $\Sigma_{h,R} \subset S_\Lambda \Longrightarrow \exists R_i \to \infty$, so that $\Sigma_{h_i,R_i} \to \Sigma \subset S_\Lambda$.

  Remark 5.4. This is due to Schoen-Simon-Yau curvature estimates for stable minimal hypersurface [8].

Asymptotics of Σ: Near ∞, volume minimizing $\Sigma = Graph_u$, i,e, Σ is given by $x^n = v(\hat{x})$, with v bounded, $|\nabla v| \leq \frac{C}{|\hat{x}|}$ and $|\nabla^2 v| \leq \frac{C}{|\hat{x}|^2}$.

- We will show it by scaling down. Take $p = (\hat{x}, x^n) \in \Sigma$, with $|\hat{x}| = 2\sigma$ for $\sigma \gg 1$. Consider Ω_σ which is the part of the slab S_Λ within the cylinder $\mathcal{C}_\sigma(\hat{x})$ centered around the line $\{(\hat{x}, t) : |\hat{x}| = 2\sigma\}$. Minimizing property of Σ and comparison $\Longrightarrow$

$$|\Sigma \cap \Omega_\sigma| \leq (1 + \epsilon(\sigma^{-1}))\omega_{n-1}\sigma^{n-1} + C\Lambda\sigma^{n-2} = \omega_{n-1}\sigma^{n-1} + O(\sigma^{-1})\sigma^{n-1}.$$

- Let $g^\sigma = \sigma^{-1}g$, then Ω_σ is scaled down to a thin slab $\sigma^{-1}\Omega_\sigma$ which has radius 1 centered at $\hat{x}$ and height $O(\sigma^{-1})$, and

$$|\sigma^{-1}(\Sigma \cap \Omega_\sigma)| \leq \omega_{n-1} + \epsilon(\sigma^{-1}).$$

  So minimal surface theory implies that $\sigma^{-1}(\Sigma \cap \Omega_\sigma)$ is a smooth graph of a function v, where $|v| \leq \sigma^{-1}\Lambda$, $|v(\hat{x})| \leq \frac{C}{\sigma}$ and $|\sigma^{-1}Dv(\hat{x})| \leq \frac{C}{\sigma}$.
- By scaling v back to the original scale, we get the result.

Step 4: $n = 3$: $\Sigma^2 \subset M^3$ is stable and asymptoticly planar. Now use stability:

$$\int_\Sigma \frac{1}{2}(R_M - \underbrace{R_\Sigma}_{=2K_\Sigma} + \|A\|^2)\varphi^2 d\mu \leq \int_\Sigma |\nabla\varphi|^2 d\mu,$$

$\forall\varphi$ of compact support.

- Σ 2-dimensional: Take v the graphical function, then near ∞,

$$g^\Sigma_{ij}(\hat{x}) = u^4(\hat{x}, v(\hat{x}))(\delta_{ij} + \frac{\partial v}{\partial x^i}\frac{\partial v}{\partial x^j}) = \delta_{ij} + O(|\hat{x}|^{-1}).$$

- Claim: dim=2, may take $\varphi \equiv 1$, and $R_M > 0$ (will be checked later).

  $\Longrightarrow \int_\Sigma K = \lim_{\sigma\to\infty} \int_{|\hat{x}|\leq\sigma} K > 0.$
- Using Gauss-Bonnet:

$$\int_{|\hat{x}|\leq\sigma} K = 2\pi\chi(\Sigma) - \int_{|\hat{x}|=\sigma} \underbrace{k_g}_{=\frac{1}{\sigma}+O(\sigma^{-2})} \to 2\pi\big(\chi(\Sigma) - 1\big) > 0, \text{ as } \sigma \to \infty.$$

  $\Longrightarrow \chi(\Sigma) > 1$ which is a contradiction since planar surface has Euler characteristic ≤ 1.

Remark 5.5. 1°. $\underline{R_M > 0}$: $R_g = 0 \Longrightarrow R_{\bar{g}} > 0$ where $\bar{g} = u^4 g$, by solving

$$Lu = -p,\ \underline{p > 0,\ \text{small}}\,, \quad u > 0,\ u \to 1,\ \text{at } \infty.$$

2°. Find $\varphi_i \Longrightarrow 1$ on compact sets and $\int_\Sigma |\nabla\varphi_i|^2 \to 0$. In fact, take

$$\varphi(|\hat{x}|) = \begin{cases} 1, & |\hat{x}| \le \sigma; \\ \frac{\log(\sigma^2) - \log(|\hat{x}|)}{\log\sigma}, & \sigma \le |\hat{x}| \le \sigma^2; \\ 0, & |\hat{x}| \ge \sigma^2. \end{cases}$$

Then

$$\int_\Sigma |\nabla\varphi|^2 \simeq \int_{\sigma \le |\hat{x}| \le \sigma^2} \frac{1}{(\log\sigma)^2 |\hat{x}|^2} d\mu \simeq \frac{2\pi}{(\log\sigma)^2} \int_\sigma^{\sigma^2} \frac{rdr}{r^2} \sim \frac{C}{\log\sigma} \to 0.$$

Step 5: $\underline{n \ge 4}$. Idea: assume (M^n, g) with $E < 0, \Longrightarrow$ find $(\Sigma^{n-1}, \bar{g})$, $\bar{R} \equiv 0$ and $\bar{E} < 0$. Using induction to reduce to $n = 3$. We will list the main steps here, with more details given in the following section.

- Since $\underline{E < 0}$, can construct Σ^{n-1} which is stable and asymptotically planar. Furthermore Σ is a graph $\{x^n = f(\hat{x})\}$ near ∞, where $|f| \le C$ and $|\nabla f| \le \frac{C}{|\hat{x}|}$. By using the minimal surface equation, we can improve $|\nabla f| \sim O(|x|^{2-n})$. Then the induced metric g^Σ has asymptotics:

$$g^\Sigma_{ij} = \underbrace{u\big(\hat{x}, f(\hat{x})\big)^{\frac{4}{n-2}}}_{\sim 1 + O(|\hat{x}|^{2-n})} (\delta_{ij} + \underbrace{\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}}_{\sim O(|x|^{4-2n})}) = \delta_{ij} + O(|\hat{x}|^{2-n})$$

  So (Σ, g^Σ) is $\underline{\text{A.F. and } E^\Sigma \equiv 0}$.
- Want to solve $L_\Sigma u_1 = 0$ on Σ, with $u_1 \to 1$ at ∞. Let $\bar{g} = u_1^{\frac{4}{n-3}} g^\Sigma$, then $\bar{R} \equiv 0$.
- Solvability: need $\underline{\lambda_0(-L_\Sigma) > 0}$.

  $\underline{\text{On } \Sigma}$: the Jacobi operator $L\varphi = \triangle_\Sigma\varphi + \frac{1}{2}(R_M - R_\Sigma + \|A\|^2)\varphi$. Stability $\Longleftrightarrow \lambda_1(-L) \ge 0$, $\forall\, \Omega$ compact in Σ, $\Longleftrightarrow -\int_\Sigma \varphi L\varphi \ge 0$, $\forall\, \varphi \in C_c(\Sigma)$.

  $\underline{R_M = 0}$, the conformal laplacian is $L_\Sigma\varphi = \triangle_\Sigma\varphi - \frac{n-3}{4(n-2)} R_\Sigma\varphi$.

  Claim: $\lambda_0(-L) \ge 0, \Longrightarrow \lambda_0(-L_\Sigma) \ge 0$, since $\frac{n-3}{4(n-2)} < \frac{1}{2}$.
- Energy Issue: Want $E(g_\Sigma) = 0 \Longrightarrow E(u^{\frac{4}{n-3}} g_\Sigma) = a < 0$, where $u_1(\hat{x})$ has expansion:

$$u_1(\hat{x}) = 1 + \frac{a}{2}|\hat{x}|^{3-n} + O(|\hat{x}|^{2-n}).$$

  Since $L_\Sigma u_1 = 0$,

$$\int_{|\hat{x}| \le \sigma} |\hat{\nabla} u_1|^2 + \frac{n-3}{4(n-2)} R_\Sigma u_1^2 = \int_{|\hat{x}| = \sigma} \underbrace{u_1}_{\sim 1} \underbrace{\frac{\partial u_1}{\partial \nu}}_{\sim \frac{3-n}{2}|\hat{x}|^{2-n} a} \to -\frac{n-3}{2} a\omega_{n-2}.$$

The left converges to $\int_\Sigma |\hat{\nabla} u_1|^2 + \frac{n-3}{4(n-2)} R_\Sigma u_1^2 \geq 0$ if the second variation of the volume of Σ satisfies $\delta^2 V(u_1, u_1) \geq 0$. Then $a \leq 0$.

Definition 5.6. The asymptotic planar Σ is called strongly stable if the second variation $\delta^2 V(\varphi, \varphi) \geq 0, \forall\, \varphi$ with $\varphi - c$ is of compact support for some $c \in \mathbb{R}^1$.

- Make special choice of Σ: Idea: Choose h_σ so that

$$|\Sigma_{h_\sigma, \sigma}| = \min_{|h| \leq \Lambda} |\Sigma_{h,\sigma}|.$$

We can prove that $|h_\sigma| < \Lambda$.

□

5.3 Several technical issues when $n \geq 4$

The reference for dimension $n \leq 7$ is: SLN (Springer Lecture Notes) # 1365.

By last section, we construct Σ^{n-1} which is a complete volume minimizing hypersurface inside the mean convex slab S_Λ. Furthermore, Σ is the graph given by: $x^n = f(\hat{x})$ outside a compact set, where $\hat{x} = (x^1, \cdots, x^{n-1})$, $|f| \leq \Lambda$ and $|\nabla f|(\hat{x}) \leq \frac{C}{|\hat{x}|}$ and $|\nabla^2 f|(\hat{x}) \leq \frac{C}{|\hat{x}|^2}$.

