Geometric variational theory and applications

Xin Zhou

MIT

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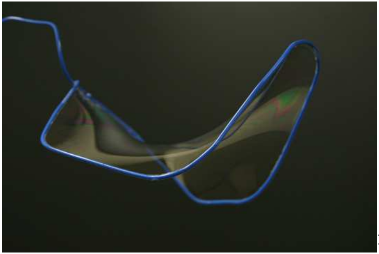
Outline

1. Introduction to min-max theory
2. My results
   - Morse index one Conjecture
   - Systolic and Waist inequalities
   - Morse indices arbitrarily large Conjecture
   - Gaussian space and Entropy Conjecture

Soap film and Plateau Problem

Given a closed curve Γ in $\mathbb{R}^3$, can we find a surface bounding Γ that minimizes area?

- Let us motivate this problem by the following famous physical experiment due to Belgian physicist Plateau:

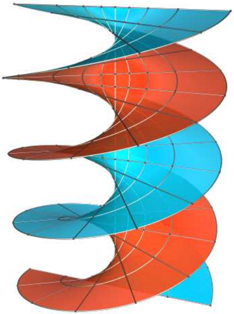
[1]

The surface tension will make the soap film to acquire least area.

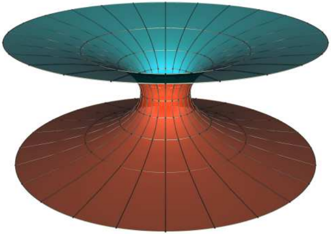
- Call the surface which locally minimizes area a minimal surface.
- Mathematically solved by Douglas, Rado in 1931— won Douglas the Fields Medal in 1936.

[1] From: http://www.math.hmc.edu/ jacobsen/demolab/soapfilm.

Examples



(a) The helicoid: $(t \cos s, t \sin s, s)$.

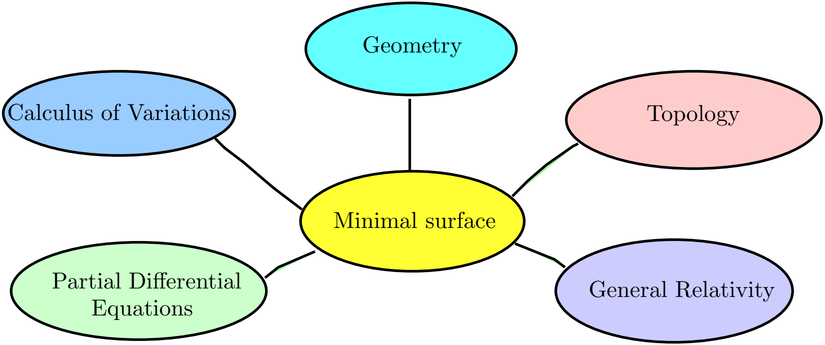


(b) The catenoid: $x^2 + y^2 = \cosh^2(z)$ [2]

[2] From: http://www.indiana.edu/∼minimal.

Why are minimal surfaces interesting?

Minimal surface is a central topic in Geometry, Analysis, Topology, Physics.

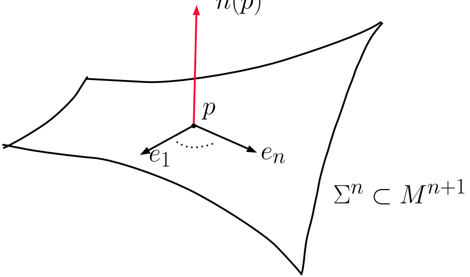


- Today I will focus on my work related to Geometry, Topology, and Calculus of Variations;

Mean curvature

Mean curvature measures the changing rate for the area of a surface.

