Introduction to the min-max theory of minimal surfaces

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Syllabus:

1. Overview, first and second variations [CM1, CM2];
2. Curvature estimates for stable minimal hypersurfaces [CM1, CM2, SSY];
3. Introduction to “varifolds” part 1 [Si];
4. Introduction to “varifolds” part 2 [Si];
5. Colding-De Lellis paper [CD] part 1;
6. Colding-De Lellis paper [CD] part 2;
7. Colding-De Lellis paper [CD] part 3;
8. Index bound of the min-max surface [MN1, Z];

References


[Si] L. Simon, Lectures on geometric measure theory, Australian National University Centre for Mathematical Analysis, Canberra, 1983.


[Z] X. Zhou, Min-max minimal hypersurface in \((M^{n+1}, g)\) with \(Ric_g > 0\) and \(2 \leq n \leq 6\). arXiv:math.DG/1210.2112, (2012).
30. Overview

- $\Sigma^k \rightarrow \mathbb{M}^n$ \textbf{k}-dim submanifold.
  - $\Sigma$ is minimal if $\Sigma$ is a critical pt of the volume functional.

- Variational theory of min submanifold

"Prototype" Baby case

Question: given $f: \mathbb{R}^2 \rightarrow \mathbb{R}^1$ find an critical pt of $f$.

Case 1:

minimize $f$

$$m_0 = \min_{x \in \Omega} f(x).$$

Case 2:

min-max

$$\min_{x \in \Omega} f(x) \text{ does not exist.}$$

$P: \mathbb{R}^2 \rightarrow \mathbb{S}^2 \quad P(\pi) \cdot P(\pi) \in \mathbb{S}^2.$
\[
\min_{p \in [P]} \max_{t \in [0,1]} f(p(t)) = \omega_0.
\rightarrow \text{ interior critical pt.}
\]

Examples of minimizing minimal surfaces.

\[
D \subset \mathbb{R}^2 \rightarrow \Omega
\]

Classical Plateau Problem: \( u \) minimizes area among all such maps.

- Douglas - Rado 1931: solution exists!
- Extension to high dim & co-dimensions
- Extension to the case of closed submanifold \( M \) in arbitrary \( \mathbb{R}^n \) closed \( M \).
  \[ \downarrow \text{geometric measure theory.} \]

Examples of min-max.

\[
S^n \rightarrow S^{n+1} \rightarrow \text{min-max, min surface.}
\]
2°. The case of $\dim = 2$ : find closed geodesics on $(S^2, g)$.

- Birkhoff. 1910s. - existence of a closed geodesic on any $(S^2, g)$.

- sweep out $S^2$ by $S^1$'s

- $\min_{\gamma \in \gamma_0} \max_{t \in \mathbb{R}} \| \dot{\gamma}(t) \|_g = \omega$.

- Final critical pt of $\omega$.

3°. Extended to high-dim and co-dim cases.

- Almgren- Pitts. 1970's. - existence of a closed minimal hypersurface in any $(\mathbb{R}^n, g)$.

- $n = 7$ - due to Schoen-Simon $(M^n, g)$. $\exists n < 7$

- Similar ideas as the above.

- $F \in (S^1) \rightarrow (n+1)$-dim generalized closed submanifold $\gamma$. $\gamma$.

- $W = \inf_{F \in [f_0, f_1]} \max_{t \in [0, 1]} \text{Area}(\gamma(t))$. $\gamma \in [0, 1]$.
Syllabus

1. First and 2nd variation.
2. Curvature estimates for stable min-hyper surface.
3. Introduction to "varifolds".
4. Introduction to "Colding-De Lellis" paper.
5. Almost minimizing and Regularity 1.
7. Index issue about min-max.
8. Sacks-Uhlenbeck's work on minimal $S^2$'s.
31. Minimal submanifold, first & second variation

- $\Sigma^k \hookrightarrow (M^n, g)$.

|\begin{align*}
\Sigma^k & : n \text{-dimensional Riemannian manifold} \\
& \text{with Riemannian metric } g \\
& \nabla : \text{covariant derivative}
\end{align*}|

|\begin{align*}
\text{\textit{X} vector field of } M & \text{ on } \Sigma \\
& = x^T : \text{tangential}
\end{align*}|

**Def.** $\nabla$ induces a covariant derivative $\nabla_\Sigma$ on $\Sigma$

and the second fundamental form $A \not\Sigma$

$\begin{align*}
X, Y \in T_p \Sigma. \\
\nabla_\Sigma X \cdot Y = (\nabla X Y)^T \\
A(X, Y) = (\nabla Y X)^T \\
\text{\textit{\nabla}} \text{ symmetric \ bilinear form}
\end{align*}$

**Def.** The \textit{mean curvature} of $\Sigma$ is defined as:

$\begin{align*}
H_\Sigma = \text{Tr} A = \frac{1}{k} \sum_{i=1}^{k} A(e_i, e_i)
\end{align*}$

$\{e_1, \ldots, e_k\}$ orthonormal base of $T_x \Sigma$

with respect to $g|_\Sigma$. 
Def: 2 is called minimal if  \( \bar{F} = 0 \)

Examples:
- \( \mathbb{R}^k \to \mathbb{R}^n \)
- \( S^k \to S^2 \) - equator
- Catenoid:
  \( y = \cosh z \)
- Helicoid:
- Clifford torus \( T^2 \to S^3 \)
  \( y^2 + z^2 = \frac{1}{2}, x^2 + 8y^2 = 1 \)

First Variational formula:
\( I^k \to M^n \) submanifold.

F: \( \Sigma \times (-\varepsilon, \varepsilon) \to M^n \) variation.

\( F(0, \cdot) = \text{id} \), \( \frac{d}{dt} \bigl|_{t=0} F(t, \cdot) = X \)

\( \Sigma + v = F^+(\Sigma) \left( = F(\Sigma, v) \right) \)
Prop: \( \partial \Sigma (x) = \frac{1}{\alpha^2 t} \text{vol}(\Sigma^t) = \int_{\Sigma} \text{div}_{\Sigma} (x) \, d\mu \)

- \( \text{div}_{\Sigma} (x) = \frac{\delta}{\alpha^2 t} \langle \partial_t x, e_i \rangle \).
- \( \{ e_i \ldots e_n \} \) o.n. basis for \( T_x \).
- \( d\mu \) -- volume element of \( \Sigma \).

Proof: Using local coordinates \( \{ x^i \} \) for \( \Sigma \).

\[ g_{ij}(x) = g \left( \left( F_i \right)_{\theta} \frac{\partial}{\partial x^i}, \left( F_j \right)_{\theta} \frac{\partial}{\partial x^j} \right) \]

\[ |\Sigma^t| = \int_{\Sigma} \sqrt{\text{det} g_{ij}(x)} \, d^nx. \]

\[ = \int_{\Sigma} \frac{\sqrt{\text{det} g_{ij}(x)}}{\sqrt{\text{det} g_{ij}(0)}} \, d\mu_x. \]

\[ \frac{d}{dt} \bigg|_{t=0} |\Sigma^t| = \int_{\Sigma} \frac{d}{dt} \bigg|_{t=0} \text{vol}_{\Sigma^t} \, d\mu_x. \]

Assume \( \{ x^i \} \) o.n. basis at point \( x \).

\[ \frac{d}{dt} \bigg|_{t=0} \sqrt{\text{det} g_{ij}(x)} = \frac{d}{dt} \bigg|_{t=0} \sqrt{\text{det} g_{ij}(0)} = \frac{1}{2} \frac{d}{dt} \bigg|_{t=0} (\text{det} g_{ij}(0)) \]

Fact: \( \frac{d}{dx} \left[ \text{det} (a_{ij}(x)) \right] = \sum_{ij} a_{ij}(x) \frac{d}{dx} a_{ij}(x) \frac{d^2 a_{ij}(x)}{dx^2} \]

\[ [a_{ij}]^{-1} \text{ -- inverse matrix of } a_{ij} \]
\[ \frac{1}{2} \sum_{i=1}^{K} \frac{d}{dt} \langle F_{x_i}, F_{x_i} \rangle = \sum_{i=1}^{K} \langle F_{t}, F_{x_i} \rangle \langle F_{t}, F_{x_i} \rangle \]

- \[ F_{x_i} = (F_0) \frac{\partial}{\partial x_i} \]
- \[ F_t = (F_0) \frac{\partial}{\partial t} \]

as \[ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial t} \] - coordinate vector field.

\[ \sum_{i=1}^{K} \frac{\partial}{\partial x_i} \langle F_{x_i}, F_t \rangle = 0 = \sum_{i=1}^{K} \frac{\partial}{\partial x_i} \langle F_{x_i}, F_t \rangle = \text{div}_I F_t \]

\[ = \text{div}_I X. \]

Q.E.D.

Write \[ X = X^T + X^N \] then

\[ \Sigma (X) = \int_\Sigma \left( \text{div}_I X^N + \text{div}_I X^T \right) \, d\Sigma. \]

\[ \text{div}_I X^N = \frac{1}{2} \sum_{i=1}^{K} \langle \nabla_e X^N, e_i \rangle = -\frac{1}{2} \sum_{i=1}^{K} \langle X^N, \nabla_e e_i \rangle. \]

\[ \nabla_e \langle X^N, e_i \rangle = \nabla_e 0 = 0. \]

\[ = -\langle X^N, H \rangle \]

\[ = -\langle X, H \rangle \]

- divergence

\[ \Sigma (X) = -\int_\Sigma \langle X, H \rangle \, d\Sigma + \int_\Sigma \langle X^T, \eta \rangle \, d\Sigma. \]

\( \eta \) unit outer normal of \((I, \partial I)\).
So $I$ is minimal ($\tilde{H} \equiv 0$) $\iff \delta \Sigma(x) = 0 \; \forall \; x$ of compact support.