1°**. Decay estimates for the graphical function f**

Propostion 5.7. For $n \geq 4$. Suppose $\Sigma^{n-1} \subset S_\Lambda$ is complete volume minimizing asymptotical planar hypersurface, which is a graph $x^n = f(\hat{x})$ near ∞, then

$$f(\hat{x}) = \alpha + O(|\hat{x}|^{3-n}), \text{ for some } \alpha \in [-\Lambda, \Lambda].$$

Proof. Write Equation for f near ∞. In fact, we use the variation formula. The volume of Σ near ∞ is:

$$V(\Omega) = \int_{\Omega \subset \mathbb{R}^{n-1}} \big[u\big(\hat{x}, f(\hat{x})\big)\big]^{\frac{2(n-1)}{n-2}} \sqrt{1 + |\nabla f|^2} d\hat{x}.$$

Using the variational formula $\frac{d}{dt}|_{t=0} V_\Omega(f + t\eta) = 0$,

$$0 = \int u^{\frac{2(n-1)}{n-2}} \Big[\frac{\nabla f \cdot \nabla \eta}{\sqrt{1 + |\nabla f|^2}} + \frac{2(n-1)}{n-2} u^{-1} \frac{\partial u}{\partial x^n} \eta \sqrt{1 + |\nabla f|^2}\Big] d\hat{x}.$$

So f should satisfy the Minimal Surface Equation:

$$\underbrace{u^{-\frac{2(n-1)}{n-2}} \partial_i \big(u^{\frac{2(n-1)}{n-2}} \frac{\partial_i f}{\sqrt{1 + |\nabla f|^2}}\big)}_{Lf} = \underbrace{\frac{2(n-1)}{n-2} u^{-1} \partial_n u \sqrt{1 + |\nabla f|^2}}_{F(\hat{x})}.$$

- Write $Lf = \sum_{i,j=1}^{n-1} a_{ij}\partial_{ij}^2 f + b_i\partial_i f$, then

$$a_{ij} = \delta_{ij} + O(|\hat{x}|^{-2}), \quad b_i = O(|\hat{x}|^{-3}).$$

- Using the expansion $u(x) = 1 + \frac{E}{2}|\hat{x}|^{2-n} + O(|x|^{1-n})$,

$$\Longrightarrow \partial_n u = \frac{2-n}{2}E|x|^{-n}x^n + O(|x|^{-n}) = O(|x|^{-n})$$

since $|x^n| \le \Lambda$, hence $F(\hat{x}) = O(|\hat{x}|^{-n})$.

- Using the elliptic theory in weighted spaces, $\Longrightarrow f(\hat{x}) = \alpha + \beta|\hat{x}|^{3-n} + l.o.t.$.

□

2°. **Conformally change** $(\Sigma, \hat{g})$ **to scalar flat** $(\Sigma, u_1^{\frac{4}{n-3}}\hat{g})$

Consider $(\Sigma, \hat{g})$, where $\hat{g}$ is the metric induced from g. In the base coordinates: $\hat{x} = (x^1, \cdots, x^{n-1})$,

$$\hat{g}_{ij} = u^{\frac{4}{n-2}}\big(\hat{x}, f(\hat{x})\big)[\delta_{ij} + \underbrace{(\partial_i f)(\partial_j f)}_{O(|\hat{x}|^{4-2n})}] = \delta_{ij} + O(|\hat{x}|^{2-n}),$$

$\Longrightarrow \hat{E} = E(\hat{g}) = 0.$

Propostion 5.8. Stability of $\Sigma \Longrightarrow \exists\, u_1 > 0$ on Σ, with $u_1 \to 1$ at ∞, s.t.

$$(*) \quad L_1 u_1 = \triangle u_1 - c(n-1)\hat{R}u_1 = 0,$$

where $c(n) = \frac{n-2}{4(n-1)}$.

Proof. Claim: $L_1 : W^{2,p}_{-q}(\Sigma) \to W^{0,p}_{-2-q}(\Sigma)$, $\frac{n-3}{2} < q < n-3$, then

$$L_1 \text{ is injective, hence isomorphism.}$$

In fact, stability $\Longrightarrow \forall \varphi \in C_c^1(\Sigma)$,

$$\int_\Sigma \frac{1}{2}(R - \hat{R} + \|A\|^2)\varphi^2 d\hat{\mu} \le \int_\Sigma |\nabla\varphi|^2 d\hat{\mu},$$

$$\underbrace{\Longrightarrow}_{\text{take } R>0} \quad \int_\Sigma -\frac{1}{2}\hat{R}\varphi^2 d\hat{\mu} < \int_\Sigma |\nabla\varphi|^2 d\hat{\mu}, \quad \text{if } \varphi \ne 0,$$

$$\Longrightarrow \quad c(n-1)\int_\Sigma -\hat{R}\varphi^2 d\hat{\mu} < \underbrace{2c(n-1)}_{<1}\int_\Sigma |\nabla\varphi|^2 d\hat{\mu} \le \int_\Sigma |\nabla\varphi|^2 d\hat{\mu},$$

$$\Longrightarrow \quad \lambda_0(-L_1) > 0, \quad \text{on compact sets.}$$

By the embedding $W^{2,p}_{-q}(\Sigma) \subset W^{1,2}(\Sigma)$ when $q > \frac{n-3}{2}$, $\forall \varphi \in W^{2,p}_{-q}(\Sigma)$, we can approximate by compactly supported function, hence

$$c(n-1)\int_\Sigma -\hat{R}\varphi^2 d\hat{\mu} < \int_\Sigma |\nabla\varphi|^2 d\hat{\mu}.$$

Using the decay estimates $\varphi(\partial\varphi) = O(|x|^{-2q-1}) = o(|x|^{-(n-2)})$, and integration by part,

$$\Longrightarrow \quad -\int_\Sigma \varphi L_1 \varphi d\hat{\mu} \geq 0,$$

" $=$ " only if $\varphi \equiv 0$, so L_1 is injective. By §3.2, L_1 is Fredholm of index 0 since $q < (n-3)$, hence L_1 is an isomorphism.

Let $v = u_1 - 1$, then $(*)$ is equivalent to

$$L_1 v = \triangle v_1 - c(n-1)\hat{R}v = c(n-1)\hat{R}.$$

By the decay of $\hat{g}$, $\hat{R} = O(|\hat{x}|^{-n}) \in W^{0,p}_{-2-q}$, so we can solve with $v \in W^{2,p}_{-2-q}$.

To show that $u_1 = 1 + v$ is positive,

1. If $u_1 < 0$ somewhere, we can take a connected component Ω of the region $\{\hat{x} : \ u_1(\hat{x}) < 0\}$. Then $u_1|_\Omega$ forms a nearly zero eigenfunction(by mollifying $u_1|_\Omega$ a little bit) of L_1 violate $\lambda_0(L_1) > 0$.
2. If $u_1 = 0$ somewhere, the it violates the Strong Maximum Principal.

□

<u>3°. **Strong stability and energy estimates for** $(\Sigma, \tilde{g} = u_1^{\frac{4}{n-3}}\hat{g})$</u>

Denote $\Sigma_h = \lim_{\rho_i \to \infty} \Sigma_{h,\rho_i}$ to be the limit for some fixed height $h \in [-\Lambda, \Lambda]$, and $\hat{g}$ the induced metric on Σ_h by (M^n, g). We can conformally change $\hat{g}$ to $\tilde{g}$ by

$$\tilde{g} = u_1^{\frac{4}{n-3}}\hat{g}.$$

Then $\tilde{g}$ is scalar flat, i.e. $\tilde{R} \equiv 0$ by equation $(*)$ above. Moreover, u_1 has the expansion:

$$u_1(\hat{x}) = 1 + \frac{\tilde{E}}{2}|\hat{x}|^{3-n} + O(|\hat{x}|^{2-n}),$$

where $\tilde{E}$ is the ADM energy of $(\Sigma, \tilde{g})$ since $\hat{g}$ has $\hat{E} = 0$.

- <u>Hope</u>: $E < 0$(for (M, g)) $\Longrightarrow \tilde{E} < 0$ (for $(\Sigma, \tilde{g})$).
- <u>Want</u>: for some h, Σ is <u>strongly stable</u>, i.e.

  $$\delta^2\Sigma(\varphi,\varphi) \geq 0, \ \forall\, \varphi, \text{ s.t. } \varphi - c \text{ is of compact support for some } c \in \mathbb{R}^1.$$

  Here $\delta^2\Sigma(\varphi,\varphi) = \int_\Sigma |\nabla\varphi|^2 - \frac{1}{2}(R - \hat{R} + \|A\|^2)\varphi^2$, where the terms $(R - \hat{R} + \|A\|^2)$ is in $L^1(\Sigma)$ by the decay estimates as in Theorem 3.4, hence makes sense for φ to approach constant near ∞.

Lemma 5.9. $\exists\, h_\rho \in (-\Lambda, \Lambda)$, *so that* $|\Sigma_{h_\rho,\rho}| = \min_{h\in[-\Lambda,\Lambda]} |\Sigma_{h,\rho}|$

Proof. First Variation Formula when $H = 0$: Let X be a smooth vector field, and F_t the flow of X. Let Σ be a smooth surface, then

$$\frac{d}{dt}\Big|_{t=0} |F_t(\Sigma)| = \int_\Sigma div_\Sigma(X) d\mu = \int_{\partial\Sigma} \langle X, \vec{\eta}\rangle d\sigma,$$

where $\vec{\eta}$ is the unit co-normal of $\partial\Sigma \subset \Sigma$. Here the second " $=$ " comes from the following. Take $e_1, \cdots, e_{n-1}$ an o.n. basis on $T\Sigma$, then $div_\Sigma(X) = \sum_{i=1}^{n-1}\langle D_{e_i}X, e_i\rangle = \sum_{i=1}^{n-1}\langle D_{e_i}X^T, e_i\rangle + \underbrace{\langle D_{e_i}X^\perp, e_i\rangle}_{=-X^\perp\cdot\vec{H}=0} = div_\Sigma(X^T)$. So the divergence Theorem gives the second " $=$ ".

Take $X = \partial_n = \frac{\partial}{\partial x_n}$ near $\partial\Sigma_{h,\rho}$, then

$$\delta\Sigma_{h,\rho}(X) = \int_{\partial\Sigma_{h,\rho}} \langle \partial_n, \vec{\eta}\rangle d\sigma.$$

1. Since $\Sigma_{\Lambda,\rho}$ lies in the slab S_Λ, the co-normal $\vec{\eta}$ is up-ward. Since g is conformally flat near ∞, the inner product of $\vec{\eta}$ and ∂_n is positive, so $\delta\Sigma_{h,\rho}(\partial_n) > 0$ for h near Λ, $\Longrightarrow$

$$|\Sigma_{h,\rho}| < |\Sigma_{\Lambda,\rho}|, \quad \text{for } h \lesssim \Lambda.$$

2. Similarly, $\Sigma_{-\Lambda,\rho}$ lies inside S_Λ, so the co-normal $\vec{\eta}$ is downward, hence $\delta\Sigma_{-\Lambda,\rho} < 0$, $\Longrightarrow$

$$|\Sigma_{h,\rho}| < |\Sigma_{-\Lambda,\rho}|, \quad \text{for } h \gtrsim -\Lambda.$$

So the minimizer of $\Sigma_{h_\rho,\rho}$ among $\{\Sigma_{h,\rho} :\ h \in [-\Lambda, \Lambda]\}$ lies strictly in S_Λ. □

Hence we have

$$\delta^2\Sigma_{h_\rho,\rho}(X, X) \geq 0, \text{ if } X = \partial_n \text{ near } \partial\Sigma_{h_\rho,\rho},$$

since $X = \partial_n$ near $\partial\Sigma_{h_\rho,\rho}$ is a valid candidate for the variation.