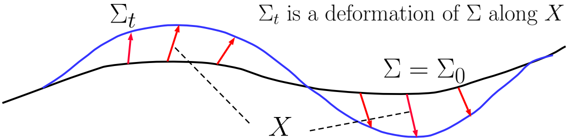
- $\Sigma^n \subset M^{n+1}$ is a hypersurface in a manifold (M^{n+1}, g);
- $p \in \Sigma$, and $\vec{n}(p) \perp T_p(\Sigma)$;
- $\{e_1, \cdots, e_n\}$ an orthonormal basis of $T_p(\Sigma)$;



$$\vec{H} = div_\Sigma(\vec{n})\vec{n} = \sum_{i=1}^{n} \langle \nabla^M_{e_i} \vec{n}, e_i \rangle \vec{n} - - - - \text{mean curvature}.$$

First variation and minimal surface

- First variation (gradient of *Area* functional):



$$\delta_\Sigma Area(X) = \frac{d}{dt}\Big|_{t=0} Area(\Sigma_t) = -\int_\Sigma \vec{H} \cdot X darea.$$

- Σ is a minimal surface if Σ is a critical point of *Area*, i.e. $\delta_\Sigma Area = 0 \Leftrightarrow \vec{H} = 0$.

Fundamental question

Question: can we find a closed minimal surface in a manifold (M, g)?

—Our discussion below is tightly related to this fundamental question.

Existence theory—minimization method

Minimize area in a topological class $[\Sigma^n]$ of a closed $\Sigma^n \subset M^{n+1}$, e.g. $[\Sigma^n] \in H_n(M^{n+1})$.

- Example: $\Sigma = T^n \subset M = T^{n+1}$

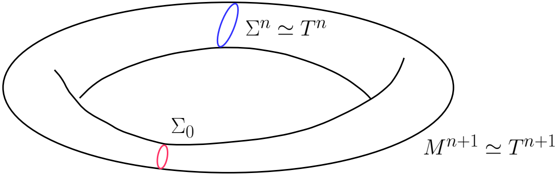
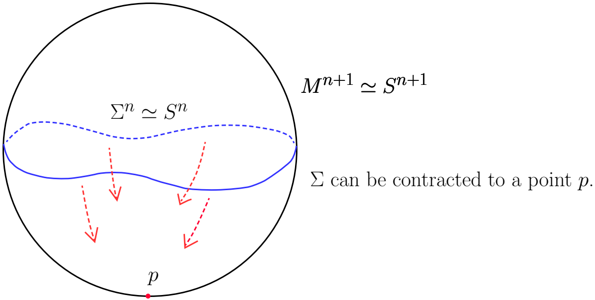


Figure: Σ_0 is an area minimizing representative in $[\Sigma^n]$.

- In general, this approach was made successful by De Giorgi, Federer, Fleming, Almgren, and Simons etc., and has led to the modern "Geometric Measure Theory".
- There exists an area minimizing $\Sigma_0 \in [\Sigma]$ which is a minimal surface.
- Answer the fundamental existence question when M^{n+1} has certain nontrivial topology, e.g. $H_n(M^{n+1})$ is nontrivial.

Existence theory—a more general case

If $H_n(M^{n+1}) = \{0\}$, the minimization method fails. For instance if $M^{n+1} \simeq \mathbb{S}^{n+1}$:

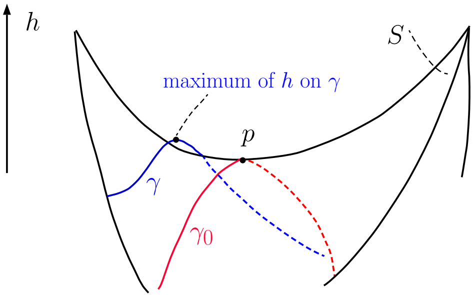


To answer the fundamental question on the existence of closed minimal surfaces for such M, we should look for more general ideas.

- Try to find saddle points instead of locally minimizers
  − − −using Morse theory for "Area" functional;
- Variational constructions for saddle points are called "Min-max theory".

Min-max characterization of saddle point

Multivariable Calculus: h is the height function on S, and p is a saddle point.