**Second Variational Formula.**

\[
\delta \Sigma(x, \dot{x}) = \frac{d^2}{dt^2} \bigg|_{t=0} [\Sigma(x, \dot{x})] = \int_\Sigma \left[ |\nabla^2 \sigma|^2 - |\langle A \times x \rangle|^2 - \sum_{i=1}^{k} R(e_i \times e_i \times e_i \times x) \right] \, dx.
\]

where $\{e_1, ..., e_k\}$ o.n. basis for $\Sigma$.

\[
|\langle A \times x \rangle|^2 = \frac{1}{2} \sum_{i=1}^{k} |A(x \times e_i)|^2.
\]

As above $\{x^1, ..., x^k\}$ local o.n. coordinates at $x \in \Sigma$.

\[
\frac{d}{dt} V(t) = \frac{1}{\sqrt{\text{det} G_{ij}(t)}}\dot{G}_{ij}(t)
\]

\[
= \frac{1}{2} \left( \text{det} G_{ij}(t) \right)^{-\frac{1}{2}} \ \dot{G}_{ij}(t) \dot{G}_{ij}(t)
\]

\[
\frac{d^2}{dt^2} \bigg|_{t=0} V(t) = -\frac{1}{4} \left( \text{det} G_{ij}(t) \right)^{-\frac{3}{2}} \ \left( \dot{G}_{ij}(t) \dot{G}_{ij}(t) \right)^2
\]

\[+ \frac{1}{2} \left( \text{det} G_{ij}(t) \right)^{-\frac{3}{2}} \ \dot{G}_{ij}(t) \dot{G}_{ij}(t) \dot{G}_{ij}(t) \dot{G}_{ij}(t)
\]

\[+ \frac{1}{2} \left( \text{det} G_{ij}(t) \right)^{-\frac{3}{2}} \ \dot{G}_{ij}(t) \dot{G}_{ij}(t) \dot{G}_{ij}(t) \dot{G}_{ij}(t)
\]
\[ \langle \hat{A} (\mathbf{x}) \rangle \big|_{t=0} = -2 \langle \mathbf{A}, \mathbf{x} \rangle = 0. \]
\[ \hat{g}_{ij}(0) = -g_{ik}g_{jl} \hat{g}_{kl}(0) \]
\[ \hat{g}_{ij} + \hat{g}_{ji}(0) = -4 \]

\[ \hat{g}_{ij}(0) = \frac{1}{2} \langle \nabla \frac{\partial \mathbf{A}}{\partial x^i}, \nabla \frac{\partial \mathbf{A}}{\partial x^j} \rangle \]
\[ = \langle \nabla \frac{\partial \mathbf{A}}{\partial x^2}, \nabla \frac{\partial \mathbf{A}}{\partial x^2} \rangle + \langle \nabla \frac{\partial \mathbf{A}}{\partial x^1}, \nabla \frac{\partial \mathbf{A}}{\partial x^1} \rangle \]
\[ = \langle \nabla \frac{\partial \mathbf{A}}{\partial x^2}, \nabla \frac{\partial \mathbf{A}}{\partial x^2} \rangle + \cdots \]
\[ = -2 \langle \mathbf{A} \left( \frac{\partial \mathbf{A}}{\partial x^2}, \frac{\partial \mathbf{A}}{\partial x^2} \right), \mathbf{x} \rangle \]

\[ \hat{g}_{ij}(0) = \hat{g}_{ij}(0) = -4 \langle \mathbf{A}, \mathbf{x} \rangle \]

\[ \hat{g}_{ij}(0) = \nabla \left( \langle \nabla \frac{\partial \mathbf{A}}{\partial x^i}, \nabla \frac{\partial \mathbf{A}}{\partial x^j} \rangle + ij \text{ reverse} \right) \]
\[ = \langle \nabla \frac{\partial \mathbf{A}}{\partial x^2}, \nabla \frac{\partial \mathbf{A}}{\partial x^2} \rangle + \langle \nabla \frac{\partial \mathbf{A}}{\partial x^1}, \nabla \frac{\partial \mathbf{A}}{\partial x^1} \rangle \]
\[ + (ij \text{ reverse term}) \]
\[ = \langle \nabla \frac{\partial \mathbf{A}}{\partial x^2}, \nabla \frac{\partial \mathbf{A}}{\partial x^2} \rangle + \langle \nabla \frac{\partial \mathbf{A}}{\partial x^1}, \nabla \frac{\partial \mathbf{A}}{\partial x^1} \rangle \]
\[ + \langle \nabla \frac{\partial \mathbf{A}}{\partial x^2}, \nabla \frac{\partial \mathbf{A}}{\partial x^1} \rangle + \langle \nabla \frac{\partial \mathbf{A}}{\partial x^1}, \nabla \frac{\partial \mathbf{A}}{\partial x^2} \rangle \]
\[ + (ij \text{ reverse term}) \]
\[
= \left< \nabla \frac{\partial F}{\partial x_i}, \dot{F} \cdot \frac{\partial F}{\partial x_i} \right> - R^m \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \cdot \frac{\partial^2 F}{\partial x_i \partial x_j} \right).
\]

\[
+ \left< \nabla \frac{\partial F}{\partial x_i}, \frac{\partial^2 F}{\partial x_j \partial x_j} \right> + \left< \nabla \frac{\partial F}{\partial x_i}, \frac{\partial^2 F}{\partial x_j \partial x_j} \right> + \text{i,j reverse terms}
\]

\[
S_0 \overset{\mathbf{j} \to \mathbf{j} (\mathbf{0})}{\Rightarrow} = \text{div} \, \dot{F} - 2 \mathbf{g}^* \mathbf{R}^m \left( \dot{F}, \mathbf{e}_i \right) \cdot \left( \dot{F}, \mathbf{e}_i \right)
\]

\[
+ 2 \left| \left< \dot{\mathbf{A}} \times \mathbf{x} \right> \right|^2 + 2 \left| \nabla \dot{\mathbf{F}} \right|^2.
\]

Combining all above:

\[
\frac{d^2}{dt^2} \mathbf{N} \bigg|_{t=0} = \text{div} \, \dot{F} - \sum_{i=1}^{k} \mathbf{R}^m \left( \mathbf{x}, \mathbf{e}_i \times \mathbf{x} \mathbf{e}_i \right) + \mathbf{K} \mathbf{A} \cdot \mathbf{x}^2
\]

\[
+ \left| \nabla \dot{\mathbf{F}} \right|^2 - 2 \left| \left< \dot{\mathbf{A}} \times \mathbf{x} \right> \right|^2.
\]

\[
\square
\]

Hyperurface case: \( \Sigma \subset M \). Assume \( \Sigma \) is "two-sided", i.e., a nontrivial normal vector field \( \mathbf{u} \).

Let \( \mathbf{x} = \varphi \mathbf{u} \), \( \varphi \in C^0(\Sigma) \).

\[
I(\varphi, \varphi) := \frac{\partial^2}{\partial x^2} (\mathbf{x}, \mathbf{x}) = -\int_{\Sigma} \varphi \mathbf{u} \cdot \varphi \mathbf{u} \, d\mathbf{u}
\]
\[ L \psi = \Delta \psi + (18 \psi^2 + \text{Ricm}(\delta \delta \nabla) \psi) \]

\[ \phi^2 \delta \Gamma (x, x) = \int \frac{1}{2} \nabla^2 \rho \left( \delta \psi \right)^2 = \int \left( \nabla^2 \psi \right)^2 - \int \rho \phi \nabla \psi \]

\[ \int \phi^2 \left( \nabla \psi \right)^2 = \int \phi \nabla \psi \phi \]

L - self-adjoint, elliptic, discrete eigenvalues \( \lambda_j \) with eigenfunction \( \psi_j \).

\[ L \psi_j + \lambda_j \psi_j = 0, \quad \lambda_1 < \lambda_2 < \lambda_3 \ldots \]

Def.: \( \Sigma \) is stable if \( \lambda > 0 \), i.e. \( L \psi, \psi > 0 \).

Morse index of \( \Sigma \) = number of negative eigenvalues counted with multiplicity.

\[ \text{Ric}^1, \quad L \psi, \psi > 0 \text{ has area} \quad (\psi, \psi) \]

\[ \text{Ric}^2, \quad \text{if} \quad \text{Ric}^2 > 0, \quad \text{no stable}\quad \Sigma^m \]
§2. Curvature estimates for stable min hypersurfaces.

Thm: \( \Sigma \hookrightarrow \mathbb{R}^n \) stable \( 2 \)-sided min hypersurf.

- \( x_0 \in \Sigma \), \( d \Sigma < \mathcal{A}B_0(\delta_0) \),
- \( \mathcal{A}B_0(\delta_0) \leq V r_0^{-1} \)
- \( n \leq 6 \).

Then \( \sup \ |A|^2 \leq (\text{Cn}. \ V) \ r_0^{-2} \).

1°. Simons' inequality.

- \( \Sigma \hookrightarrow \mathbb{R}^n \) min hypersurface, \( 2 \)-sided
- \( h \sim \text{hij} \) second fundamental form
- \( \{e_1, \ldots, e_n\} \) or. frame.

The rough Laplacian:

\[ \Delta h_{ij} = \sum_{k=1}^{n-1} h_{ij} \ k k. \]

Prop: \( \Delta h_{ij} + \partial^2 h_{ij} = 0 \), \( 0 \leq i, j \leq n-1 \).

Proof: Gauss eq: \( R_{ijkl} = R_{ijkl}^{\text{sym}} + h_{ik}h_{jl} - h_{ij}h_{kl} \).
Codazzi - eq.

\[ R^{m}_{ij;k} = h_{ik;j} - h_{ij;k} = 0 \]

Ricci identity (for symmetric 2-tensor)

\[ h_{ij,kl} - h_{ij,kl} = \sum_{p=1}^{n} h_{pi} R_{pjkl} + \frac{1}{2} h_{pk} R_{pjk}. \]

Now:

\[ \Delta h_{ij} = h_{ij,kk} = h_{ik,jk}. \]

\[ = h_{ik,j} + \sum_{p=1}^{n} h_{pi} R_{pjik} + \frac{1}{2} h_{pk} R_{pijk}. \]

\[ = h_{ik,j} + \sum_{p=1}^{n} h_{pi} (h_{pj} h_{ik} - h_{pk} h_{ij}) \]

\[ + h_{pk} (h_{pj} h_{ik} - h_{pk} h_{ij}). \]

\[ = -1h_{i}^{2} h_{j} + (-h_{ip} h_{pk} h_{ij} + h_{ik} h_{k} h_{ij}). \]

\[ = -1h_{i}^{2} h_{j}. \]

Cor.: \[ \frac{1}{2} \Delta_{s} |A|^{2} = |D|^{2} - |A|^{4}. \]

\[ \Delta_{s} |A|^{2} = (h_{ij} h_{ij})_{kk} = 2h_{ij,k} h_{ij;k} + 2h_{ij} h_{ij,kk}. \]
Prop \text{ Set } L \varphi = \Delta_{n} \varphi + \| \varphi \|^{2} \varphi \text{ - stability operator.}

then \text{ } | \varphi | L(\varphi) \geq \frac{2}{n+1} | \nabla \varphi |^{2}.

\text{If } \frac{1}{2} \Delta_{n} | \varphi |^{2} = | \nabla \varphi |^{2} - | \varphi |^{4}.

\Rightarrow \text{ } (\varphi | L(\varphi) = | \nabla \varphi |^{2} - | \nabla \varphi |^{2}.

\text{Trick, Choose o.n. eigenbasis } \{ e_{1} \ldots e_{n} \} \text{ for } \nabla^{2}.