Now let $\Sigma = \lim_{\rho_i\to} \Sigma_{h_{\rho_i},\rho_i}$, which is asymptotically planar and stable.

Lemma 5.10. Σ *is strongly stable.*

Proof. Take any $X = \partial_n$ near ∞, we claim:

$$\lim_{i\to\infty} \delta^2\Sigma_i(X, X) = \delta^2\Sigma(X, X) \geq 0.$$

- Take $F_t(x) = x + t\partial_n$ near ∞.
- $d\mu = u^{\frac{2(n-1)}{n-2}} d\mu_0$ near ∞, where $d\mu_0 = dx^1 \cdots dx^n$, then $\frac{d^2}{dt^2} d\mu_t = (\frac{d^2}{dt^2}\big|_{t=0} u^{\frac{2(n-1)}{n-2}}) d\mu_0$.
- $u(x) = 1 + O(|x|^{2-n})$, so $\partial_n^2 u = O(|x|^{-n})$, $\Longrightarrow \partial_n^2 u^{\frac{2(n-1)}{n-2}} = O(|x|^{-n})$.
- By volume comparison, $|\Sigma_i \cap \mathcal{B}_\sigma| \leq C\sigma^{n-1}$.

So $\delta^2\Sigma_i|_{\mathbb{R}^n\setminus B_R}(X,X)$ is uniformly small for R large.

Let ν be the unite normal of Σ, then

$$\nu = u^{-\frac{2}{n-2}}\frac{(-\nabla f,1)}{\sqrt{1+|\nabla f|^2}} = \partial_n + O(|\hat{x}|^{2-n}),$$

since $u = 1+O(|x|^{2-n})$ and $|\nabla f| = O(|\hat{x}|^{2-n})$. Since 2nd variation $\delta^2\Sigma$ are non-negative on $W^{1,2}$ terms, so $\delta^2\Sigma(\nu,\nu)\geq 0$. So Σ is strongly stable, i.e. $X=\varphi\nu$, with $\varphi=1$ near ∞, then

$$\delta^2\Sigma(\varphi,\varphi)\geq 0.$$

□

Lemma 5.11. $\tilde{E}<0$.

Proof. Take $\varphi = u_1 = 1+O(|\hat{x}|^{3-n})$, then by strong stability

$$\int_\Sigma (|\nabla u_1|^2 + c(n-1)\hat{R}u_1^2)d\hat{\mu} > 0.$$

While the left hand side is

$$\lim_{\rho\to\infty}\int_{B_\rho}(|\nabla u_1|^2 + c(n-1)\hat{R}u_1^2)d\hat{\mu} = \lim_{\rho\to\infty}\int_{\partial B_\rho} u_1\frac{\partial u_1}{\partial\nu}d\sigma = -\frac{n-3}{2}\tilde{E}\omega_{n-1} > 0,$$

where in the second “ = ” we use the equation $L_1u_1=0$, and the last “ = we use $\partial u_1 = -\frac{n-3}{2}|\hat{x}|^{2-n}$ near ∞. So $\tilde{E}<0$. □

5.4 Rigidity part of Theorem 5.3

Proof. We know $E\geq 0$. If $E=0$, let us show the rigidity step by step.

Step 1: $R_g\equiv 0$. If $R_g>0$ somewhere, solve

$$\triangle u - c(n)R_g u = 0, \qquad u\to 1 \text{ at } \infty.$$

- Let $\hat{g} = u^{\frac{4}{n-2}}g$, then $\hat{R}\equiv 0$.
- u has expansion: $u(x) = 1+\frac{\hat{E}}{2}|x|^{2-n} + l.o.t.$.
- Claim: $\hat{E}<0$, since

$$0=\int_{B_\rho}\big(\triangle u - c(n)R_g u\big) = \int_{\partial B_\rho}\underbrace{\frac{\partial u}{\partial\nu}}_{\sim\frac{2-n}{2}\hat{E}\rho^{1-n}} d\sigma - c(n)\int_{B_\rho}R_g u d\mu,$$

$\rho\to\infty$:

$$0 = -\frac{n-2}{2}|S^{n-1}|\hat{E} - c(n)\int_M R_g u d\mu.$$

$\Longrightarrow \hat{E}<0$ if $R_g>0$ somewhere.

Step 2: $Ric_g \equiv 0$.

Let σ be a smooth symmetric $(0,2)$ tensor with compact support. Denote $g_\epsilon = g + \epsilon\sigma$, since $R_g \equiv 0$, then $R_\epsilon = O(\epsilon)$ and has compact support. Moreover $\int_M |R_\epsilon|^{n/2} d\mu_\epsilon$ is small if ϵ is small enough. Solve

$$\triangle_\epsilon u_\epsilon - c(n)R_\epsilon u_\epsilon = 0, \quad u_\epsilon \to 1 \text{ at } \infty.$$

- $\hat{g}_\epsilon = u_\epsilon^{\frac{4}{n-2}} g_\epsilon$, then $\hat{g}_\epsilon$ has $\hat{R}_\epsilon = 0$.
- Positive Energy Theorem $\Longrightarrow \hat{E}_\epsilon \geq 0$ and $\hat{E}_0 = E = 0$, with $u_0 = 1$. Since $g_\epsilon = g$ near ∞,

$$\hat{E}_\epsilon = \frac{2c(n)}{(n-2)|S^{n-1}|} \int_M R_\epsilon u_\epsilon d\mu_\epsilon.$$

- So $\hat{E}_\epsilon$ has a critical point when $\epsilon = 0$,

$$0 = \frac{d}{dt}\Big|_{\epsilon=0} \hat{E}_\epsilon = C\int_M (\frac{d}{d\epsilon}\Big|_{\epsilon=0} R_\epsilon) d\mu + C\int_M \underbrace{R_0}_{=0} \frac{d}{d\epsilon}\Big|_{\epsilon=0}(u_\epsilon d\mu_\epsilon)$$

$$= -C\int_M \langle \sigma, Ric_g\rangle d\mu, \quad \forall \sigma \text{ of compact support.}$$

$\Longrightarrow Ric \equiv 0$.

Step 3: When $n = 3$, $Ric \equiv 0 \Longrightarrow g$ is flat $\Longrightarrow (M^n, g) \cong \mathbb{R}^3$.

Step 4: $n \geq 4$. Let $\{x^1, \cdots, x^n\}$ be coordinates of (M^n, g) near ∞. We can extend to the interior to get a mapping $\{x^1, \cdots, x^n\} : M^n \to \mathbb{R}^n$.

- Let $f^i = \triangle_g x^i = g^{jk}\Gamma^i_{jk} = O(|x|^{-q_1})$, where $q_1 > 1 + \frac{n-2}{2} = \frac{n}{2} = 2$. Solve

$$\triangle v^i = f^i, \quad v^i \to 0 \text{ at } \infty.$$

$\Longrightarrow v^i = O(|x|^{-q_2})$ for $q_2 = q_1 - 2 > 0$.
- Let $\hat{x}^i = x^i - v^i$, then $\triangle \hat{x}^i = 0$. Let $\hat{F} = (\hat{x}^1, \cdots, \hat{x}^n) : M \to \mathbb{R}^n$. Then from the asymptotics, $(\hat{x}^1, \cdots, \hat{x}^n)$ is a harmonic A.F. coordinates near ∞.
- Let $\hat{g}_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$, then under harmonic coordinates

$$0 = \hat{R}_{ij} = -\frac{1}{2}\triangle_{\hat{g}}(\hat{g}_{ij}) + Q(\partial\hat{g}, \partial\hat{g}),$$

where $\partial\hat{g} \cong \partial g$ decays faster than $|x|^{-\frac{n}{2}}$, hence $Q(\partial\hat{g}, \partial\hat{g})$ decays faster than $|x|^{-n}$.

$$\Longrightarrow \quad \hat{g}_{ij} = \delta_{ij} + c_{ij}|\hat{x}|^{2-n} + O(|\hat{x}|^{1-n}).$$

- Claim: $c_{ij} = 0, \forall\, i, j$.

  Pf: May rotate the coordinates $\hat{x}^i$ to assume $c_{ij} = c_i \delta_{ij}$. Then $\hat{g}_{ij} = \delta_{ij} + c_i \delta_{ij} |\hat{x}|^{2-n} + l.o.t.$ and $\hat{g}^{ij} = \delta^{ij} - c_i \delta_{ij} |\hat{x}|^{2-n} + l.o.t.$. Then $\sqrt{det(\hat{g})} = 1 + \frac{1}{2}(\sum_{i=1}^n c_i)|\hat{x}|^{2-n} + l.o.t.$. Using the harmonic condition

$$0 = \triangle x^i = \frac{1}{\sqrt{\hat{g}}} \partial_j (\sqrt{\hat{g}} \hat{g}^{ij}) \cong \frac{\sum c_i}{2} \partial_j |\hat{x}|^{2-n} - c_i \delta_{ij} \partial_j |\hat{x}|^{2-n}$$

$$= (n-2)(c_i - \frac{1}{2} \sum_{k=1}^n c_k) \frac{\delta_{ij} \hat{x}^j}{|\hat{x}|^n}.$$

  $\Longrightarrow c_i = \frac{1}{2} \sum_{k=1}^n c_k$, for $i = 1, \cdots, n$, $\Longrightarrow \sum_i c_i = \frac{n}{2} \sum_i c_i$, $\Longrightarrow \sum_i c_i = 0$, $\Longrightarrow c_i = 0$ for $i = 1, \cdots, n$.
- Hence $\hat{g}_{ij} = \delta_{ij} + O(|\hat{x}|^{1-n})$.
- Claim: $\hat{F}$ is an isometry.