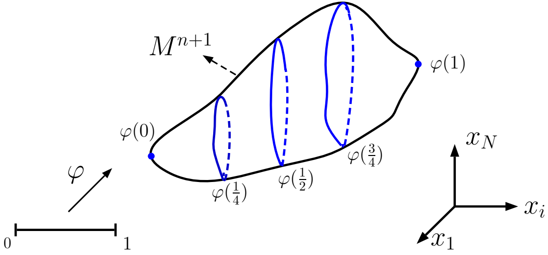


$$h(p) = \max_{t\in[0,1]} h(\gamma_0(t)) = \min_{\gamma\in[\gamma_0]} \max_{t\in[0,1]} h(\gamma(t))$$

Now change

- $S - -- > \mathcal{Z}_n(M^{n+1})$ the space of closed hypersurfaces in $M - -\infty$ dimensional,
- $h - -- > Area$,
- Topology of $\mathcal{Z}_n(M^{n+1})$ gives critical points (saddle points) of $Area$.

Almgren-Pitts theory 1

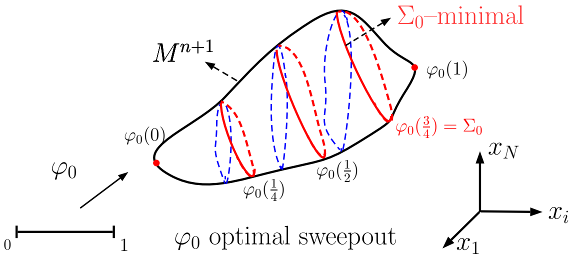


- $\varphi : [0, 1] \to \mathcal{Z}_n(M^{n+1})$, $---$"sweepout";
- Min-max value—"width":

$$W = \inf \left\{ \max_{t \in [0,1]} Area(\phi(t)) : \phi \text{ is a sweepout} \right\};$$

- The question now is to find a critical point associated with W.

Almgren-Pitts theory 2



- Almgren-Pitts 1970s, and Schoen-Simon 1981:

$$W = Area(\Sigma_0), \qquad \Sigma_0 \text{ is a closed minimal hypersurface.}$$

Corollary: every closed manifold admits a closed minimal hypersurface.

—My results in the followings are closely related to this beautiful and powerful theory.

Min-max and Willmore Conjecture

- T. Willmore made this conjecture in 1965: for a closed surface $\Sigma^2 \subset \mathbb{R}^3$, $genus(\Sigma) \geq 1$,
$$W(\Sigma) = \int_\Sigma |H|^2 dArea \geq 2\pi^2,$$
" $=$ " only if Σ is conformal to the Clifford torus$= \mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$;

- Marques-Neves ('12) solved this conjecture using Almgren-Pitts min-max theory.

Min-max theory via PDE

Harmonic map equations:

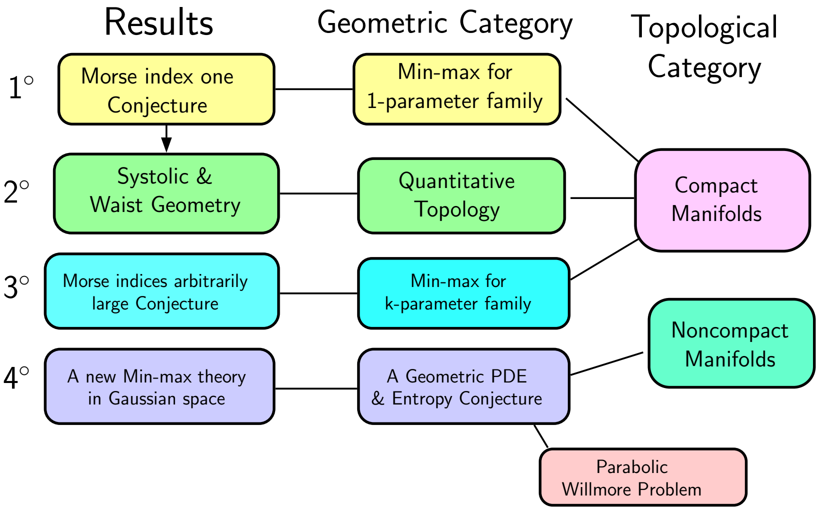
$$\triangle u = A(\nabla u, \nabla u).$$

—Semi-linear PDE system; A is a quadratic term.

Min-max theory via harmonic maps:

- Sacks-Uhlenbeck: perturbation approach;
- Micallef-Moore: sphere theorem;
- Colding-Minicozzi, myself: a new direct variational approach.