\text{st. } \varphi_{ij} = \varphi i \delta_{ij}.

| \nabla \varphi |^{2} = \sum_{i,j,k} (\sum_{i} \varphi_{ij} \varphi_{jk})^{2} = \sum_{i} (\sum_{i} \varphi_{ii})^{2} \frac{1}{| \varphi |^{2}}.

\text{ } \leq \frac{2}{n+1} \sum_{i} (\sum_{i} \varphi_{ii})^{2} \frac{1}{| \varphi |^{2}} = \frac{2}{n+1} \sum_{i} \varphi_{ii}^{2}. \frac{2}{n+1} \sum_{i} \varphi_{ii}^{2}.

\text{ } = \sum_{i} \varphi_{ii}^{2} + \sum_{i} \varphi_{ii}^{2} \left( - \sum_{j \neq i} \frac{\varphi_{ij}^{2}}{\varphi_{ii}} \right)^{2}.

= \sum_{i} \varphi_{ii}^{2} + \sum_{i} \varphi_{ii}^{2} \left( - \sum_{j \neq i} \frac{\varphi_{ij}^{2}}{\varphi_{ii}} \right)^{2}.
\[
\begin{align*}
&= \sum_{i \neq k} h_{i,k}^2 + (n-2) \sum_{i \neq j} h_{j,i}^2 \\
&= \mathfrak{O} (n^{-1}) \sum_{i \neq k} h_{i,k}^2 \\
&= \frac{n-1}{2} \sum_{i \neq j} \left( h_{i,j}^2 + h_{j,i}^2 \right) \\
\Rightarrow \quad &\left( 1 + \frac{2}{n-1} \right) (\partial A) \le \sum_{i \neq k} h_{i,k}^2 + \sum_{i \neq j} h_{i,j}^2 + \sum_{i \neq j} h_{j,i}^2 \\
&\le \sum_{i \neq k} h_{i,k}^2 = (\partial A)^2.
\end{align*}
\]

\[ L^2 \text{- curvature estimates} \]

\[ \text{Proof: } \sum_{i} \le Q^n \text{ stable min-hyper. } L^2 \text{- sided.} \]

\[ \forall \rho \in \left[ 2, 2 + \sqrt{n-1} \right) \]

\[ \int_{\Sigma} |\partial^{2p} \varphi|^2 \le C(n,p) \int_{\Sigma} |\partial^{p} \varphi|^2. \]

\[ \forall \varphi \in C^1(\Sigma). \]

\[ \varphi \in L^{(\varphi, 0)} \]

\[ \begin{align*}
&\varphi \in L^{(\varphi, 0)} = \varphi \partial A \left( \Delta (\varphi A A) + \varphi |\varphi A| \right) \\
&= \varphi |\varphi A| \left( \Delta \varphi A A + 2 \partial A \varphi A A + \varphi A A A A + \varphi^2 A A A \right) \\
&= \varphi |\varphi A|^2 \Delta \varphi + 2 \varphi |\varphi A| \varphi A A A + \frac{2}{n-1} \left( \varphi^2 |\varphi A| \right)^2 \\
&\ge \varphi |\varphi A|^2 \Delta \varphi + 2 \varphi |\varphi A| \varphi A A A + \frac{2}{n-1} \varphi^2 |\varphi A|^2.
\end{align*} \]

Integrate on \( \Sigma \).
Together with stability inequality

\[ \frac{2}{n-1} \sum \varphi^2 |\nabla \psi|^{2} \leq - \sum \varphi \varphi_{\psi} \varphi_{\psi} - 2 \sum \varphi \psi_{\psi} \varphi_{\psi} \psi_{\psi} \]

\[ \geq \frac{\varepsilon}{\theta} + \sum \nabla \cdot (\varphi \psi \varphi_{\psi}) \cdot \nabla \psi - 2 \sum \varphi \psi_{\psi} \varphi_{\psi} \psi_{\psi} \]

\[ = \left[ \sum \varphi \varphi_{\psi} \varphi_{\psi}^{2} + 2 \varphi \psi_{\psi} \varphi_{\psi} \psi_{\psi} \right] - 2 \varphi \psi_{\psi} \varphi_{\psi} \psi_{\psi} \]

\[ \Rightarrow \frac{2}{n-1} \sum \varphi^2 |\nabla \psi|^{2} \leq \sum \varphi \varphi_{\psi} \varphi_{\psi}^{2} \]

* Change \( \varphi \rightarrow \varphi |\theta|^{q} \).

\[ \Rightarrow \frac{2}{n-1} \sum \varphi^{2} |\theta|^{2q} |\nabla |\theta| |^{2} = \sum \theta \theta_{\varphi}^{2} \left[ \varphi \varphi_{\varphi} |^{q} + \theta \theta_{\varphi} \theta_{\varphi} |^{q} \right]^{2} \]

\[ \leq \left( q^{2} + 3 \right) \sum \varphi^{2} |\theta|^{2q} |\nabla |\theta| |^{2} \]

\[ + \left( 1 + \frac{1}{q} \right) \sum \theta^{2} |\theta|^{2q+2} |\theta|^{2} \]

If \( q < \sqrt{2} \), then

\[ \sum \varphi^{2} |\theta|^{2q} |\nabla |\theta| |^{2} \leq C(n, q) \sum \theta^{2} |\theta|^{2q+2} |\theta|^{2} \]

Set \( \varphi = q+2 \):

\[ \sum \varphi^{2} |\theta|^{2q+2} |\nabla |\theta| |^{2} \leq C(n, p) \sum \left( |\theta|^{2q+2} |\theta|^{2} \right) \]

* Change \( \varphi \rightarrow \varphi |\theta|^{q+1} \).

* Let \( \varphi \rightarrow \varphi |\theta|^{q+1} \) in stability inequality.
\[ \sum_1 |\varphi|^2 \leq \sum_1 |\varphi|^2 \]
\[ \Rightarrow \sum_1 |\varphi|^2 \leq \sum_1 |A(\varphi|18\varphi_{\perp}|^2)|^2 \]
\[ \leq 2 \sum_1 |A\varphi|^2 |\varphi|^2 + |\varphi|^2 |A\varphi|^2 |^2 \]
\[ \Rightarrow \sum_1 |\varphi|^2 \leq C(n,p) \int_\Sigma (|\varphi|^2)^2 d\rho \varphi \]
Replace \( \varphi \) by \( \varphi^p \)
\[ \Rightarrow \sum_1 |\varphi|^2 \leq C(n,p) \int_\Sigma (|\varphi|^2)^2 p |\varphi|^2 d\rho \varphi \]
\[ \leq C(n,p) \left( \int_\Sigma |\varphi|^2 \right)^{2/p} \left( \int_\Sigma |\varphi|^2 \right)^{1-p} \]
\[ \Rightarrow \left( \int_\Sigma |\varphi|^2 \right)^{2/p} \leq C(n,p) \int_\Sigma |\varphi|^2 \]
\[ A \quad p < 2 + \sqrt{\frac{2}{n-1}} \]

**Poincare estimates**

**Theorem**
\[ \Sigma^{n-1} \subset R^n \text{ stable min-hypersurface}, \quad 2\text{-sided} \]
\[ x_0 \in \Sigma, \quad \Sigma < x \subset B_{\rho_0}(x_0) \]
\[ |\Sigma \cap B_{\rho_0}(x_0)| \leq V_{\rho_0}^{n-1} \]
\[ n \leq 6 \]
Then, \( \sup_{\sum \lambda B_{\lambda} \subseteq U} |A|^2 \leq C(\ln U) r_0^{-2} \).

Take \( 2p \geq n+1 \), i.e.,

\[ n+1 < 4 + 2 \sqrt{\frac{2}{n+1}} \Rightarrow n \leq 6 \]

i.e.,

\[ \sum |A|^{2p} \varphi \leq C(n) \sum |\lambda B_{\lambda}|^{2p} \varphi. \]

Take \( \varphi(x) = \varphi(x_0, x_0) = \frac{1}{r_0} \) in \( B_{2r_0} \times x_0 \).

Let \( \varphi = \text{dist}(x, x_0) \):

\[ 18 \varphi | \leq \frac{C}{r_0} \]

then

\[ \sum \lambda B_{\lambda} \subseteq U \quad |A|^2 \leq C \sum \lambda B_{\lambda} \frac{1}{r_0^{2p}} \text{ dist} \]

\[ \leq C(n,p) \frac{\text{vol}(B_{r_0}(x_0) \setminus \Sigma)}{r_0^{2p}} \]

\[ \leq C(n,p) \frac{r_0^{-n}}{r_0^{2p}} \]

Using Simon's inequality:

\[ \frac{1}{2} |A|^2 + |A|^4 \geq 0 \quad (\Rightarrow \Delta |A|^2 + |A|^4 \geq 0) \]

A Moser iteration. (on \( U \))

\[ \sup_{\sum \lambda B_{\lambda} \subseteq U} |A|^2 \leq C \left( \frac{1}{r_0^{-n}} \sum |\lambda B_{\lambda}|^{4} \text{ dist} \right)^{1/2} \]
where $c$ depends on $n + 8 \geq p - (n + 1)$,

\[
\sum_{i}^{\infty} \frac{1}{r_i^{n-1}} \int_{B_{r_i}(x_i)} |\Theta_i| \, du_i \leq c.
\]

Let $p = q$,

\[
\sum_{i}^{\infty} \frac{1}{r_i^{n-1}} \int_{B_{r_i}(x_i)} |\Theta_i| \, du_i \leq c(n, p, V) \cdot \frac{r_i^{n-1}}{r_i^4 \cdot r_i^m} \to \frac{c}{r_i^4}.
\]

\[\Rightarrow \sup_{\Sigma_i} |\Theta_i|^2 \leq \frac{c}{r_i^2}.
\]

\[\text{Rk1: Need to check the Sobolev inequality on } \Sigma_i.
\]

\[\text{Rk2: When } \Sigma_i \subset M^n, \text{ all the above estimates similarly work, except that there are curvature terms of } M^n \text{ appearing } \text{[ESS]}.
\]

\[\text{Rk3: Uniform curvature estimates + uniform volume estimates } \Rightarrow \text{ compactness.}
\]

\[\text{i.e., } \Sigma_i \subset \Sigma \quad \text{and } c \subset \Sigma.
\]

\[\sup_{i} \|\Theta_i\| < C < \infty \quad \sup_{i} \text{Area}(\Sigma_i \cap K) \leq C\]

\[\Rightarrow \Sigma_i \to \Sigma_\infty.
\]
$L_{4.5} \quad I^2 \leftrightarrow M^3 \quad \Sigma^2 \text{ stable, \ 2-sided.}$

always have volume growth estimates, hence curvature estimates.
3. Introduction to the theory of Varifold.