  Pf: Using the Bochner formula,

$$\frac{1}{2} \int_{B_\rho} \triangle_{\hat{g}} |\nabla \hat{x}^i|^2_{\hat{g}} = \int_{B_\rho} |\nabla \nabla x^i|^2_{\hat{g}} + \underbrace{Ric(\nabla \hat{x}, \nabla \hat{x})}_{=0}.$$

  Here $|\nabla \hat{x}^i|^2_{\hat{g}} = \hat{g}^{ii} = 1 + O(|\hat{x}|^{1-n})$, so using the divergence theorem, LHS (left hand side)= $\frac{1}{2} \int_{\partial B_\rho} \underbrace{\frac{\partial |\nabla \hat{x}^i|^2}{\partial r}}_{\sim O(\rho^{-n})} d\sigma \to 0$ when $\rho \to \infty$. $\Longrightarrow \nabla \nabla \hat{x}^i \equiv 0$, which means that $\nabla \hat{x}^i$ is parallel vector field.

$$\Longrightarrow \hat{g}^{ij} = \langle \nabla \hat{x}^i, \nabla \hat{x}^j \rangle \equiv \delta_{ij}, \Longrightarrow \text{ isometry.}$$

□

6 Marginally outer trapped surface (MOTS)

6.1 Introduction to MOTS

Space-time Case: (M^n, g, h) A.F. satisfies (D.E.C.): $\mu \geq |J|$, with the ADM energy-momentum vector (E, P) a 4-vector.

Positive Energy Theorem: $E \geq 0$, and "$E = 0$" only if (M, g, h) is isometrically embedded in $\mathbb{R}^{n,1}$, with g the induced metric and h the 2nd f.f..

Positive Mass Theorem: $E \geq |P|$, and "$E = |P|$" only if (M, g, h) is isometrically embedded in $\mathbb{R}^{n,1}$, with g the induced metric and h the 2nd f.f.

Remark 6.1. $3 \le n \le 7$ case was proved by Eichmair-Huang-Lee-Schoen [5].

Recall: $\Sigma^{n-1} \subset M$ is Outer-trapped if $\theta = div_\Sigma(\nu + e_0) = -(H_\Sigma + Tr_g h) \le 0$; and a MOTS if $\theta \equiv 0$ on Σ.

Stability Criterion: Σ with $H = 0$, then stability $\iff \delta^2\Sigma(\varphi,\varphi) \ge 0$, $\forall\ \varphi$ compactly supported. $\delta^2\Sigma(\varphi,\varphi) = -\int_\Sigma \varphi\mathcal{L}\varphi d\mu$, where $\mathcal{L} = \triangle - Q$, with $Q = Ric(\nu,\nu) + \|A\|^2$.

Propostion 6.2. Stability $\iff \exists\ u > 0$ with $\mathcal{L}u \le 0$.

Remark 6.3. Let us discuss the motivation of the stability criterion. Consider Σ_t, $t \in (-\epsilon, \epsilon)$ a local foliation of Σ along $u(x)\nu(x)$. Let θ_t be the expansion of Σ_t. If $|\Sigma_t|$ is increasing for $t > 0$, while $|\Sigma_t|$ decreasing for $t < 0$, we should have

$$\theta_t < 0, \text{ when } t < 0; \quad \theta_t > 0, \text{ when } t > 0.$$

Hence

$$-\mathcal{L}u = \frac{d}{dt}\Big|_{t=0}\theta_t \ge 0.$$

In this case, $\implies \mathcal{L}u \le 0$, $\implies \Sigma$ is stable.

Proof. (of Proposition 6.2) ($\implies$) We only talk about the case Σ is compact. Σ stable $\implies \exists\ u > 0$ the lowest eigen-function, such that $-\mathcal{L}u = \lambda_0 u \le 0$.

($\impliedby$) Assume $u > 0$, $\mathcal{L}u = \triangle u + Qu \le 0$. Let $w = \log u$, then

$$\triangle w = \frac{\triangle u}{u} - |\nabla w|^2 \le -Q - |\nabla w|^2.$$

Given $\varphi \in C_c^1(\Sigma)$, multiply the above by φ^2 and integration:

$$\int_\Sigma \varphi^2\triangle w (= -\int_\Sigma 2\varphi\langle\nabla\varphi,\nabla w\rangle) \le \int_\Sigma(-Q - |\nabla w|^2)\varphi^2,$$

$$\implies \quad \int_\Sigma Q\varphi^2 \le -\int_\Sigma |\nabla w|^2\varphi^2 + \underbrace{2\varphi\langle\nabla\varphi,\nabla w\rangle}_{\le|\nabla\varphi|^2+|\nabla w|^2\varphi^2},$$

$$\implies \quad \int_\Sigma Q\varphi^2 \le \int_\Sigma |\nabla\varphi|^2, \iff -\int_\Sigma \varphi\mathcal{L}\varphi \ge 0.$$

□

Stable MOTS:

Given $\Sigma^{n-1} \subset (M^n, g, h)$. Take $\{e_1, \cdots, e_{n-1}\}$ an o.n. basis of Σ^{n-1}, and $\nu = e_n$ the unit normal of Σ in M. Denote e_0 the unit normal of (M, g, h) in some ambient space-time $\mathcal{S}$.

Definition 6.4. Σ is a stable MOTS, if $\theta \equiv 0$ and $\exists\, u > 0$, such that $\mathcal{L}u \le 0$. Here

$$\mathcal{L}\varphi = -\frac{d}{dt}\Big|_{t=0}\theta_t = \triangle\varphi - 2\langle X, \nabla\varphi\rangle - (Q + div_\Sigma X - \|X\|^2)\varphi,$$

- $Q = \frac{1}{2}R_\Sigma - (\mu + J(\nu)) - \frac{1}{2}\|\chi\|^2$,
- $\chi = A + h|_\Sigma$,
- $X = Tan_\Sigma(D_\nu e_0) = -\sum_{i=1}^{n-1} h_{in}e_i$.

We call such u a *test function*.

6.2 Property of stable MOTS

Propostion 6.5. If Σ is a stable MOTS, then $\forall\, \Omega \subset \Sigma$, $\lambda(-L, \Omega) \ge 0$. Here

$$L\varphi = \triangle\varphi + \frac{1}{2}(-R_\Sigma + \|\chi\|^2)\varphi,$$

where $\chi = A + h|_\Sigma$, with A the 2nd f.f. of Σ in M. Or equivalently,

$$\frac{1}{2}\int_\Sigma(-R_\Sigma + \|\chi\|^2)\varphi^2 d\mu \le \int_\Sigma |\nabla\varphi|^2 d\mu, \quad \forall\, \varphi \in C_0^1(\Sigma).$$

Proof. Take $w = \log u$ with u given by the definition, then $\triangle w = \frac{\triangle u}{u} - |\nabla w|^2$. Using $\mathcal{L}u \le 0$, then

$$\triangle w \le 2\langle X, \nabla w\rangle + (Q + div_\Sigma X - \|X\|^2) - \|\nabla w\|^2.$$

Multiply the above with φ^2, for $\varphi \in C_0^1(\Sigma)$, integrate on Σ and use divergence theorem,

$$-2\int_\Sigma \varphi\langle\nabla\varphi, \nabla w\rangle d\mu \le -2\int_\Sigma \varphi\langle\nabla\varphi, X\rangle d\mu + 2\int_\Sigma\langle X, \nabla w\rangle\varphi^2 d\mu$$

$$+\int_\Sigma (Q - \|X\|^2)\varphi^2 - \int_\Sigma \|\nabla w\|^2\varphi^2 d\mu.$$

$$\Longrightarrow\ -2\int_\Sigma \varphi\langle\nabla\varphi, \nabla w\rangle d\mu \le -\int_\Sigma \|X - \nabla w\|^2\varphi^2 - 2\int_\Sigma \varphi\langle\nabla\varphi, X\rangle + \int_\Sigma Q\varphi^2,$$

$$\Longrightarrow\ -\int_\Sigma Q\varphi^2 d\mu \le -2\int_\Sigma \varphi\langle X - \nabla w, \nabla\varphi\rangle d\mu - \int_\Sigma \|X - \nabla w\|^2\varphi^2 d\mu$$

$$\le 2\int_\Sigma |\varphi|\|X - \nabla w\||\nabla\varphi| d\mu - \int_\Sigma \|X - \nabla w\|^2\varphi^2 d\mu$$

$$\le \int_\Sigma |\nabla\varphi|^2 d\mu.$$

The (D.E.C.) dominant energy condition means that $Q \le \frac{1}{2}(R_\Sigma - \|\chi\|^2)$. Hence we finish the proof.

□

Assume: Σ is a asymptotic planar MOTS (in some slabs) and $n \geq 4$.

Definition 6.6. Σ is a strongly stable MOTS, if Σ is a stable MOTS, and the test function $u = 1 + O(|x|^{2-n})$, $\triangle u = O(|x|^{-n})$.

Propostion 6.7. If Σ is strongly stable, then

$$\frac{1}{2}\int_\Sigma (-R_\Sigma + \|\chi\|^2)\varphi^2 \leq \int_\Sigma |\nabla\varphi|^2 d\mu, \quad \forall\, \varphi \in C_0^1(\Sigma), \text{ or } \varphi - c \in C_0^1(\Sigma).$$

Proof. In the proof above, we need to take care of integration by part:

$$\int_\Sigma \varphi^2 \triangle w = -2\int_\Sigma \varphi\langle\nabla\varphi, \nabla w\rangle, \;\&\; \int_\Sigma \varphi^2 div_\Sigma X = -2\int_\Sigma \varphi\langle\nabla\varphi, X\rangle.$$

Here $w = \log u = O(|x|^{2-n})$, $\nabla w = O(|x|^{1-n})$ and $\triangle w = \frac{\triangle u}{u} - \frac{|\nabla u|^2}{u^2} = O(|x|^{-n})$. Then

- $$\int_\Sigma \varphi^2 \triangle w = \lim_{\rho\to\infty}\int_{B_\rho}\varphi^2\triangle w = \lim_{\rho\to\infty}\int_{\partial B_\rho}\varphi^2 \underbrace{\frac{\partial w}{\partial \nu}}_{\sim O(\rho^{1-n})} \underbrace{d\sigma}_{O(\rho^{2-n})} -2\int_{B_\rho}\varphi\langle\nabla\varphi,\nabla w\rangle$$
  $$= \lim_{\rho\to\infty} -2\int_{B_\rho}\varphi\langle\nabla\varphi,\nabla w\rangle = -2\int_\Sigma \varphi\langle\nabla\varphi,\nabla w\rangle.$$
- Using the decay of the data (g, h) under harmonic asymptotics, i.e. $g = \delta + O(|x|^{2-n})$ and $h = O(|x|^{1-n})$, $\Longrightarrow X = \sum_{i=1}^{n-1} h_{i,n}e_i = O(|x|^{1-n})$. Hence
  $$\int_{\partial B_\rho}\varphi^2\langle X,\nu\rangle d\sigma = \int_{\partial B_\rho} 1\cdot O(\rho^{1-n})\cdot O(\rho^{n-2})d|S^{n-2}| \to 0.$$

□

Results known about stable MOTS:

- Σ: compact stable MOTS; (M, g, h): (D.E.C.), then Σ is Yamabe-positive.
- $n = 3$: (M, g, h) (D.E.C.), $\nexists$ A.P. (asymptotically planar) stable MOTS.