My results along this direction



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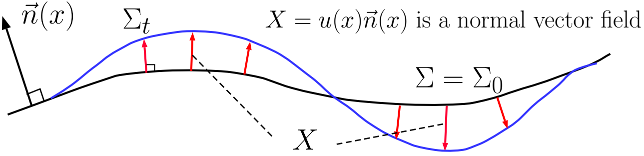
Morse index

Morse index controls the geometry of the minimal surface

- Second variation (Hessian of *Area*):

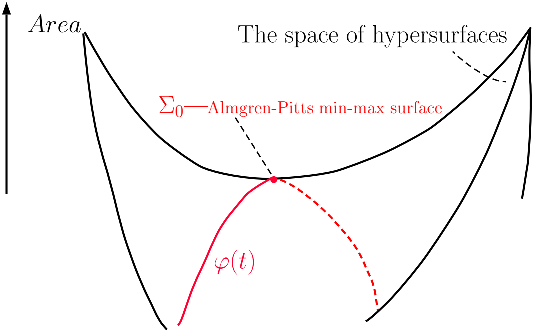


$$\frac{d^2}{dt^2}\Big|_{t=0} Area(\Sigma_t) = -\int_\Sigma (uL_\Sigma u) dArea;$$

- $L_\Sigma u = \triangle_\Sigma u + \left(Ric^M(n,n) + |A|^2\right) u$—Jacobi operator;
- Morse index: $ind(\Sigma) = \#$ of negative eigenvalues of L_Σ.

Morse index for Almgren-Pitts min-max hypersurface

Conjecture: Morse index of $\Sigma_0 = 1$.



$$Area(\Sigma_0) = \inf\left\{ \max_{t\in[0,1]} Area(\phi(t)) : \phi \text{ is homotopic to } \varphi \right\};$$

Characterization of min-max hypersurface

Let Σ_0 be the Almgren-Pitts min-max hypersurface in (M^{n+1}, g),

Theorem

(2012, 2015) If $Ric_g > 0$, then Σ_0

- *has multiplicity 1 and Morse index 1, when it is orientable;*
- *Moreover, it has least area among all closed minimal hypersurfaces.*

Remark

- *$Ric_g > 0$ is a natural geometric condition when $H_n(M^{n+1}) = \{0\}$;*
- *I also have a full version of this theorem including the case when Σ_0 is non-orientable.*

Corollary

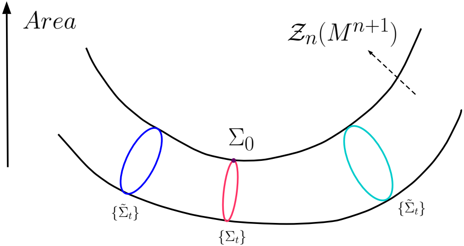
- *The min-max surface in $\mathbb{S}^3$ is the equator $\mathbb{S}^2$. $\mathbb{S}^2$ has index 1 and least area.*
- *The min-max surface in $\mathbb{RP}^3$ is the $\mathbb{Z}_2$-quotient of the Cllifford torus.*

Why is this problem hard and deep?

1. From the point of view of Analysis, the Area functional is far from being coercive, and can barely controls the geometry of a surface.
    - For instance, the convergence to the min-max hypersurface from an approximation sequence is very weak (in the sense of varifolds);
2. From the point of view of Geometry, the setup of Almgren-Pitts is not geometrically friendly. It is non-trivial to apply the Almgren-Pitts theory to geometric objects.
    - For instance, no one knew how to show that two sweepouts were homotopic to each other before;
    - I invented an Identification Theorem to handle this issue—Key novel idea.

Main ideas

To show Σ_0 has Morse index one: we prove that the "red" direction below is the only direction one can decrease the area of Σ_0.



*1) Show that Σ_0 lies in a "good" continuous family $\{\Sigma_t\}$, i.e. $Area(\Sigma_0) = \max_t Area(\Sigma_t)$
—by geometric method and a min-max theory for manifold with boundary;

*2) Show that any other $\{\tilde{\Sigma}_t\}$ is homotopic to $\{\Sigma_t\}$
—achieved by my "Identification Theorem".