1. Hausdorff measure $H^n$ on $\mathbb{R}^{n+k}$.

\[ H^n_\delta(A) = \lim_{\delta \to 0} \frac{1}{\delta^n} \int_A \frac{1}{\delta^n} \left( \text{diam} C_i \right)^n. \]

\[ A = \bigcup C_i \]

\[ H^n(A) = \lim_{\delta \to 0} H^n_\delta(A). \]

---

Density, $\mu^n$ measure on $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k}$.

\[ \Theta^n_\mu (M, x) = \lim_{\rho \to 0} \frac{\mu(B_r(x))}{H^n(B_r(x))} \cdot \frac{1}{\rho^n} \]

\[ \Theta^n_\mu (M, x) = \lim_{\rho \to 0} \frac{\mu(B_r(x))}{\rho^n} \cdot \frac{1}{\rho^n} \]

If $\Theta^n_{\bar{\mu}}(M, x) = \Theta^n_\mu (M, x)$.

\[ \Theta^n_{\bar{\mu}}(M, x) = \Theta^n_\mu (M, x) \]

$M^n = n$-dim Riem manifold.

$\mathcal{G}_k(M^n) = \text{Grassmannian bundle of un-oriented } k\text{-planes over } M.$
Def. A \( k \)-varifold \( V \) on \( U \) is a Radon measure on \( G_k(U) \). Write \( V_k(U) \) to be the set of all \( k \)-varifolds on \( U \).

Given \( V \in V_k(U) \), there is a Radon measure \( \mu_V \) on \( U \), defined by:

\[
\mu_V(B) = V(z^-(B)),
\]

where \( z : G_k(U) \rightarrow U \),

\[
(z, s) \rightarrow s.
\]

We call such \( \mu_V \) weight of \( V \).

The mass \( m(V) \) is \( m(V) = \mu_V(U) \).

Rectifiability:

Def. A \( H^k \)-measurable set \( M \subset U^n \) is said to be countably \( k \)-rectifiable if \( M = \bigcup_{j=0}^{\infty} M_j \)

where \( H^k(M_0) = 0 \), \( M_j \subset F_j(A_j) \), \( j \geq 1 \).

\( F_j : A_j \subset \mathbb{R}^k \rightarrow \mathbb{R}^n \) Lipschitz.

Lemma. Given \( M \) a countably \( k \)-rectifiable set,

\( \delta \) locally \( k \) integrable function on \( M \).
Can define a $k$-varifold $V = V(M, \theta)$ as

$$V(M, \theta)(A) = \left( H^k(\theta) \left( M \cap x^{-1}(B(x, \mathbb{R}^n) \cap A) \right) \right)_{AC \mathcal{G}_k(M)}$$

**Def.** Given $M$ a $H^k$-measurable set, $0 > 0$ locally $H^m$-integrable function on $M$.

Say $P$, $k$-dim space linear subspace, is an "approximate tangent plane" for $M$ at $x$ w.r.t. $\theta$ if

$$\lim_{\lambda \to 0} \int_{x + \lambda P} f(y) \theta(x + \lambda y) \, d\lambda = \theta(x) \int_{P} f(y) \, dy \quad \forall f \in C_c^\infty(\mathbb{R}^m)$$

where $\pi, \lambda : \mathbb{R}^n \to \mathbb{R}^m$, $\pi(x, y) = \frac{y - x}{\lambda} x.$

**Thm.** "$M$ is $H^k$-measurable. $M$ is countably-$k$-rectifiable if and only if $\exists \theta$ locally $H^m$ integrable on $M$ and $\int_{T_xM} \text{app tangent plane } H^m$-a.e. $M$.

"Go back to (lemma) to define. Varifold"

**Question:** When is a general $V \in V(M)$ rectifiable?
3. \textbf{First variation of varifolds.}

- \textbf{Push-forward:} \( \Psi : U \rightarrow U' \subset \mathbb{R}^n \) diffeomorphism.
  
  \( \Psi^* V \in \mathcal{V}_k(U') \) is defined by:
  
  \[
  (\Psi^* V)(B) = \int_{\Psi^{-1}(B)} J \Psi(x,s) \, dV(x,s).
  \]

  \( \theta \in \mathcal{G}_k(U) \). \( \Psi : \mathcal{G}_k(U) \rightarrow \mathcal{G}_k(U') \)

  \[
  J\Psi(x,s) = \text{volume Jacobian of } d\Psi_x|_s.
  \]

  \[
  = \det [(d\Psi_x|_s)^* \circ d\Psi_x|_s]^{\frac{1}{2}}.
  \]

- \textbf{First variation:} \( x \in X(U) \) vector field supported in \( U \).
  
  \( \Psi_t = \text{flow of } X \).

  i.e., \( \frac{d}{dt} \Psi_t(x) = X(\Psi_t(x)) \).

  then:

  \[
  \partial_t V(x) := \frac{d}{dt} \Psi_t^* (\Psi_t^* V).
  \]

  \[
  = \frac{d}{dt} \int_{\mathcal{G}_k(U)} J \Psi_t(x,s) \, dV(x,s).
  \]

\textbf{Prop.:}

\[
\partial_t V(x) = \sum_{\omega \in \mathcal{G}_k(U)} \text{div}_S X \, dV(x,s)
\]

where \( \text{div}_S X = \sum_{i=1}^k \langle D_{e_i} X, e_i \rangle \) in basis of \( S \).
\[ y_t(x) = x + t X(x) + o(t). \]

\[ D_t y_t = 2_t x + D_t X + o(t). \]

\[ (D_t y_t)_t^n = 2_t x^n + D_t x^n + o(t). \]

\[ (\text{det} \{ \partial y_t / \partial s \} \cdot (\partial y_t / \partial s) \}) = (\text{det} \{ \partial y / \partial s \} \cdot (\partial y / \partial s) \}). \]

\[ = \left( \frac{\partial_i + t D_{ii} X + o(t)}{n} \right) \left( \frac{\partial_i + t D_{ii} X + o(t)}{n} \right) \]

\[ = \delta_{ij} + t \left( 2_i \cdot D_{ij} X + 2_j \cdot D_{ii} X \right) + o(t). \]

\[ \text{det} \left[ (\partial y_t / \partial s) \cdot (\partial y_t / \partial s) \right] \]

\[ = 1 + t Tr \left( 2_i \cdot D_{ij} X + 2_j \cdot D_{ii} X \right) + o(t) \]

\[ = 1 + 2t \left( \text{det} \{ y_t / \partial s \} \cdot (\partial y_t / \partial s) \right) \]

\[ = \text{det} \{ y_t / \partial s \} \cdot (\partial y_t / \partial s) \]
Monotonicity Formula:

Thus, \( V - (k - \text{vanfled}) = V_k(u) \) stationary.

Then \( B_0 \sim (0) = B_0(0) \sim U \).

Then \( \frac{1}{Dr} \mu(B_0) = \frac{1}{Dr} \mu(B_0) - \int_{B_0} \frac{1}{Dr} |Dr \text{v}| \text{v} \text{d}u \).

\( = D_r^+\text{v} = (Dr)^+ \text{v} \text{w.r.t.} S \).

Take \( \text{v} = u(r) \text{v} \text{, } u(0) = 0 \).

\( \text{div}_S \text{v} = u'(r) \sum_{i=1}^{k} \text{v}.e_i + u(r) \sum_{i=1}^{k} \sqrt{g} \text{v}.e_i \).

\( = u'(r) \frac{k}{r} \text{v}.e_i + k u(r) \).

\( \text{div}_S \text{v} = u(r)(|Dr| + k u(r)). \)

Stationary:

\( 0 = \delta V(\text{v}) = \int_{S} \text{div}_S \text{v} \text{v} \text{d}u \text{v} \text{S} \).

\( = \int_{S} (u(r)|Dr| + k u(r)) \text{d}V \text{v} \text{S} \).

Trick:

\( u(r) = \phi \left( \frac{r}{p} \right) . \)

\( \phi = \int_{0}^{1} m B_{\text{v}} \phi \text{c} \text{o} \)
\[
\begin{align*}
\mu(r) &= \phi \left( \frac{r}{p} \right) - \frac{1}{p} \cdot \frac{d}{dp} \left( \phi \left( \frac{r}{p} \right) \right) - \frac{r}{p^2} \\
\Rightarrow \quad r \cdot \mu(r) &= -p \cdot \frac{d}{dp} \left( \phi \left( \frac{r}{p} \right) \right) \\
\Rightarrow \quad 0 &= \int_{\Omega_n \cup \Omega} \left( -p \cdot \frac{2}{\sigma_p} \left[ \phi \left( \frac{r}{p} \right) \right] |D_s r|^2 + k \phi \left( \frac{r}{p} \right) \right) dV_{RIS} \\
\quad \text{Write } |D_s r|^2 &= 1 - |D^r_s r|^2 \\
\Rightarrow \quad k \int_{\Omega_n \cup \Omega} \phi \left( \frac{r}{p} \right) dV_{RIS} - p^2 \frac{d}{dp} \left( \int_{\Omega_n \cup \Omega} \phi \left( \frac{r}{p} \right) dV_{RIS} \right) \\
\quad &= -p \cdot \frac{d}{dp} \int_{\Omega_n \cup \Omega} \phi \left( \frac{r}{p} \right) |D^r_s r|^2 dV_{RIS} \\
\left( \phi \right)_{p \rightarrow p+1} \\
\frac{k}{p^{k+1}} \int_{\Omega_n \cup \Omega} \phi \left( \frac{r}{p} \right) dV_{RIS} - \frac{1}{p^k} \frac{d}{dp} \int_{\Omega_n \cup \Omega} \phi \left( \frac{r}{p} \right) dV_{RIS} \\
\quad &= -\frac{1}{p^k} \frac{d}{dp} \int_{\Omega_n \cup \Omega} \phi \left( \frac{r}{p} \right) |D^r_s r|^2 dV_{RIS} \\
\Rightarrow \quad \frac{d}{dp} \left[ \frac{1}{p^k} \int_{\Omega_n \cup \Omega} \phi \left( \frac{r}{p} \right) dV_{RIS} \right] \\
\quad &= -\frac{1}{p^k} \frac{d}{dp} \int_{\Omega_n \cup \Omega} \phi \left( \frac{r}{p} \right) |D^r_s r|^2 dV_{RIS} \\
\text{Let } \phi \rightarrow \chi_{[0,1]} \\
\Rightarrow \quad \frac{1}{p^k} \text{AV}(B_0) - \frac{1}{p^k} \text{AV}(B_5) &= \int_{B_0 \setminus B_5} \frac{1}{p^k} |D^r_s r|^2 dV_{RIS}
\end{align*}
\]
Let \( \gamma : \mathbb{R} \to \mathbb{R} \) be \( \gamma(t) = \frac{y - x}{\lambda} \).