  (By Gauss-Bonnet Theorem as in the time symmetric case.)
- (M, g, h) (D.E.C.), $\nexists$ A.P. strongly stable MOTS.

  (The existence of such $\Sigma \Longrightarrow \exists\, u > 0$, $R(u^{\frac{4}{n-2}}g) = 0$, with $u(x) = 1 + \frac{a}{2}|x|^{3-n} + l.o.t.$, $\Longrightarrow E(u^{\frac{4}{n-2}}g) < 0$, contradiction to Positive Energy Theorem.)

6.3 Jang equation and MOTS

Given initial data set (M^n, g, h), let $(\hat{M} = M \times \mathbb{R}^1, \hat{g}, \hat{h})$, where $\hat{g} = g + dt^2$, $\hat{h} = \pi^* h$, with $\pi : M \times \mathbb{R}^1 \to M$ given by $\pi(p,t) = p$. In local coordinates $\{x^1, \cdots, x^n\}$ on M, $\hat{h}_{ij} = \sum_{i,j=1}^n h_{ij} dx^i dx^j$.

- Given a function $f : M \to \mathbb{R}^1$ and the graph $G = graph_f = \{(x, f(x)) : x \in M\}$.
- Denote ∇ to be the connection on M and $\hat{M}$, and $\bar{\nabla}$ the connection on G.
- Jang equation: $\underline{H_G + Tr_G \hat{h} \equiv 0}$, where H_G the mean curvature of $G \subset M \times \mathbb{R}^1$. Hence Jang equation is just the MOTS equation $\theta_G = H_G + Tr_G \hat{h} = 0$ for graph G.
- The upper-ward unit normal is $\nu = \frac{(-\nabla f, 1)}{\sqrt{1+|\nabla f|^2}}$.
- The position vector $X = \big(x, f(x)\big)$; 2nd f.f. $A_{ij}(x) = \langle \nabla_i \nabla_j X, \nu \rangle = \frac{\nabla_i \nabla_j f}{\sqrt{1+|\nabla f|^2}}$.

  (The second equality is because:

$$A_{ij}(x) = \langle \nabla_{\partial_i + f_i \partial t}(\partial_j + f_j \partial_t), \nu \rangle = \langle \nabla_{\partial_i}(\partial_j + f_j \partial_t), \nu \rangle$$

$$+ \underbrace{f_i \langle \nabla_{\partial_t}(\partial_j + f_j \partial_t), \nu \rangle}_{=0,\text{ since } \partial_t \text{ is parallel.}} = \langle \nabla_i \nabla_j X, \nu \rangle.$$

So the Jang equation is

$$(J.E.): \quad \sum_{i,j=1}^n \bar{g}^{ij} \Big(\frac{\nabla_i \nabla_j f}{\sqrt{1+|\nabla f|^2}} + \hat{h}_{ij} \Big) = 0, \tag{6.1}$$

where the induced metric is $\bar{g}_{ij} = g_{ij} + \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}$.

(J.E.) and trivial data:

Suppose M^n is embedded in $\mathbb{R}^{n,1}$, i.e. $M^n \subset \mathbb{R}^{n,1}$, and M^n is space-like and Asymptotic Flat. Let $\{y^0, y^1, \cdots, y^n\}$ be the flat coordinates of $\mathbb{R}^{n,1}$. Assume M^n is given by a graph $y^0 = f(y^1, \cdots, y^n)$, with $|\partial f| < 1$(space-like) and $f \to 0$ at ∞. Then the induced metric and 2nd f.f. (g, h) are given by:

$$\begin{cases} g_{ij} = \delta_{ij} - (\partial_i f)(\partial_j f), \\ h_{ij} = -\frac{\partial_i \partial_j f}{\sqrt{1-|\partial f|^2}}. \end{cases} \quad \to \text{trivial data characterization.}$$

Propostion 6.8. Initial data (M^n, g, h) is trivial $\iff \exists\, f$ solution of $(J.E.)$, with $f \to 0$ at ∞, such that

$$G = graph_f \text{ is isometric to } \mathbb{R}^n; \quad \chi = A_G + \hat{h}|_G \equiv 0.$$

Proof. (" $\Longleftarrow$ ") Let $\{y^1, \cdots, y^n\}$ be the Euclidean coordinates on $G \cong \mathbb{R}^n$. We can push down $\{y^1, \cdots, y^n\}$ to be coordinates on M by the graphical diffeomorphism $\pi : G \to M$. Then $\bar{g}_{ij} = \delta_{ij} = g_{ij} + f_{y^i} f_{y^j}$. Hence

$$g_{ij} = \delta_{ij} - f_{y^i} f_{y^j} \text{ is the metric induced of Graph in } \mathbb{R}^{n,1}.$$

Now construct a mapping

$$\varphi : G \to \mathbb{R}^{1,n}, \text{ by } \varphi : y \in G \to \big(f(y), y\big) \in \mathbb{R}^{1,n},$$

we know that the induced metric is just g_{ij}.

$$\text{What left to } \underline{\text{show}} \text{ is:} \quad h_{ij} = -\frac{\partial_i \partial_j f}{\sqrt{1 - |\partial f|^2}}.$$

Since $\chi = A_G + \hat{h}|_G \equiv 0$, $A_G = -\hat{h}_G$. Take an o.n. basis $\{e_1, \cdots, e_n\}$ for G, and view f as a function on G, i.e. $f|_G = t|_G$, where t is the time function, then $\bar{\nabla}_j f = \langle e_j, \frac{\partial}{\partial t} \rangle$. Moreover,

$$\bar{\nabla}_i \bar{\nabla}_j f = \langle \nabla_{e_i} e_j, \frac{\partial}{\partial t} \rangle = \underbrace{(\bar{\nabla}_{e_i} e_j)}_{=0} + A_{ij} \langle \nu, \frac{\partial}{\partial t} \rangle = A_{ij} \frac{1}{\sqrt{1 + |\nabla f|^2}}.$$

Hence $\underline{A_{ij} = \sqrt{1 + |\nabla f|^2} \bar{\nabla}_i \bar{\nabla}_j f}$.

$$\underline{\text{Claim}}: \quad \sqrt{1 + |\nabla f|^2} = \frac{1}{\sqrt{1 - |\bar{\nabla} f|^2}}.$$

To show the claim, consider $\pi : G \to M$, with $\pi^{-1}(x) = (x, f(x))$, so $det\big[(\pi^{-1})_*\big] = \sqrt{1 + |\nabla f|^2}$. On the other hand, since $\bar{g}_{ij} = \delta_{ij}$, $det(\pi_*) = \sqrt{det(g)}$. Furthermore since $g_{ij} = \delta_{ij} - (\partial_i f)(\partial_j f)$, $\sqrt{det(g)} = \sqrt{1 - |\partial f|^1} = \sqrt{1 - |\bar{\nabla} f|^2}$. Combing above,

$$h_{ij} = -A_G = -\sqrt{1 + |\nabla f|^2} \bar{\nabla}_i \bar{\nabla}_j f = -\frac{\bar{\nabla}_i \bar{\nabla}_j f}{\sqrt{1 - |\bar{\nabla} f|^2}} = -\frac{\partial_i \partial_j f}{\sqrt{1 - |\partial f|^2}}.$$

□

7 Space-time Positive Energy Theorem

Theorem 7.1. (Schoen-Yau [10]) (D.E.C.) $\Longrightarrow E \geq 0$. $E = 0$ only if (M, g, h) is trivial.

Proof. Assume can solve (J.E.) for f and $\Sigma = \{(x, f(x)) : x \in M\}$ is the graph. Under the decay of the initial data $g = \delta + O(|x|^{-p})$ and $h = O(|x|^{-1-p})$ for $p > \frac{n-2}{2}$, the decay of f is

$$f(x) = O(|x|^{1-p}), \quad \partial f = O(|x|^{-p}),$$

by (J.E.). So the induced metric $\bar{g}_{ij} = g_{ij} + f_i f_j = g_{ij} + O(|x|^{-2p})$, with $2p > n - 2$, so $\underline{\bar{E} = E}$.

- **Claim**: Σ is strongly stable.

  Pf: If $v = \langle \nu, \frac{\partial}{\partial t} \rangle$, then $v = \frac{1}{\sqrt{1+|\nabla f|^2}}$. Hence $v > 0$ and $v = 1 + O(|x|^{-2p})$.