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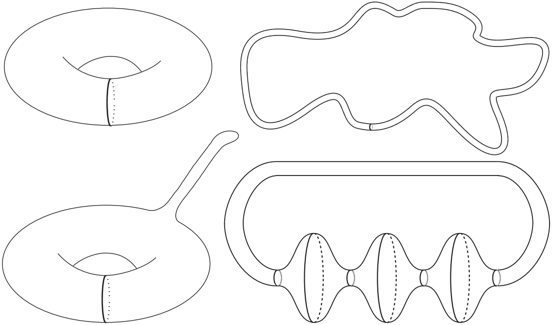
Size of manifold—a case on two torus

Question: how to measure the size of a closed Riemannian manifold?
—diameter, volume...

Given any shape on a two torus,



[3]

we can always find **a non-contractible closed curve** of length $\leq CArea(T^2, g)^{1/2}$:

[3] From: L. Guth, Metaphors in systolic geometry.

Systolic inequality—general cases

Systole: (M^n, g) closed,

- $Sys(M)$: the Systole is the length of the shortest non-contractible closed curve in (M, g).

Systolic inequality:

- (Gromov 1983): If M^n is aspherical, e.g. $M^n \simeq T^n$, then

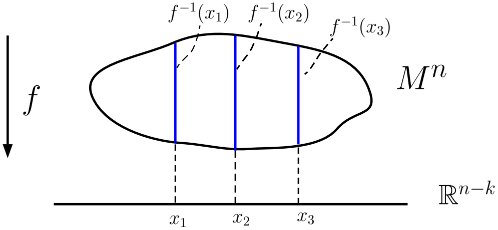
$$Sys(M, g) \leq C(n) vol(M, g)^{1/n},$$

  —note that $vol(M, g)^{1/n}$ scales as 1-dimensional;
- This inequality is purely topological;
- $Sys(M, g)$ is a good 1-dimensional size for such manifolds.

Size of manifold—waist

How to define a 1-dimensional size in general, e,g. when $\pi_1(M) = \{0\}$?

- The k-waist:



$$waist_k(M^n, g) = \inf\{\max_{x \in \mathbb{R}^{n-k}} vol_k(f^{-1}(x)) : f : M^n \to \mathbb{R}^{n-k} \text{ proper function}\};$$

- When $k = 1$, $waist_1(M^n, g)$ is a 1-dimensional size of (M^n, g).

Question:

$$waist_1(M^n, g) \leq C(n) vol(M^n, g)^{1/n}?$$

—This is a strong generalization of the systolic inequality.

Waist volume inequality

- The waist volume inequality has been studied extensively when $n = 2$;
- The case when $n \geq 3$ is much harder and largely open;

Conjecture (L. Guth): can we find $f : (M^3, g) \to \mathbb{R}^2$, such that

$$length(f^{-1}(x)) \leq C vol(M, g)^{1/3}?$$

Theorem

(w/ Y. Liokumovich 2015) If $Ric^M > 0$, then the conjecture is true.

Remark

- *It is the first time such inequalities can be proven in dimension above 2.*

Dimension reduction argument

Idea: dimension reduction argument

1. By my geometric characterization of Almgren-Pitts min-max surface, sweep out M^3 by 2-surfaces $\{\Sigma_t\}$, such that

$$\max_t Area(\Sigma_t) \leq Cvol(M)^{2/3}, \quad genus(\Sigma_t) \leq 3$$

2. Then we developed stronger versions of results when $n = 2$; sweep out $\{\Sigma_t\}$ simultaneously by 1-cycles with

$$\text{length} \leq C\big(genus(\Sigma_t)\big) Area(\Sigma_t)^{1/2} \leq C' vol(M)^{1/3}.$$

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Multi-parameter family and Infinitely many minimal hypersurfaces

- Yau (1982) conjectured that every closed 3-manifold (M^3, g) admits ∞ many closed minimal surfaces.
- Marques-Neves (2013) confirmed this when $Ric_g > 0$:

  1. Almgren: $\mathcal{Z}_2(M^3, \mathbb{Z}_2) \cong \mathbb{RP}^\infty$;
  2. Consider $\Phi_k : \mathbb{RP}^k \to \mathcal{Z}_2(M^3, \mathbb{Z}_2)$;
  3. Let
  $$W_k = \inf_{\phi \sim \Phi_k} \max_{x \in \mathbb{RP}^k} Area(\phi(x));$$
  4. Min-max theory applied to W_k gives (not explicitly) minimal surfaces $\{\Sigma_k\}_{k \in \mathbb{N}}$.