**Definition:** Manifold tangent. Let \( \mathbf{V}_x \) be the set of all \( c = \lim_{t \to 0} \gamma(t) \). Assume \( x \in \mathcal{U} \). It follows that \( \mathbf{V}_x \) is the tangent space of \( \mathcal{U} \) at \( x \).

**Claim:** \( \frac{\Delta c(x_p)}{w_k \rho_k} = \vartheta_0 \) (for a.e. \( \rho \))

\[
\Delta c(x_p) = \lim_{\lambda \to 0} \frac{\lambda}{w_k \rho_k} \Rightarrow \frac{\Delta c(x_p)}{w_k \rho_k} = \vartheta_0
\]

By monotonicity formula,

\[
\vartheta_0 = \mathbf{Q}(\vartheta_0) = \frac{\Delta c(x_p)}{w_k \rho_k} - \int_{w_k \rho_k}^{1} \frac{D_p r^2}{G_k(r)} \, dr
\]

\( \Rightarrow |D_p r| > 2 \) for all \( r > 0 \).
\[ \begin{align*}
\text{i.e., } & \exists \alpha > 5 \text{ for } C \text{-a.e. } (x_0, s) \in G_2(B) \\
\text{"Lemma" : } & \frac{1}{\lambda^k} \mathcal{L}(\mathcal{V}_0 \times A) = \mathcal{L}(A).
\end{align*} \]

**Rectifiability Thm:** \( V \in V_{\text{loc}}(U) \text{ stationary.} \)

if \( \Theta(J, u(x)) > 0 \) for \( u \) a.e. \( x \in U \) then \( U \) is rectifiable, i.e. \( V = V(M, 0) \)

\[ \begin{align*}
& \quad \text{\( M \text{-}\mathcal{H}^k \text{-measurable, countably-}\mathcal{H}^k \text{-rectifiable} \)} \\
& \quad \text{\( \Theta \text{-locally } \mathcal{H}^k \text{ integrable} \)}
\end{align*} \]

\[ \begin{align*}
\text{If, } \Theta \text{ satisfies, } & \Theta(J, u(x)) \geq 1 \text{ for } u \text{-a.e. } x \in \mathbb{R}^n, \\
\text{then } C \text{ is a "cone,"} & \text{ i.e. } \eta \circ \xi C = C.
\end{align*} \]

\[ \begin{align*}
\text{\( V \in V_{\text{loc}}(U) \text{ stationary.} \)
\end{align*} \]

\[ \text{\( \text{supp}(V) \subset M^k \subset U \text{ smooth. } \Rightarrow V = \mathcal{F}_0 \circ V_{\text{lim}} \)} \]

**Compactness Thm:**

\[ \begin{align*}
& \{ V_i \subset V_{\text{loc}}(U) \text{ stationary.} \} \\
& \text{sup} \{ \Theta(J, u(w) : i \} < \infty \text{ a.e. } w \in U \} \\
\Rightarrow & \ V_i \rightarrow V \rightarrow \text{rectifiable.}
\end{align*} \]
1. **Remarks** (for the case $U \in M^n$).

Can embed $M^n \rightarrow \mathbb{R}^{n+k}$ then $V \in \text{Val}(U)$

$\mathcal{B}$ stationary $\iff$

$$
\delta V(x) = \int_{\partial V(x)} \nu(x) \text{d}V(x)
$$

$$
= -\int_{\partial V(x)} \left( \text{Tr}_x A^m(x) \right) \text{d}V(x).
$$

- $A^m$ and ff. of $M \rightarrow \mathbb{R}^{n+k}$

$$
\text{Tr}_x A^m = \frac{1}{k} \left( A^m(\epsilon_i, \epsilon_i) \right)
$$

- $\{ \epsilon_1, \ldots, \epsilon_k \}$ orthonormal basis for $x$.

- Can redo Monotonicity

- Variance (defined on $\mathbb{R}^{n+k}$)

- All others works.
8° Maximum Principal

Lemma (Thm). \( V \in V_{2}(U) \). \( U \subset \mathbb{R}^{n} \). starting.

\( 0 < t < s \). \( B_{t}(0) \subseteq B_{s}(0) \subseteq U \).

\( \text{II}V_{I} (A_{0}(t,s)) \neq 0 \). \( (A_{0}(t,s) = B_{s} \setminus B_{t}) \).

then \( \forall x \in \text{spt II}V_{I} \land \exists B_{+} \)

\[ (B(x, r) \setminus B_{+}) \land \text{spt II}V_{I} \neq \emptyset \]

\( r > 0 \)

\( \quad \exists \varepsilon > 0 \). set.

\[ (B(x, \varepsilon) \setminus B_{+}) \land \text{spt II}V_{I} = \emptyset \]

Want to construct a variational v.f. \( \overline{X} \) to get contradiction.

Take \( \frac{1}{2} < r < h + r + \varepsilon \).

\( P = -\overline{X} \)

\( K = B_{+}(0) \setminus B_{r}(x) \)

\( \subseteq B(x, \varepsilon) \)

\( \& B_{r}(x) \setminus B(x, \varepsilon) \subseteq B(r, \varepsilon) \).
Would like to take \( X = \bar{x} - \bar{p} \).

Cutoff function:

\[ 0 \leq f \leq 1, \quad \text{supp} f \subset B(\bar{x}, r), \quad f = 0 \text{ on } \kappa \]

\[ 0 \leq g_{18}(s) \leq 1, \quad g_{18}(s) = 0 \quad \text{if} \quad s \geq t + \delta t \]
\[ g_{18}(s) = 0 \quad \text{if} \quad s < 0 \]

\[ \text{let} \quad \bar{x}_{(s)} = g_{18}(\bar{x} - \bar{p}_1) \cdot f_{18} \cdot (\bar{x} - \bar{p}) \]
\[ \in \mathcal{A}(\kappa) \]

\[ 0 = \text{div} (\bar{X}) = \int_{\partial \kappa (\kappa)} \text{div}_s (\bar{X}) \cdot d\nu (x, s) \]

\[ \text{div}_s (\bar{X}) = g_{18}(\bar{x} - \bar{p}_1) \cdot P_s (x - \bar{p}) + g_{18}(\bar{x} - \bar{p}_1) \cdot P_s (x - \bar{p}) \cdot k \]
\[ + g_{18}(\bar{x} - \bar{p}_1) \cdot f_{18} \cdot k \]
\[ I = 0 \]

Claim: \( I = 0 \) on \( \text{supp} \nu \).

\[ g = 0 \quad \text{if} \quad |\bar{x} - \bar{p}_1| > r, \quad \Rightarrow 0 \text{ on } \kappa \]

\( \text{supp} \nu \subset B_{18}(\bar{x}) \cap B_{10}(\kappa) \)
\[ 1P = 0 \text{ on } \kappa \]
\[
\begin{align*}
\text{only } k \text{ left.} \\
\Rightarrow f &= 1.
\end{align*}
\]

\[
\begin{align*}
0 &= \int_{B_2(0,1)} g(x-\bar{x}_1, f(x)) \, dV(x, y) \\
&\quad + \int_{B_2(0,2)} g(x-\bar{x}_2) \, dV(x, y) \\
&\quad + \int_{B_2(0,3)} g(x-\bar{x}_3, f(x)) \, dV(x, y) \\
&\quad + \int_{B_2(0,4)} g(x-\bar{x}_4) \, dV(x, y) \\
&\quad + \int_{B_2(0,5)} g(x-\bar{x}_5) \, dV(x, y) \\
&\quad + \int_{B_2(0,6)} g(x-\bar{x}_6) \, dV(x, y) \\
&\quad + \int_{B_2(0,7)} g(x-\bar{x}_7) \, dV(x, y) \\
&\quad + \int_{B_2(0,8)} g(x-\bar{x}_8) \, dV(x, y) \\
&\quad + \int_{B_2(0,9)} g(x-\bar{x}_9) \, dV(x, y) \\
&\quad + \int_{B_2(0,10)} g(x-\bar{x}_{10}) \, dV(x, y).
\end{align*}
\]

\[
\begin{align*}
k \ll 1(k) &> 0. \\
\Rightarrow 0 &\in E.
\end{align*}
\]

Rk: Also works for \( U \subset M \subset \mathbb{R}^{n+k} \) if restrict to "\( U \)" small.
Sand type. Then for stationary varifold.

\( \text{Lemma.} \quad V \in \mathcal{V}_k(W) \) stationary, integer rectifiable.

(i.e. \( V = V(M, \theta) \), \( \theta : M \to \mathbb{Z}^+ \).

1. \( \forall x \in U, \quad B_\rho(x) \subset U \).

2. \( \mathcal{T} = \{ y \in \text{spf}(W) : \mathcal{T} \text{ transversal to } \partial B_{\rho(x)} \} \).

Then \( \mathcal{T} \) is dense in \( \text{spf}(W) \cap B_\rho(x) \).

\[ \begin{array}{c}
\text{If false, then } \exists y \in B_\rho(x) \cap \text{spf}(W) \quad \text{and } y \ll 1.
\end{array} \]

\( \Gamma = \{ \bar{z} \in M : \partial \Lambda_{B_{\rho(x)}}(x) \} \).

Assume \( B_{\rho(x)} \subset B_{\rho(x)} \). Let \( (x, \gamma, \rho) \) -plan curve on \( B_{\rho(x)} \).

Take \( f \) cutoff function on \( B_{\rho(x)} \).

\[ \begin{array}{c}
\text{Let } \chi = \frac{f}{f} \geq \frac{1}{\rho}.
\end{array} \]
\[ \delta V(X) = \int_{\mathfrak{m}(B^+_\mathfrak{y})} \text{d} \nu_{\text{tan}} X \mathfrak{y} \otimes \text{d}(\nu^2 \Omega) \]

\[ \text{d} \nu_{\text{tan}} X \mathfrak{y} = \frac{1}{2} \left< \text{Rei}(X \mathfrak{y}) , e_i \right> \text{ for } \mathfrak{y} \in \mathfrak{m} \]

\[ = \frac{1}{2} \left< \text{Rei}(\frac{\partial \mathfrak{y}}{\partial y}), e_i \right> \]

\[ \Rightarrow \delta V(X) = \int_{\mathfrak{m}(B^+_\mathfrak{y})} \frac{\frac{\partial \mathfrak{y}}{\partial y}}{r} \text{d}(\nu^2 \Omega) \]

\[ \Rightarrow c \| \nabla (B_{x^2 y^2}) \| > 0. \]

Set-up & main result.

- \( M \) = closed 3-mfld.
- \( \text{Diff}_0 (M) = \text{diffeomorphisms} \)
  of \( M \) = id.