  Notice that vertical translation of the graph is invariant. Let $\Sigma_t = \{(x, f(x)+t) : \ x \in M\}$. Then $\mathcal{L}v = \frac{d}{dt}\big|_{t=0} \theta_{\Sigma_t} = 0$. Hence

$$\mathcal{L}v = 0, \ v > 0, \ v = 1 + O(|x|^{-2p}), 2p > n-2, \Longrightarrow \ \Sigma \text{ is strongly stable.}$$

- Let $\tilde{g} = u^{\frac{4}{n-2}} \bar{g}$ on Σ. We want $\tilde{R} \equiv 0$. Here $|\bar{R}| \sim |R| + O(|x|^{-2p}) = O(|x|^{-n-\alpha})$ by A.F. conditions. Hence by Proposition 6.7 and similar ideas as in Step 2 of Chap **??**, $\exists\, u > 0$, which solves:

$$\bar{\triangle} u - c(n)\bar{R}u = 0 \ \text{ on } \Sigma, \quad u \to 1 \text{ at } \infty.$$

- By strong stability, we get $\frac{1}{2}\int_\Sigma (-\bar{R} + \|\chi\|^2) u^2 \overline{d\mu} \le \int_\Sigma |\bar{\nabla} u|^2 \overline{d\mu}$.

$$\Longrightarrow \quad (2 - c(n)^{-1}) \int_\Sigma |\bar{\nabla} u|^2 + \int_\Sigma (-\bar{R} + \|\chi\|^2) u^2 \le c(n)^{-1} \int_\Sigma |\bar{\nabla} u|^2$$

$$= c(n)^{-1} \lim_{\rho\to\infty} \int_{B_\rho} |\bar{\nabla} u|^2 = c(n)^{-1} \lim_{\rho\to\infty} (-\int_{B_\rho} u\bar{\triangle} u + \int_{\partial B_\rho} u \bar{\nabla}_\eta u d\sigma).$$

  Using the equality $\bar{\triangle} u - c(n)\bar{R}u = 0$, $\Longrightarrow$

$$\big(2 - c(n)^{-1}\big) \int_\Sigma |\bar{\nabla} u|^2 + \int_\Sigma \|\chi\|^2 u^2 \le c(n)^{-1} \lim_{\rho\to\infty} \int_{\partial B_\rho} u\bar{\nabla}_\eta u d\sigma = -(n-2)c(n)^{-1}|S^{n-1}|a,$$

  where $u = 1 + a|x|^{2-n} + l.o.t.$. So $a < 0$, hence

$$\tilde{E} = \bar{E} + a = E + a \le E.$$

- If $E = 0$, then $a = 0$,

$$\big(2 - c(n)^{-1}\big) \int_\Sigma |\bar{\nabla} u|^2 + \int_\Sigma \|\chi\|^2 u^2 = 0,$$

  $\Longrightarrow u \equiv 1$, and $\chi \equiv 0$. Hence $\tilde{E} = 0$, $\to \tilde{g} = \bar{g}$ is flat. So $(\Sigma, \bar{g})$ is isometric to $\mathbb{R}^n$. $\Longrightarrow$ (M, g, h) is trivial.

□

Remark 7.2. In fact, we must allow the solution f of (J.E.) to blow-up. f blows up to $+\infty$ at the cylinder over stable future MOTS, and blows down to $-\infty$ to stable past MOTS.

8 Space-time Positive Mass Theorem

(M^n, g, h) A.F.. E and P are energy and linear momentum.

Theorem 8.1. (Eichmair, Huang, Lee, Schoen [5]) For $3 \leq n \leq 7$, if D.E.C. holds, then $E \geq |P|$.

<u>**Density Theorem**</u>: If $E < |P|$, then $\exists$ $(\tilde{g}, \tilde{h})$ satisfying <u>strict</u> D.E.C. (i.e. $\tilde{\mu} > |\tilde{J}|$), and $(\tilde{g}, \tilde{h})$ are in harmonic asymptotics, and $\tilde{E} < |\tilde{P}|$.

<u>**W.L.O.G.**</u>: Assume Strict D.E.C. and Harmonic asymptotics (Definition 4.6):

$$\begin{cases} g = u^{\frac{4}{n-2}}\delta, \\ \pi = u^{\frac{2}{n-2}}\big(L_Y\delta - (div_\delta Y)\delta\big), \end{cases} \quad \text{near } \infty.$$

By harmonic expansion,

$$u(x) = 1 + \frac{E}{2}|x|^{2-n} + O(|x|^{1-n}),$$
$$Y_i(x) = -\frac{n-1}{n-2}P_i|x|^{2-n} + O(|x|^{1-n}).$$

Propostion 8.2. If $E < |P|$ and we choose coordinates such that $\vec{P} = (0, \cdots, 0, |P|)$, then the slab $S_\Lambda = \{x = (\hat{x}, x_n) : \ |x_n| \leq \Lambda\}$ is a trapped region.

Proof. Consider $\Sigma_\Lambda = \{x_n = \Lambda\}$. The expansion is $\theta = -(H_\Sigma + Tr_\Sigma h)$. First, as in step 2 of Theorem 5.3, we can compute the mean curvature of Σ_Λ as,

$$H_\Sigma = -\frac{2(n-1)}{n-2}u^{-\frac{n}{n-2}}\partial_n u = (n-1)E\frac{x_n}{|x|^n} + O(|x|^{-n}).$$

Then we calculate $Tr_\Sigma h$ near ∞. $Tr\pi = (Trh) - n(Trh) = (1-n)(Trh)$, then

$$h = \pi + (Trh)g = \pi - \frac{1}{n-1}(Tr\pi)g = u^{\frac{2}{n-2}}\big(L_Y\delta - (div_\delta Y)\delta\big) + \frac{n-2}{n-1}u^{\frac{2}{n-2}}(div_\delta Y)\delta$$
$$= u^{\frac{2}{n-2}}\big(L_Y\delta - \frac{1}{n-1}(div_\delta Y)\delta\big).$$

Since g is conformally flat near ∞,

$$Tr_\Sigma h = Tr_\Sigma\big\{u^{\frac{2}{n-2}}[Y_{i,j} + Y_{j,i} - \frac{1}{n-1}(\sum_{k=1}^n Y_{k,k})\delta_{ij}]\big\}, \quad \text{for } 1 \leq i,j \leq n-1$$
$$= -\frac{n-1}{n-1}Y_{n,n} + O(|x|^{-n}), \quad \text{since } Y_i(x) = O(|x|^{1-n}), \text{ for } 1 \leq i \leq n-1$$
$$= -(n-1)|P|\frac{x_n}{|x|^n} + O(|x|^{-n}), \quad \text{since } Y_n(x) = -\frac{n-1}{n-2}|P||x|^{2-n} + l.o.t..$$

Adding together,

$$\theta = -(H_\Sigma + Tr_\Sigma h) = -(n-1)(E - |P|)\frac{x_n}{|x|^n} + O(|x|^{-n}).$$

So

- $x_n = +\Lambda \gg 1, \Longrightarrow \theta_\Lambda > 0;$
- $x_n = -\Lambda \ll -1, \Longrightarrow \theta_{-\Lambda} < 0.$

□

Constructing stable MOTS: As in Step 3 of Theorem 5.3, we consider the boundary $\Gamma_{h,\rho} = \{x_n = h\} \cap \{|\hat{x}| = \rho\}$. Solve the Dirichlet Problem for stable MOTS with boundary $\Gamma_{h,\rho}$, we get

$$\Sigma_{h,\rho} - -\text{stable MOTS}, \quad \text{with } \partial\Sigma_{h,\rho} = \Gamma_{h,\rho}.$$

- Consider the region which is the intersection between the intersection of the trapped slab S_Λ and the cylinder $\mathcal{C}_\rho = \{x = (\hat{x}, x_n) : \ |\hat{x}| \le \rho\}$. The upper and lower boundary is trapped by the previous Proposition. The side boundary $\mathcal{C}_\rho$ is also trapped because the mean curvature decays as $-H_{\mathcal{C}} \sim \frac{1}{\rho}$, and $|Tr_{\mathcal{C}} h| \sim o(\frac{1}{\rho^{n/2}})$, hence $-H > |Tr_{\mathcal{C}} h|$. So the whole region inside S_Λ and the cylinder $\mathcal{C}_\rho$ is a trapped region. Hence $\Sigma_{h,\rho}$ lies entirely inside the region inside S_Λ and the cylinder $\mathcal{C}_\rho$.
- Now taking limits as $\rho \to \infty$,

$$\Sigma_h = \lim_{\rho\to\infty} \Sigma_{h,\rho}, \quad \text{(need local volume and 2nd f.f. bounds).}$$

- Volume estimates for stable MOTS: Recall that the MOTS Σ is the blow-up sets of the MOTS equation:

$$\bar{g}^{ij}\big(\frac{D^2_{ij} f}{\sqrt{1+|\nabla f|^2}} + \hat{h}_{ij}\big) = div_M\big(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\big) + \bar{g}^{ij}\hat{h}_{ij} = 0,$$

where $\bar{g}_{ij} = g_{ij} + f_{x^i} f_{x^j}$, $\bar{g}^{ij} = g^{ij} - \frac{\nabla^i f \nabla^j f}{1+|\nabla f|^2}$. Denote $\eta = \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$, then

$$|\eta| \le 1, \quad \eta = -\nu, \text{ on } \Sigma.$$

Given a ball B_ρ in M, Σ cuts B_ρ into two connected components, and consider one of the connected component Ω. Using divergence Theorem,

$$\int_\Omega -\bar{g}^{ij}\hat{h}_{ij} = \int_\Omega div_M \eta = \int_{\partial\Omega} \eta\cdot\nu = \int_{\Sigma\cap\Omega} \eta\cdot(-\nu) + \int_{\partial\Omega\cap\Sigma} \eta\cdot\nu.$$

Using the above fact, and noting that $\bar{g}^{ij}\hat{h}_{ij}$ is bounded, $\Longrightarrow$

$$\int_{\Sigma\cap\Omega} \eta\cdot(-\nu) = |\Sigma\cap B_\rho|, \ \int_{\partial\Omega\cap\Sigma} \eta\cdot\nu \ge -|\partial B_\rho|, \ \text{ and } \int_\Omega -\bar{g}^{ij}\hat{h}_{ij} \le \big(\sup_{B_\rho}|h|\big) vol(B_\rho).$$

Putting them together,

$$\Longrightarrow |\Sigma \cap B_\rho| \le |\partial B_\rho| + \big(\sup_{B_\rho} |h|\big) vol(B_\rho).$$

Hence Σ_h exists by volume estimates and curvature estimates (see the original paper). By similar arguments as in step 3 of Theorem 5.3, we have

- Σ_h is a graph of $x^n = f(x^1, \cdots, x^{n-1})$ near ∞;
- $|f| \le \Lambda$;
- $|\nabla f| \le C/|x|$.

Claim: When $n \ge 4$, $f(x) = \alpha + O\big(|x|^{(3-n)+\delta}\big)$ by the MOTS equation similar as in Step 1 of §5.3.