Almost nothing is known about these minimal surfaces, as the construction is implicit!

Minimal surfaces of arbitrarily high Morse index

Conjecture (Marques-Neves): In a "generic" manifold, $Ind(\Sigma_k) \to \infty$.

Theorem

(w/ H. Li 2015) The conjecture is true.

Finiteness theorem for minimal surfaces in a generic metric

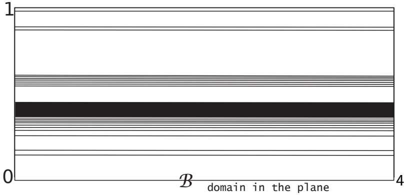
Theorem

(w/ H. Li 2015) There are only finitely many closed, embedded minimal surfaces in (M^3, g) with Morse index $ind(\Sigma) \leq N$, if g is "generic" with $Ric_g > 0$.

Main idea:

- Assume $\exists$ a sequence of closed embedded minimal surfaces $\{\Sigma_i\}$ with $ind(\Sigma_i) \leq N$;
- If $\Sigma_i \to \Sigma_\infty$, Σ_i minimal, then Σ_∞ inherits an infinitesimal symmetry—not "generic".

Key point: Σ_i converge to a minimal lamination $\mathcal{L}$ away from N singular points:



[4] From: David Hoffman.

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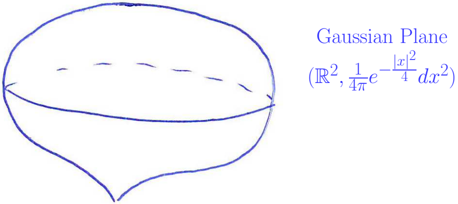
Minimal surface in Gaussian space

- $(\mathbb{R}^3, \frac{1}{4\pi}e^{-\frac{|x|^2}{4}}dx^2)$: Gaussian probability space, incomplete, noncompact, and singular space:



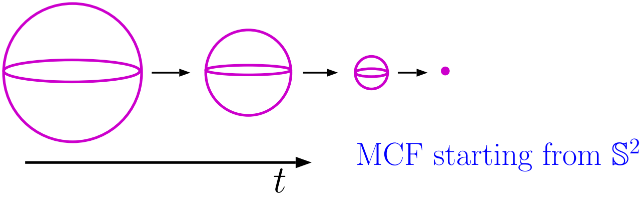
- Gaussian area of $\Sigma^2 \subset \mathbb{R}^3$: $F(\Sigma^2) = \int_\Sigma \frac{1}{4\pi}e^{-\frac{|x|^2}{4}}dx^2$, $F(\mathbb{R}^2) = 1$;
- Minimal surface w.r.t. F is called "Gaussian minimal surface";
- Examples: $\mathbb{R}^2$, $\mathbb{S}^2(2)$, $\mathbb{S}^1(\sqrt{2}) \times \mathbb{R}$;
- All Gaussian minimal surfaces are saddle points.

Gaussian minimal surface and Mean Curvature Flow

$\{\Sigma_t^2 \subset \mathbb{R}^3, t < 0\}$ evolves by Mean Curvature Flow (MCF), if

$$(\frac{d}{dt}x)^\perp = \vec{H};$$

—this is a nonlinear parabolic PDE.



- Gaussian minimal surface Σ corresponds to a self-similar solution of the MCF, i.e. $\{\Sigma_t = \sqrt{-t}\Sigma, t < 0\}$ solves MCF;
- Any blow-up solution of MCF is self-similar, and all singularities are modeled by self-similar solutions.
- Gaussian minimal surfaces are also called self-shrinkers.