\( \mathcal{K} = \text{the set of isotopies} \)

\[
\forall \tau \in \mathcal{K}, \quad \forall \omega \in \mathcal{C}^0 (\tau \times \mathcal{K}, \text{Diff}_0 (M)),
\]

\[
\begin{array}{c}
\forall \tau \in \mathcal{K}, \quad \forall \omega \in \mathcal{C}^0 (\tau \times \mathcal{K}, \text{Diff}_0 (M)), \\
4 \in \mathcal{C}^0 ([0,1] \times \mathcal{K}, \mathcal{K}), \\
4(0 \cdot \tau) = \text{id} \\
4(\tau \cdot \omega) \in \text{Diff}_0 (M)
\end{array}
\]

- A smooth family of surfaces in \( \mathcal{K} \) of \( \mathcal{K} \):

\[
\forall \tau \in \mathcal{K}, \quad \forall \omega \in \mathcal{C}^0 (\tau \times \mathcal{K}, \text{Diff}_0 (M)),
\]

\[
\begin{array}{c}
\forall \tau \in \mathcal{K}, \quad \forall \omega \in \mathcal{C}^0 (\tau \times \mathcal{K}, \text{Diff}_0 (M)), \\
\forall \tau \in \mathcal{K}, \quad \forall \omega \in \mathcal{C}^0 (\tau \times \mathcal{K}, \text{Diff}_0 (M)), \\
\exists \tau, \omega \in \mathcal{C}^0 ([0,1] \times \mathcal{K}, \text{Diff}_0 (M)), \\
\tau(0 \cdot \tau) = \text{id} \\
\omega(\tau \cdot \omega) \in \text{Diff}_0 (M)
\end{array}
\]

A smooth family of surfaces in \( \mathcal{K} \):

\[
\forall \tau \in \mathcal{K}, \quad \forall \omega \in \mathcal{C}^0 (\tau \times \mathcal{K}, \text{Diff}_0 (M)),
\]

\[
\begin{array}{c}
\forall \tau \in \mathcal{K}, \quad \forall \omega \in \mathcal{C}^0 (\tau \times \mathcal{K}, \text{Diff}_0 (M)), \\
\tau \in \mathcal{K}, \quad \forall \omega \in \mathcal{C}^0 (\tau \times \mathcal{K}, \text{Diff}_0 (M)), \\
\exists \tau, \omega \in \mathcal{C}^0 ([0,1] \times \mathcal{K}, \text{Diff}_0 (M)), \\
\tau(0 \cdot \tau) = \text{id} \\
\omega(\tau \cdot \omega) \in \text{Diff}_0 (M)
\end{array}
\]

**Def 1.1:** A family \( \{ \Sigma_t \}_{t \in [0,1]} \) is continuous if:

- (c1) \( \Sigma_t \) cont. w.r.t. \( t \),
- (c2) \( \Sigma_t \rightarrow \Sigma_{t_0} \) in Hausdorff when \( t \rightarrow t_0 \).

**Def 1.2:** A family \( \{ \Sigma_t \}_{t \in [0,1]} \) of \( \mathcal{K} \)-measurable subsets of \( M \) is a generalized (smooth) family (of surfaces) if there exists a finite set \( T \subset [0,1] \),

\[
\begin{array}{c}
\exists \text{finite set } P \subset M, \\
\exists \text{finite set } P \subset M, \\
\exists \text{finite set } P \subset M, \\
\exists \text{finite set } P \subset M,
\end{array}
\]

\[
\begin{array}{c}
\forall \epsilon > 0, \quad \exists \delta > 0, \\
\forall \epsilon > 0, \quad \exists \delta > 0, \\
\forall \epsilon > 0, \quad \exists \delta > 0, \\
\forall \epsilon > 0, \quad \exists \delta > 0,
\end{array}
\]

\[
\begin{array}{c}
\forall \epsilon > 0, \quad \exists \delta > 0, \\
\forall \epsilon > 0, \quad \exists \delta > 0, \\
\forall \epsilon > 0, \quad \exists \delta > 0, \\
\forall \epsilon > 0, \quad \exists \delta > 0,
\end{array}
\]
Given a generalized family \( \{ \Sigma_t \} \), one can generate a new generalized family as follows:

\[ \forall t \in \mathbb{R} \quad (i.e. \, t \in \mathbb{R}) \Rightarrow \quad \Sigma_t = \Sigma_t(\Sigma_t^+). \]

\( \Sigma_t^+ \) is the family generated by isotopies.

**Def.** A set \( \Lambda \) of generalized family is said to be **saturated** if it is closed under isotopies.

\( R_k \). Require \( \forall \Lambda. \quad k \in \mathbb{N} \cup \{ 0 \} \quad \text{integer.} \quad \forall t \in \text{set P of bad pts. for any } \Sigma_t \in \Lambda.

\[ \text{has } \# \leq N(\Lambda). \]

**Def.** Given \( \{ \Sigma_t \} \subset \Lambda \)

\[ \mathcal{F}(\{ \Sigma_t \}) := \max_{t \in \{0,1\}} H^2(\Sigma_t). \]

\( m(\Lambda) := \inf_{\Sigma_t \in \Lambda} \mathcal{F}(\{ \Sigma_t \}) = m(\Lambda) \max_{t \in \{0,1\}} H^2(\Sigma_t) \).

\[ \text{maximal slice. (not min.)} \]

\[ \text{min.} \]

- If \( \lim_{t \to 0} \mathcal{F}(\{ \Sigma_t \}) = m(\Lambda) \quad \{ \{ \Sigma_t \} \} \quad \text{minimizing seq.} \subset \Lambda \).

- If \( \forall t \quad \{ \Sigma_t \} \subset \Lambda \quad \text{all } \{ \Sigma_t \} \quad \text{min-max seq.} \subset \Lambda \).
Question: When is \( m_0(A) > 0 \).

Prop.: \( M \) - closed, \((3, \text{ inf})\). \( f \) is Morse function, \( f: M \to \mathbb{R} \).

then the level sets \( \{ I_t = \{ f = t \} \} \) is a generalized family.

Moreover, let \( A = \) the smallest saturated set containing \( \{ I_t \} \), then \( m_0(A) > 0 \).

Proof:

\[ A = \{ \{ I_t \} + \varepsilon \mid \varepsilon \geq 0 \} \]  

\[ T_t = (t + I_t), \; t \in C([0,1] \times M, M) \]

\[ \text{let} \; U_t = f^{-1}(I_t, I_t) = \{ x \mid f(x) < t \} \]

then \( V_t = 4(t + U_t) \quad \Rightarrow AV_t = T_t \)

\[ U_0 = \phi, \; U_1 = M \quad \Rightarrow V_0 = \phi, \; V_1 = M \]

\[ \exists S_0 \in C(0, 1) \quad \Rightarrow \quad \text{vol} (V_{S_0}) = \frac{\text{vol}(M)}{2} \]

Isoperimetric inequality: \( \Rightarrow \)

\[ \frac{\text{vol}(M)}{2} = \text{vol} (V_{S_0}) \leq C(M) \left( H^2(T_{S_0}) \right)^{\frac{3}{2}} \]

\[ \Rightarrow \quad \frac{f(t S_0)}{3} = \max_{t \in [0,1]} H^2(T_t) \geq H^2(T_{S_0}) \geq \left( \frac{\text{vol}(M)}{2C(M)} \right)^{\frac{2}{3}} \]

Thus, \( M \) - closed, \((3, \text{ inf})\). A saturated set (of generalized family) \( m_0(A) > 0 \). Then \( \exists \min_{\text{max}} \exists \sigma \{ x \in \mathbb{R} \mid \text{vol} \} \) now.
\[
\max_{\Sigma_t} H^2(\Sigma_t) \leq \max_{\Sigma_t} H^2(\Sigma_t') \quad \text{s.t.} \quad \Sigma_t' \to \Sigma_t
\]

\[
\Sigma^n_{\Sigma_t} : H^2(\Sigma^n_{\Sigma_t}) \to \text{varifold}
\]

\[
\forall \xi, \Sigma^n_{\Sigma_t} = \frac{1}{\sum_i n_i \Sigma_i}
\]

2. Sketch of proof.

Step 1. Take \( \{\Sigma^n_t\} \) minimizing sequence.

\[
\max_{\Sigma_t} H^2(\Sigma_t) \leq \max_{\Sigma_t} H^2(\Sigma_t') \quad \text{s.t.} \quad \Sigma_t' \to \Sigma_t
\]

- Easy to show that \( \Sigma_t^k \to \Sigma_0 \) stationary for some \( \{\Sigma^k_t\} \).
- May still exists \( \Sigma_t^k \to \) stationary.

\[
\Sigma_0 \to \text{varifold}
\]

\[
\Sigma_0 \text{ has the same area as } \Sigma_0
\]

- Tightening to get rid of bad slices.

Prop 1: A minimizing sequence \( \{\Sigma^n_t\} \subset \text{varifold} \)

\[
\text{s.t. evey min-max seg } \{\Sigma^n_t\} \text{ subconcave to stationary varifold}
\]

- Idea: functional analysis type argument in the setting of varifolds.
Step 2: “Almost minimizing”.

- Regularity theory for (stationary) hypersurfaces.
  \[ \Sigma^{n+1} \subset M^n, \quad n \in \mathbb{N}, \quad \Sigma^{n+1} \text{ volume minimizing} \]
  \[ \implies \Sigma \text{ smooth} \]
- Need some kind of minimizing.

**Def.** Given \( \varepsilon > 0 \), \( U \) open \( \subset M \), \( \Sigma^2 \) is \( 2 \)-a.m. in \( U \)

\[ f \not\in A^2(U), \quad \forall t \]
\[ H^2(f(t, \Sigma)) \leq H^2(\Sigma) + \varepsilon/8 \]
\[ H^2(f(1, \Sigma)) \leq H^2(\Sigma) - \varepsilon \]

A seq \( \{\Sigma^n\} \) is \( 2 \)-a.m. in \( U \) if each \( \Sigma^n \) is \( 3^n \)-a.m. in \( U \) for some \( \varepsilon_n \to 0 \)

**Prop.** \( \exists \rho \subset M \to \mathbb{R}^+ \) and a min-max seq \( \{\Sigma^i\} \)

s.t.
- \( \{\Sigma^i\} \) a.m. in every \( B_n \) centered at \( x \) & \( \rho \)
- \( \Sigma^i \) smooth in \( B_n \) for \( j \) large
- \( \Sigma^i \to \Sigma \) stable in \( M \)

Step 3. If \( \{\Sigma^i\} \) is a.m. in \( \Delta_n \), \( \Delta \) \( \nu \)-stable manifold.

\[ \|\nu\|_n = 1/2^i \|\nu\|_M, \quad \nu = \nu^i \text{ on } M \setminus \Delta_n \]
\[ \nu^i \text{ stable minimal surf in } \Delta_n \]
Step 4: Construct $V'$. 