Case $n = 3$: $\Sigma^2 \subset M^2$ stable MOTS, $\Sigma \subset S_\Lambda$. Using Proposition 6.5,

$$\frac{1}{2} \int_\Sigma -R_\Sigma \varphi^2 d\mu < \int_\Sigma |\nabla \varphi|^2 d\mu, \quad \forall\, \varphi \in C^1_c(\Sigma), \varphi \ne 0.$$

Using the log cutoff trick, $\exists\ \varphi_i (\in C^1_c) \to 1$ on compact subset of Σ, and $\int_\Sigma |\nabla \varphi_i|^2 \to 0$. Using Gauss-Bonnet,

$$\Longrightarrow\ 2\pi\chi(\Sigma) - \lim_\rho \int_{\Sigma \cap \partial B_\rho} k_g = \int_\Sigma K d\mu = \lim_{\rho \to \infty} \int_{\Sigma \cap B_\rho} K d\mu > 0.$$

Since $\lim_\rho \int_{\Sigma \cap \partial B_\rho} k_g = 2\pi, \Longrightarrow \chi(\Sigma) > 1$, contradiction to Σ not compact.

Case $n \ge 4$: Need to find strongly stable MOTS Σ, which is asymptotically planar. Let $\bar{g}$ be the induced metric, i.e.

$$\bar{g}_{ij} = u^{\frac{4}{n-2}} (\delta_{ij} + f_{x_i} f_{x_j}) = \delta_{ij} + O(|x|^{2-n}).$$

- So $(\Sigma, \bar{g})$ has energy $\bar{E} = 0$.
- Using the stability, $\exists\ u > 0$, $u \to 1$ at ∞, such that $\triangle u - c(n)\bar{R}u = 0$ (see Step 2 in §5.3). Hence consider the conformal change $\hat{g} = u^{\frac{4}{n-3}} \bar{g}$, $\hat{R} = 0$ is scalar flat.
- Furthermore, $\hat{E} < 0$ by strong stability (see Step 3 in §5.3), hence a contradiction by induction.

Strongly stability: In the Riemannian case (Theorem 5.3), in order to get a strongly stable minimal surface, we take a volume minimizer $\Sigma_{h_\rho, \rho}$ among all $\{\Sigma_{h,\rho} :\ \rho \in (-\Lambda, \Lambda)\}$, then $\delta^2 |\Sigma_\rho|(X, X) \ge 0$ for $X = \partial_n$ near $\partial \Sigma_\rho$. There

$$\frac{d}{dh} |\Sigma_{h,\rho}| = \int_{\Sigma_{h,\rho}} div_\Sigma X d\mu = \int_{\partial \Sigma_{h,\rho}} \langle X, \eta \rangle d\sigma,$$

where η is the unit outer co-normal for $\Sigma_{h,\rho}$.

In the MOTS case, for fixed ρ, we similarly consider

$$F(h) = \int_{\Gamma_{\rho,h}} \langle \partial_n, \eta \rangle d\sigma,$$

where η is the unit outer co-normal for the stable MOTS $\Sigma_{h,\rho}$.

- Want to find h_ρ, $F'(h_\rho) \geq 0$.
- By the barrier arguments as step 3 in §5.3, $F(\Lambda) > 0 > F(-\Lambda)$. This is because $\langle \partial_n, \eta \rangle > 0$ on $\Gamma_{\Lambda,\rho}$; and $\langle \partial_n, \eta \rangle < 0$ on $\Gamma_{-\Lambda,\rho}$.
- $F(h)$ may have jumps.
- If F is not continuous at h_0, then

$$\lim_{h \nearrow h_0} F(h) > \lim_{h \searrow h_0} F(h).$$

  This is because: If h_0 is a jump, let $\overline{\Sigma}_{h_0,\rho}$ be the lower bound of un-trapped surfaces, and $\underline{\Sigma}_{h_0,\rho}$ be the upper bound of trapped surfaces. Then $\overline{\Sigma}_{h_0,\rho}$ lies above $\underline{\Sigma}_{h_0,\rho}$. Here $\underline{\Sigma}_{h_0,\rho}$ and $\overline{\Sigma}_{h_0,\rho}$ are upper and lower MOTS. Hence $F(h_0)\big|_{\underline{\Sigma}_{h_0,\rho}} > F(h_0)\big|_{\overline{\Sigma}_{h_0,\rho}}$, since the unit co-normal of $\underline{\Sigma}_{h_0,\rho}$ is "above" that of $\overline{\Sigma}_{h_0,\rho}$.
- Let $h_\rho = \inf\{h : F(f) > 0\}$. Then F is continuous at h_ρ, and $F(h_\rho) = 0$.
- $\Sigma_{h_\rho,\rho}$ satisfies the strongly stability conditions (details referred to the original paper).

9 Penrose inequality

9.1 Motivation and statement

- Schwartzchild: when $\underline{n = 3}$, is given by $(\mathbb{R}^3 \setminus \{0\}, g_0)$,

$$g_0 = (1 + \frac{m}{2|x|})^4 \delta_{ij}.$$

  When $r = |x| = \frac{m}{2}$, we have a horizon(minimal surface) $\Sigma = \{|x| = \frac{m}{2}\}$. Consider the restricted metric on Σ: $g_0|_\Sigma = 16 d\xi_{m/2}$, where $d\xi_{m/2}$ is the standard metric on $\{|x| = \frac{m}{2}\} \subset \mathbb{R}^3$. Hence

$$A = Area(\Sigma) = 16 \cdot 4\pi \cdot (\frac{m}{2})^2 = 16\pi m^2.$$

  So we have the equality connecting ADM mass m and horizon area:

$$\underline{m = \sqrt{\frac{A}{16\pi}}}.$$

- Consider an arbitrary 3-dimensional initial data set (M, g, h), with $\Sigma = \partial M$. Assume Σ is a MOTS, and M contains no other compact MOTS, then the **Penrose Inequality** states that:

$$\underline{\text{D.E.C.}} \Longrightarrow \quad m \geq \sqrt{\frac{A}{16\pi}},$$

and equality holds only if $(M, g, h) \hookrightarrow \mathcal{S}^4$, where $\mathcal{S}^4$ is the Schwartzchild space-time.

- In case $\underline{h = 0}$, the statement reduces to **Riemannian Penrose inequality**:

$$\mathbf{R_g \geq 0}, \Longrightarrow \quad \mathbf{m} \geq \sqrt{\frac{\mathbf{A}}{\mathbf{16}\pi}};\ \text{“ = ” only for } (\mathbb{R}^3 \setminus \mathbf{B_{m/2}}, \mathbf{g_0}).$$

- The Schwartzchild space-time is $(\mathcal{S}^4, g_S)$, where $\mathcal{S}^4 = \mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, and $g_S = -V(|x|)^2 dt^2 + g_0 = -(1 - \frac{2m}{|x|})^2 dt^2 + \underbrace{(1 - \frac{2m}{|x|})^2 dr^2 + r^2 d\xi^2}_{g_0}$.

  Note that $V \equiv 0$ on $\mathbb{R}$. Now we construct a graph (M^3, g, h) outside $\mathbb{R} \times \Sigma$, where $M = graph_f = \{(t, x) \in \mathcal{S}^4 :\ t = f(x)\}$, $g = g_S|_{graph} = g_0 - V^2 df \otimes df$, and h is the induced 2nd f.f.. Hence we can construct an example where $|E|^2 - |P|^2 = m^2$, which has strict inequality in what we proposed. i.e. $E > \sqrt{\frac{A}{16\pi}}$.

<u>Results in $h = 0$ case (Riemannian Penrose Inequality (R.P.I.))</u>:

In the times symmetric ($h \equiv 0$) case (M, g),

- MOTS reduces to minimal surface, and D.E.C. (dominant energy condition) reduces to $R_g \geq 0$.

There are two approaches to the time symmetric case. One is the Inverse Mean Curvature Flow method by G. Huisken and T. Ilmenan [6], and the other is the Conformal Flow of Metric method by H. Bray [3]. Let us discuss them separately in the following.

9.2 Inverse Mean Curvature Flow ($n = 3$)

This method is proposed by R. Geroch, P. Jang and R. Wald, and eventually solved by Huisken and Ilmenan.

In this case, the only MOTS Σ is then a minimal surface. They proposed to find a family of 2-surfaces $\{\Sigma_t\}_{t=0}^{\infty}$ with $\Sigma_t \simeq S^2$ starting from $\Sigma_0 = \Sigma$, and evolves to infinity in the AF end.

- <u>Hawking Mass</u>: Given a 2-surface $\Sigma \simeq S^2 \subset M$, the Hawking mass is defined as:

$$m_H(\Sigma) = \sqrt{\frac{A}{16\pi}} \int_\Sigma (K - \frac{1}{4} H^2) d\mu = \sqrt{\frac{A}{16\pi}} (4\pi - \frac{1}{4} \int_\Sigma H^2 da).$$

Since Σ_0 is minimal, i.e. $H = 0$, $m_H(\Sigma_0) = 4\pi\sqrt{\frac{A}{16\pi}}$, which is just $4\pi m$ is the Schwartzchild metric.

The Hawking mass of the large sphere $S_\rho = \{|x| = \rho\}$ tends to the ADM mass, i.e. $4\pi m = \lim_{\rho\to\infty} m_H(S_\rho)$.

- There exists a flow $\{\Sigma_t\}_{t=0}^{\infty}$, such that $\frac{d}{dt}m_H(\Sigma_t) \geq 0$, and " $= 0$" only for the $t = 0$ slice in the Schwartzchild $\mathcal{S}^4$.
- The *Inverse Mean Curvature Flow* (*$1/H$ flow*) is an evolution $F : S^2 \times [0, \infty) \to M$, defined by:
$$\frac{dF}{dt} = \frac{1}{H}\vec{\nu}, \quad F_0(S^2) = \Sigma. \tag{9.1}$$
where $H > 0$ is the mean curvature for $\Sigma_t = F_t(S^2) = \partial\Omega_t$, which is a boundary for some region Ω_t, and $\vec{\nu}$ is the unit outer normal of Σ_t.