Entropy Conjecture

- Entropy:
$$\lambda(\Sigma) = \sup_{x\in\mathbb{R}^3, t>0} F(\Sigma_{x,t}), \quad \Sigma_{x,t} = t(\Sigma - x).$$
  1. $\lambda(\Sigma) \geq 1$,
  2. If $\{\Sigma_t\}$ solves MCF, then $\lambda(\Sigma_t)$ is non-increasing.
- Entropy measures the complexity of a surface.

Conjecture (Colding-Ilmanen-Minicozzi-White):

$$\lambda(\Sigma) \geq \lambda(\mathbb{S}^2(2)), \quad \text{if } \Sigma \text{ is a closed surface in } \mathbb{R}^3.$$

Theorem

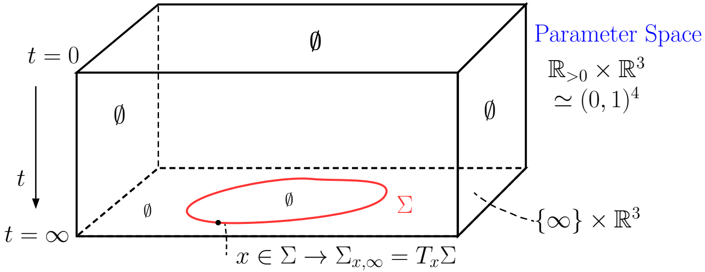
(w/ D. Ketover 2015) if $\Sigma \ncong \mathbb{T}^2$, then $\lambda(\Sigma) \geq \lambda(\mathbb{S}^2)$.

—-We use min-max method to solve a problem of parabolic PDE.

Canonical family

- Given Σ closed, consider $\{\Sigma_{x,t} = t(\Sigma - x)\}_{(x,t)\in\mathbb{R}^3\times\mathbb{R}_{>0}\simeq(0,1)^4}$;

$$\Sigma_{x,t}|_{\partial I^4} = \begin{cases} \emptyset, & \text{if } t \to 0, \text{ or if } |x| \to \infty, \text{ or if } t \to \infty, \text{ but } x \notin \Sigma \\ T_x\Sigma, & \text{if } t \to \infty, \text{ and } x \in \Sigma \end{cases}.$$



Boundary map: $x \in \Sigma \to T_x\Sigma$ — — — Gauss map with degree $= 1 - genus(\Sigma)$!

- Make $\{\Sigma_{x,t}\}$ continuous by blow-up method, and get $\{\tilde{\Sigma}_\nu\}_{\nu\in I^4}$ —— canonical family.

Entropy and min-max theory

- Consider the min-max for the canonical family:

$$\lambda(\Sigma) = \max_{\nu \in I^4} F(\tilde{\Sigma}_\nu) \geq inf_{\{\Sigma'_\nu\} \simeq \{\tilde{\Sigma}_\nu\}} \max_{\nu \in I^4} F(\Sigma'_\nu) = W,$$

here $\{\Sigma'_\nu\}_{\nu \in I^4} \in \left[\{\tilde{\Sigma}_\nu\}_{\nu \in I^4}\right] \in \pi_4\big(\mathcal{Z}_2(\mathbb{R}^3), \text{affine planes}\big)$.

Theorem

(w/ D. Ketover 2015) If genus$(\Sigma) \neq 1$, then $W > 1$.

Theorem

(w/ D. Ketover 2015) $W = n_0 F(\Sigma_0)$, where $n_0 \in \mathbb{N}$, and Σ_0 is a smooth, embedded, Gaussian minimal surface.

- $ind(\Sigma_0) \leq 4$, then $\Sigma_0 = \mathbb{S}^2, \mathbb{S}^1 \times \mathbb{R}, \mathbb{R}^2$;
- $\lambda(\Sigma) \geq W \geq \min\{\lambda(\mathbb{S}^2) < \lambda(\mathbb{S}^1 \times \mathbb{R}), \lambda(\mathbb{R}^2) = 1\}$.

Concluding Remarks

- The Geometric Variational Theory, especially the min-max theory of minimal surfaces, have achieved many celebrated results recently in many branched of mathematics;
- It is still a fast growing subject to explore, which has huge potential applications in the future. A large portion of my near future research plan will be focused on this subject.

Thank you!