1. Take $V'$ where $\Sigma^m \subset V$. 

2. Fix $n$-large $\Sigma^m \subset \mathbb{R}^n$ at:

$$L_\omega \cdot H(\Sigma \times \mathbb{R}^n) = \inf_{\varphi \in \mathcal{G}} H(\varphi(\Sigma, \mathbb{R}^n)) \left\{ \begin{array}{l}
T(\mathbb{R}^n) = \{ \varphi \in \mathcal{G}(\mathbb{R}^n) : H(\varphi(\Sigma, \mathbb{R}^n)) \\ H(\varphi(\Sigma, \mathbb{R}^n)) \leq H(\Sigma \times \mathbb{R}^n) + \frac{1}{\alpha n} \}
\end{array} \right.

3. Claim: $\Sigma^m \rightarrow W^d$ smooth, stable, min in $\Lambda$. 

4. Hence $\Sigma^m \rightarrow V'$. 

Proof of Claim: Take $B_{\alpha n} \subset \Sigma^m$. 

Show that minimal isotopy problem in $B_{\alpha n}$ w.r.t. $\Sigma^m$:

$$\leq T(\mathbb{R}^n) - \text{problem}.$$
3. Tightening.

\[ X = \{ V \in V_2(M) : A^i(V) \leq 4m_0 \} \quad \text{— weak topo} \]

\[ V_\infty = \text{stationary varifold in } X. \]

Prop.: \( \exists \{ T_\infty^n \} \subset A \). \( f : \{ T_\infty^n \} \) is min-max, then \( \exists (T_\infty^n, V_\infty) \).

Proof. Want: \( f : X \rightarrow T_\infty^n \quad V \text{ starting} \quad \gamma(V) = V \quad \text{not-stable} \quad \gamma \text{ clears mass} \).

\[ \{ \{ T_\infty^n \} \} \xrightarrow{\gamma} \{ \{ T_\infty^n \} \}. \]

\( \forall \epsilon > 0, \exists \delta > 0, N > 0 \) s.t.

\[ \text{if } \quad \gamma^n(T_\infty^n) > m_0 - \delta \quad \Rightarrow \quad \gamma^n(T_\infty^n, V_\infty^0) < \epsilon. \]

---

**Step 1.** A map \( X \rightarrow \mathcal{A}(M) \).

\( + \nu \) — generated from \( X \in \mathcal{A}(M) \) as 1-parameter family of diffeos.

\[ V \mapsto \mathcal{N}_\nu \]

\( \forall k \in \mathbb{Z} \quad \mathcal{N}_\nu = \{ V \in X : \frac{1}{2^k+1} \leq \gamma(V, V_\infty) \leq \frac{1}{2^k} \} \)

Claim: \( \forall V \in X \quad \exists c(k) > 0 \quad \forall V \in \mathcal{N}_\nu \quad \exists X \nu \neq X \nu (m) \)

\[ \| X \nu \|_{L^p} \leq 1, \quad \delta^V(X \nu) \leq -c(k). \]

(i.e. by contradiction argument.)

Want: \( V \mapsto \mathcal{N}_\nu \) continuous.

\( \forall V \in \mathcal{N}_\nu \quad \exists r = r(V) > 0, \quad \text{s.t. if } \mathcal{W} = U(V) \)

\[ \delta^W(X \nu) \leq -c(k/2) \quad \delta^V(X) = \int_{\partial_{\nu \in X}} c(k/2) \chi \text{ div} \nu \text{ div}(X) \]
\( k \). Find \( \{ U_i^k \}_{i=1}^{\infty} \) (finite union) \& \( \{ X_i^k \}_{i=0}^{\infty} \) such that:

- Each \( U_i^k \) concentric ball to \( U_i^k \) of half radius cover \( V_0 \).
- \( \partial U_i^k \cap U_j^k = \emptyset \) if \( |i-j| = 2 \).
- \( \{ U_i^k \} \) locally finite covering of \( X \setminus D_0 \). 
- \( \forall \theta \in H^0 \rightarrow \sum_i \psi_i^k (v) (X_i^k) \), \( x \rightarrow \theta (m) \) const \( ||H_0||_{L^\infty} < 1 \).

**Step 2.** \( X \rightarrow \text{Homotopy} \:

\( V \in U_k, \ r(V) = \text{smallest radius of } U_i^k, \ r(V) > |k| \),

\( \forall W \subset U(V), \ \partial W (H(V)) < -\frac{1}{2} \min \{ c(k+1), c(k), c(k+1) \} \).

\( \exists g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), \( r \rightarrow \mathcal{R}^+ \).

\( T : \mathbb{R}^+ \rightarrow \mathcal{R}^+ \), \( g \rightarrow \mathcal{R}^+ \).

\( \exists G(x, t) \) \( m \rightarrow m \) \( \frac{G(x, t)}{g} = T(v) \).

**Claim**

- \( T : \mathbb{R}^+ \rightarrow \mathcal{R}^+ \), \( G : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \).
- \( T(v) = \partial (V \cup D_0) > 0 \), \( \forall V \rightarrow V' = \partial (T(v) \cdot V) \).

\( ||v'||(m) \leq ||v||(m) - G(v) \).

- \( G(s) \rightarrow 0 \) as \( s \rightarrow 0 \).

**In fact:** \( \forall V, \ r(V) = r (\partial (V \cup D_0)), \ \forall T = T (\partial (V \cup D_0)) \)

- \( v^T (x) \rightarrow v^T (x) \neq V, \ t \in [0, T] \).

then

\( H(v) \rightarrow \vec{v}^T (t) \neq V, \ t \in [0, T] \).
\( \|\Phi(T)\#V\|_W(M) - \|V\|_W(M) \leq \int_0^T (\partial V_+ (t)) (H_V) \)
\( \leq -T \int G (\partial (V_+ W_0)) \leq G (\partial (V_+ W_0)) \)

**Step 3:** Construct \( \{T^n_+\} \)

- Let \( V(t, \cdot) = \mathbb{E}_V (T(\partial (V_+ W_0) t, \cdot)) \)
- \( \mathbb{E}_V: \mathbb{R}^+ \to \mathbb{R}^+ \), const \( \mathbb{E}(0) = 0 \)
- \( V = \mathbb{E}_V (1, \cdot) \# V \), \( \|V\|_W(M) \leq \|V\|_W(M) - L (\partial (V_+ W_0)) \)

\[ \{\Sigma^n_+\} \to \{T^n_+\}, \quad T^n_+ = \mathbb{E}_{\Sigma^n_+} (1, \cdot) \Sigma^n_+ \]

\( H^2 (T^n_+) \leq H^2 (\Sigma^n_+) - L (\partial (\Sigma^n_+ W_0)) \]

- \( \mathbb{E}_+ \Sigma^n_+ \) generated by \( H_+ = T (\partial (\Sigma^n_+ W_0)) H_+ \)

\[ \tilde{h}_+ = \mathbb{E}_+ T \to \mathbb{A}(M, \text{smooth}) \]

- \( \mathbb{A}_+ \) flow generated by \( \tilde{h}_+ \)

\[ T^n_+ = \mathbb{A}_+ (1, \cdot) \Sigma^n_+ \]

\[ H^2 (\mathbb{A}_+ T^n_+) \leq H^2 (\Sigma^n_+) - \frac{1}{2} L [\partial (\Sigma^n_+ W_0)] \]

**Thus**

\[ m_0 - \delta \leq H^2 (T^n_+) \leq H^2 (\Sigma^n_+) - \frac{1}{2} L [\partial (\Sigma^n_+ W_0)] \]

\[ \leq m_0 + \frac{\delta}{2} - \frac{1}{2} L [\partial (\Sigma^n_+ W_0)] \]

\[ \Rightarrow L [\partial (\Sigma^n_+ W_0)] \leq 2 (\delta + \frac{\delta}{2}) \]

\[ \Rightarrow \partial (\Sigma^n_+ W_0) \leq \delta \]

\( \Rightarrow \)
Almost minimizing.

$\{x^n\}$ - minimizing seq from $3^\circ$.

Prop1: $x: M \rightarrow \mathbb{R}^\mathbb{N}$ & min-max seq $\{x^j\}$ at

- $\{x^j\}$ a.m. in every $\mathbb{A} = \mathbb{A}_{(x^n, x^j)}$, $\forall x \in M$;
- $x^j$ is $C^0$ in $\mathbb{A}$ for $j$ large;
- $x^j \rightarrow V$ stating varifold $\checkmark$ by $3^\circ$.

Def: $x$ is ε-a.m. in $(U_1, U_2)$ if $x: M \rightarrow \mathbb{R}^\mathbb{N}$
in one of $(U_1, U_2)$.

$\mathcal{E}_0 = \{ (U_1, U_2) : U_1 \subset M$ domain, $d(U_1, U_2) > 2 \max \{ \text{diam}(U_1) \} \}^{\mathbb{N}}_\leq 1.2$

Prop2: $\exists \text{ min-max seq } \{x^L\} = \{x^{n(1)}\}$

- $x^L \rightarrow$ stating varifold
- $x^L$ is $\alpha$-a.m. in every $(U_1, U_2) \in \mathcal{E}_0$.

Prop2 $\Rightarrow$ Prop1: If Prop1 is false, then $x \cdot \forall r > 0$, $\exists \text{ } r_1 < r_2 < r$.

St. $\{x^j\}$ is not a.m. in $B_{r_1, r_2}$.

Fix $r_1, r_2$. Choose $x_r \in (r_1, r_2)$, $\forall x_r \in \mathbb{A}_{(x^n, x^j)}$ & $\Rightarrow x^j \text{ is not a.m. in } B_{r_1, r_2}$.

\[ \Rightarrow x^L \text{ not a.m. in } (B_{r_1, r_2}, B_{r_1, r_2}') \Rightarrow \checkmark \]

2. $x \in P$ singular at for $\{x^n\} \rightarrow P$.

- If $x \in P$, In $\Lambda$, $B_n$ is $C^0$ for $n \gg 1$.
- If $x \notin P$, In $\Lambda$, $B_n$ is $C^0$ for small radii.

Proof of Prop2: Take $\{x^n\}$, $d(\mathbb{A}^n) < \delta + \frac{1}{n}$.

Claim. $\forall \zeta > 0$. $\exists n > \zeta$, $\forall n > \zeta$, $x^n = x^m$ is $\frac{1}{\zeta}$-a.m.

in any $(U_1, U_2) \in \mathcal{E}_0$. $\Rightarrow H^1(x^n) \approx \text{mo} - \frac{1}{\zeta}$.
If not, \( k_n = \{ t \in [0, 1], H^2(\Sigma^+ t) > k_n - \frac{\epsilon}{n} \} \).