Assume we have a smooth solution to the $1/H$-flow (9.1), let us calculate the evolution of the Hawking mass $m_H(\Sigma_t)$.

$$\frac{\frac{d}{dt}m_H(\Sigma_t)}{m_H(\Sigma_t)} = \frac{1}{2}\frac{A'}{A} - \frac{1}{4}\frac{\int_{\Sigma_t} 2H \cdot H' d\mu + H^2(d\mu)'}{4\pi - \frac{1}{4}\int_\Sigma H^2 d\mu}$$

Along the IMCF, we have

$$(d\mu_t)' = H \cdot |\frac{dF}{dt}| d\mu_t = d\mu_t;$$

$$A' = \frac{d}{dt}A(\Sigma_t) = \int_{\Sigma_t} (d\mu)' = A;$$

and

$$H' = \frac{\partial}{\partial t}H = L_{\Sigma_t}|\frac{dF}{dt}| = -\triangle_{\Sigma_t}\frac{1}{H} - \big(Ric(\nu,\nu) + |A|^2\big)\frac{1}{H},$$

where L_{Σ_t} is the stability operator for the 2nd variation. The Gauss equation on Σ_t gives us:

$$Ric(\nu,\nu) = \frac{1}{2}(R_M - \sum_{i,j=1}^{2} R^M_{jiij}) = \frac{1}{2}(R_M - R_\Sigma + H^2 - |A|^2).$$

Hence using $\underline{R_M \geq 0}$,

$$H \cdot H' = -H\triangle\frac{1}{H} - \big(Ric(\nu,\nu) + |A|^2\big) \leq -H\triangle\frac{1}{H} - \frac{1}{2}(-R_\Sigma + H^2 + |A|^2).$$

Using $R_\Sigma = 2\kappa$ with κ the Gaussian curvature and the Gauss-Bonnet theorem,

$$\int_{\Sigma_t} 2H \cdot H' d\mu + H^2(d\mu)' \leq \int_{\Sigma_t} [-2H\triangle\frac{1}{H} + \underbrace{R_{\Sigma_t}}_{=2\kappa} - H^2 - |A|^2 + H^2] d\mu$$

$$= 8\pi + \int_{\Sigma_t} -2\frac{|\nabla H|^2}{H^2} - |A|^2 \leq 8\pi - \frac{1}{2}\int_{\Sigma_t} H^2.$$

Plug in to the formula for $\frac{d}{dt}m_H(\Sigma_t)$,

$$\frac{\frac{d}{dt}m_H(\Sigma_t)}{m_H(\Sigma_t)} \geq \frac{1}{2} - \frac{1}{4}\cdot\frac{8\pi - \frac{1}{2}\int_{\Sigma_t} H^2}{4\pi - \frac{1}{4}\int_{\Sigma} H^2 d\mu} = 0.$$

Furthermore, "$= 0$" only if $H = const$ and $A = \frac{1}{2}Hg$.

- The monotonicity $\frac{d}{dt}m_H(\Sigma_t) \geq 0$ along a smooth solution of the IMCF is called *Geroch monotonicity formula*.
- However, smooth solution of IMCF does not always exist. A counterexample is a thing torus, which can be mean convex, i.e. $H > 0$, but the IMCF develops singularity at finite time.
- Huisken-Ilmenan develops a weak solution to IMCF. They proposed a level set flow of a function u, such that $\Sigma_t = \{x \in M : u(x) = t\}$. Hence the IMCF equation (9.1) becomes:

$$div(\frac{\nabla u}{|\nabla u|}) = |\nabla u|, \quad \Sigma = \{u(x) = 0\}.$$

- Note: If Σ_t exists globally, each Σ_t is outer minimizing.

  In fact, $div(\frac{\nabla u}{|\nabla u|}) \geq 0$ by definition. Consider any other surface $\tilde{\Sigma}$ enclosing Σ_t, i.e. there exists a region $\Omega \subset M$, such that $\partial\Omega = \tilde{\Sigma} - \Sigma_t$. Using the divergence theorem for $div(\frac{\nabla u}{|\nabla u|})$ in Ω, and observing that the unit outer normal of Σ_t is just $\frac{\nabla u}{|\nabla u|}$,

$$0 \leq \int_{\Omega} div(\frac{\nabla u}{|\nabla u|}) = \int_{\tilde{\Sigma}} \langle \frac{\nabla u}{|\nabla u|}, \tilde{\nu}\rangle - |\Sigma_t| \leq |\tilde{\Sigma_t}| - |\Sigma_t|.$$

- In Huisken-Ilmeman's weak solution, if Σ_{t_0} is not outer minimizing anymore, the weak flows just replaces it by the outer-minimizing hall $\tilde{\Sigma}_{t_0}$.

Remark 9.1. This method could not solve the full problem in the case of multi-blackholes.

9.3 H. Bray's Conformal Flow of Metrics

- Bray's method works for R.P.I. for disconnected MOTS, and the case $3 \leq n \leq 7$ (with D. Lee [4]).
- Bray's method uses Positive Mass Theorem, while Huisken-Ilmanen's method can give another proof of Positive Mass Theorem.

Assume the initial data set (M, g_0) is a asymptotically flat, with an outmost horizon boundary $\Sigma_0 = \partial M$. Assume furthermore g_0 is harmonic flat near ∞, i.e. $g_0 = v^4\delta$, where δ is the standard metric and v is a harmonic function in the asymptotically flat coordinates $x = \{x^i\}_{i=1}^3$. Hence

$$v = 1 + \frac{m_0}{2|x|} + O(|x|^{-2}),$$

where m_0 is the ADM mass of (M, g_0).

The idea is to find a flow of the metrics which are conformal to the original one. We want to construct a flow like follows:

- Let $g_t = u_t^4 g_0$, where $u_t > 0$ is a positive function defined in the below.
- Let Σ_t be the outer-minimizing minimal surface for g_t.
- $R_{g_t} \equiv 0$ in $M \setminus \Sigma_t$.

Key Properties:

- $A(t) = Area(\Sigma_t, g_t) \equiv A(0) = Area(\Sigma)$;
- The ADM $m(t)$ mass of g_t is non-increasing;
- For t large enough, (M, g_t, Σ_t) is diffeomorphic to a Schwartzchild solution.

These properties imply the Riemannian Penrose Inequality.

Flow is defined by: First define v_t as

$$\begin{cases} \triangle_{g_0} v_t = 0, & \text{on } M \setminus \Sigma_t; \\ v_t = 0, & \text{on } \Sigma_t; \\ v_t(x) \to -e^{-t}, & \text{as } x \to \infty. \end{cases}$$

Assume $v_t \equiv 0$ inside Σ_t, then v_t is super harmonic, i.e. $\lambda v_t \le 0$.

Define u_t by $\frac{d}{dt}u_t = v_t$, $u_0 \equiv 1$, then $u_t(x) = 1 + \int_0^t v_s(x)ds$. Hence u_t is harmonic on $M \setminus \Sigma_t$, and super-harmonic, i.e. $\triangle u_t \le 0$ on M.

The existence theory of this flow is referred to H. Bray's paper [3]. Now let us check some of the key properties.

1. $Area(t)$ constant:

$$\frac{d}{dt}\Big|_{t=t_0} Area(\Sigma_t, g_t) = \underbrace{\delta Area(\Sigma_t, g_{t_0})}_{I_1=0} + \underbrace{\int_{\Sigma_{t_0}} \frac{d}{dt}\Big|_{t=t_0}(da_t)}_{I_2=0} = 0.$$

$I_1 = 0$ because Σ_{t_0} is a minimal surface in (M, g_{t_0}); $I_2 = 0$ is due to the fact that $\frac{d}{dt}g_t\big|_{\Sigma_t} = 4\frac{d}{dt}u_t\big|_{\Sigma_t} u_t^3 g_0 = 0$, since $\frac{d}{dt}u_t\big|_{\Sigma_t} = v_t\big|_{\Sigma_t} \equiv 0$.

2. $m(t)$ non-increasing:

- For the metric $u^4\delta$, where $u(x) = a + \frac{b}{|x|} + O(|x|^{-2})$ near ∞, the ADM mass $m\big(u^4\sum(dx_i)^2\big) = 2ab$.

  Proof: $u^4\sum(dx_i)^2 = a^4\big(1+\frac{b}{a|x|}+O(|x|^{-2})\big)^4(dx_i)^2$. Let $y_i = a^2x_i$, then metric changes to $\big(1+\frac{ab}{|y|}+O(|y|^{-2})\big)^4|dy|^2$.
- $g_t = (u_tv)^4\delta$ near ∞. Since u_t is harmonic near ∞,

$$u_t(x) = \alpha(t) + \frac{\beta(t)}{|x|} + O(|x|^{-2}).$$

  Counting the expansion for v above,

$$(u_tv)(x) = \alpha(t) + \frac{\big(\beta(t)+\alpha(t)\frac{m}{2}\big)}{|x|} + O(|x|^{-2}).$$

  So the AMD mass

$$m(t) = 2\alpha(t)\big[\beta(t)+\alpha(t)\frac{m_0}{2}\big] = 2\alpha(t)\beta(t)+\alpha^2(t)m_0.$$

  To show $m'(t)\le 0$, we only need to show $m'(0)\le 0$.
- $\frac{d}{dt}\big|_{t=0}u_t(x) = v_0(x) = \alpha'(0) + \frac{\beta'(0)}{|x|} + l.o.t.$. Where

$$\begin{cases} \triangle v_0 = 0, & \text{on } M\setminus\Sigma_0; \\ v_0 = 0, & \text{on } \Sigma_t; \\ v_0(x)\to -1, & \text{at } \infty. \end{cases}$$

  So $\alpha'(0) = -1$. Using $u_0\equiv 0 \Longrightarrow \alpha(0)=1$, $\beta(0)=0$, hence

$$m'(0) = -2\beta(0) + 2\alpha(0)\beta'(0) - 2m_0 = \underline{2\beta'(0)-2m_0}.$$

- To show that $m'(0)\le 0$, we double $(M, \partial M = \Sigma)$ along ∂M to get $\tilde{M} = M\cup|_\Sigma M_{-1}$, where $M_{-1}\simeq M$. Do an odd reflection for v_0, such that $v_0(x) = -v_0(x^*)$, where x^* is the reflection of x under $M_{-1}\to M$. Let

$$\tilde{g} = (\frac{1-v_0}{2})^4 g_0.$$

  Here $\frac{1-v_0}{2}\to 1$ at ∞, and $\frac{1-v_0(x)}{2} = 1 - \frac{\beta'(0)}{2|x|} + O(|x|^{-2})$ as $x\to+\infty$. This conformal change collapses the infinity ∞ of (M_{-1}, g_0) to a single point. Then we get a complete manifold with a single asymptotically flat end $(\tilde{M},\tilde{g})$. The harmonicity of v_0 implies that $\tilde{R}=0$. Applying the Positive Mass Theorem, $\Longrightarrow \tilde{m}\ge 0$. Since $\tilde{m} = m_0 - \beta'(0)$ by the expansion of $\frac{1-v_0}{2}$, $\Longrightarrow$ $m'(0) = 2(\beta'(0)-m_0)\le 0$.

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