Then \( A = n > L, A \in k_n, \exists (U_1, U_2) \text{ s.t. } \Sigma^+ \text{ is not a } X-\text{ann in } (U_1, U_2) \).

\[ \exists \, \psi_+^2 \in \mathcal{M}(U_2) \text{ s.t. } \]
\[ H^2(4_i^2 (1, \Sigma^+)) \leq H^2(\Sigma^+) - \frac{1}{2L} \]
\[ H^2(4_i^1 (2, \Sigma^+)) \leq H^2(\Sigma^+) + \frac{1}{4L} \quad \text{for all j}. \]

\( k_n, t \in k_n \), as \( \{ \Sigma^+ \} \) is count. w.r.t. "+", \( \aleph \) is increasing s.t.
\[ H^2(4_i^1 (1, \Sigma^+)) \leq H^2(\Sigma^+) - \frac{1}{2L} \]
\[ H^2(4_i^2 (1, \Sigma^+)) \leq H^2(\Sigma^+) + \frac{1}{4L} \quad \text{for all j}. \]

So can find a cover of \( k_n \). \[ I_1, \ldots, I_r \]
\[ (U_1, U_2) \ldots (U_r, U_2) \in \mathcal{C}_0 \]
\[ (4_i^1, 4_i^2) \ldots (4_i^1, 4_i^2) \]

\[ I_k \text{ covers } k_n, \exists \lambda I_k = \emptyset \quad 1 \not\subset \not\subset k \]
\[ 4_i^2 \in \mathcal{M}(U_2) \]
\[ H^2(4_i^2 (1, \Sigma^+)) \leq H^2(\Sigma^+) - \frac{1}{2L} \quad \text{for all j}. \]
\[ H^2(4_i^2 (2, \Sigma^+)) \leq H^2(\Sigma^+) + \frac{1}{4L} \quad \text{for all j}. \]

---

**Step 1**: Define \( \{ I_k \} \).

\[ \rightarrow \]
\[ I_1, \ldots, I_r \text{ intervals in } [0, 1] \]
\[ V_1, \ldots, V_r \in \mathcal{M}(U_1) \]
\[ \psi_1, \ldots, \psi_r \in \mathcal{M}(U_2) \]

Want \( I_1, \ldots, I_r \) intersect only nearby intervals.

\[ v_i \not\subset \not\subset v_{i+1} = \emptyset \text{ if } I_i \cap I_{i+1} = \emptyset \]
\[ H^2(\psi_i (1, \Sigma^+)) \leq H^2(\Sigma^+) - \frac{1}{2L} \quad \text{for all j}. \]
\[ H^2(\psi_i (2, \Sigma^+)) \leq H^2(\Sigma^+) + \frac{1}{4L} \quad \text{for all j}. \]

**Lemma**: If \( (U_i, U_i^1) (V_i, V_i^1) \in \mathcal{C}_0 \), then \( i \not\subset \not\subset j \in [1, 2] \).
\[ \langle u_i, v_i \rangle > 0 \]
If \( \lambda_1 \lambda_2 = \emptyset \), \( J_1 = \emptyset \), \( V_1 = U_1 \).

If \( \lambda_1 \lambda_2 \neq \emptyset \), take \( U_1, U_2 \) at \( d(U_1, U_2) > 0 \).
\( J_1 = I_1 \), \( V_1 = U_1 \).

If \( \lambda_2 \lambda_3 = \emptyset \), \( J_2 = I_2 \), \( V_2 = U_2 \).

If \( \lambda_1 \lambda_3 \neq \emptyset \), if \( d(U_2, U_3) > 0 \), \( J_3 = I_3 \), \( V_3 = U_3 \).
\( J_3 = \emptyset \), \( V_3 = U_3 \),

Need to separate \( I_2 \) to \( J_2 \) \( V \) \( J_3 \).

\( V_2 = U_2 \), \( U_3 = \text{the other} \).

And so on.

\[ \text{Step 2.} \quad \text{Take} \quad \psi_i : \mathfrak{M}_1 \rightarrow \{0, 1\}. \quad \text{Supported in} \quad J_i. \]

If \( s \in K \), at least one \( \psi_i(s) = 1 \).

\( \forall t \in \{0, 1\}. \quad \text{Inel}^+ = \{ i \in \{1, \ldots, K \} : \psi_i(t) \neq 0 \}. \quad \text{rel} J_i \).

\( \mathcal{T}_+ \) \{ \psi_i(y_i(t), \Sigma^+_t) \} \quad \text{in} \quad V_i, \quad i \in \text{Inel}^+ \)

at least two \( \psi_i(t) \neq 0 \).

The inequality \( \lambda_i \wedge \lambda_j = \emptyset \).

\[ \text{Step 3.} \quad \text{If} \quad t \notin K. \quad h^2(\mathcal{T}_+) \leq h^2(\Sigma^+_t) + \frac{1}{2L} \leq m_0 - \frac{1}{2L} \]

If \( t \in K. \quad h^2(\mathcal{T}_+) \leq h^2(\Sigma^+_t) + \frac{1}{4L} - \frac{1}{2L} \leq m_0 - \frac{1}{4L} \)

\( \Rightarrow \text{in contradiction!} \)
5° Regularity

**Def.** \( U \in U_2(M) \) stating. \( U \subset M \) open. \( V' \in U_2(M) \) is a "replacement" of \( V \) in \( U \) if:

1. \( V' \) stating
2. \( \|V'\|_m = \|V\|_m \) \( V'_x = V_x \) \( \forall x \in U \)
3. \( V' \perp U \) - stable minimal surface. I.e., \( (\exists \omega < \varepsilon) \)

**Def.** "\( V \) has "good replacement property" in \( U \) if:

a) \( \exists \ v : U \rightarrow \mathbb{R}^+ \). \( V \) has a "replacement" \( V' \) in any \( \mathbb{R}^n \setminus \{x\} \)

b) \( V' \) has a "replacement" \( V'' \) in \( \mathbb{R}^n \setminus \{x\} \)

Prop. 3. If "\( V' \in U_2(M) \) stating has "good replacement property" in \( U \subset M \), then \( V \) is a \( C^\infty \) min surface in \( U \).

Prop. 4. "\( V \) as above. If \( \exists \ v_1 : M \rightarrow \mathbb{R}^+ \) s.t. \( V \) has a replacement.

in any \( \mathbb{R}^n \setminus \{x\} \), then:

1. \( V \) is integer rectifiable in \( U \).
2. \( \Theta(V, x) \geq 1 \) \( \forall x \in \text{Int} \setminus U \).
3. \( T(V, x) \cdot e \text{ Tang}(V, x) \) is a multiple of a plane.

**Proof.** \( \forall x \in \text{Int} \setminus U \) \( \Rightarrow \Theta(V, x) > 0 \).

\( \forall \ v > 0 \) small. \( \mathbb{R}^n \) \( \setminus \{x \} \in \text{Int} \setminus U \), \( V_\leq \) replacement of \( V \) at \( \mathcal{M} \).

Claim: \( \text{pt} \setminus U \cap \mathbb{R}^n \setminus \{x \} \neq \emptyset \).

or can shrink the inner radius to touch \( V' \) from outside.
Show integer - multiplicity.

Take $C \in \text{VarTan}(V, x)$. $C = \lim_{r \to 0} V_{x, y}$. $V_{r}$ = replacement of $V$ in $\Delta n(x, r, 2r)$.

$C' = \lim_{r \to 0} V_{r}^{'}$ - starting

- smooth stable in $\Delta n(0, 1.2)$

Thus $C = C'$ on $B(0) \cup B(1.2)$. $V_{r}^{'}$ - smooth in $\Delta n(0, 1.2)$

$\Rightarrow C = C'$ on $B(0) \cup B(1.2)$.

Now $\frac{1}{\sigma^2} = \frac{1}{\sigma^2} = \text{const}$ for $\sigma \in (0, 1) \cup (2, \infty)$

$\Rightarrow C = \text{n. plane}$.

Proof of Prop 6.3.

$V_{r}^{'}$ = replacement of $V$ in $\Delta n(x, r, 2r)$ + $\Delta n(x, 1)$

$v_{r}^{'} \Delta n(x, r, 2r) = \Sigma '$ $C^2$-surface (maybe disconnected)

$V_{r}^{''}$ = replacement of $V_{r}^{'}$ in $\Delta n(x, r, 2r)$

Take $\gamma \in (r, 2r)$ s.t. $\Sigma ^{'} \cap B(x)$ transversely.
$V'' = \text{replacement of } V' \text{ in } \mathbb{R}^n (x, y, z)$ \hspace{2cm} 0 < \delta < r$

$V'' \setminus B_n(x; \delta + t) = \Sigma'' \text{ \hspace{2cm} c^r \text{-surface}}$

\textbf{Step 1:} \hspace{2cm} \text{Want to show} \hspace{2cm} $\Sigma' = \Sigma'' \text{ in } \mathbb{R}^n (x, y, t)$

0. \hspace{2cm} \text{Take} \hspace{2cm} $y \in \Sigma' \setminus B_n(x; \delta)$, \hspace{2cm} & \hspace{2cm} $B_n(y) \cap \Sigma' = \emptyset$.

2. \hspace{2cm} \text{Fix} \hspace{2cm} $\bar{z} \in V = \Sigma' \setminus B_n(x; \delta)$.

\hspace{2cm} \textbf{Claim:} \hspace{2cm} $\text{TV} (\bar{z}, V'') = \{ \bar{z} \in \Sigma' \}$.

\hspace{2cm} $V''$ \hspace{2cm} \text{contains} \hspace{2cm} $\bar{z}$ \hspace{2cm} \text{and} \hspace{2cm} $\Sigma''$ \hspace{2cm} \text{in} \hspace{2cm} $B_n(y) \setminus \{ \bar{z} \}$.

\hspace{2cm} $\text{TV} (\bar{z}, V'') = \{ \text{planes} \}$.

\hspace{2cm} $\tau (\bar{z}) = \text{unit normal of } \Sigma'$ \hspace{2cm} \text{at} \hspace{2cm} $\bar{z}$.

\hspace{2cm} $\text{Claim:} \hspace{2cm} \lim_{\bar{z} \rightarrow \Sigma''} \frac{(\bar{z} - \bar{z}) \cdot \tau (\bar{z})}{|\bar{z} - \bar{z}|} = 0$.

\hspace{2cm} \text{If not, } \exists \, B(r, r, \Sigma'') \neq \emptyset.

\hspace{2cm} \text{short} \hspace{2cm} $(\Sigma' \setminus B(r, r, \Sigma'')) \geq C \cdot r^2 > 0$.

\hspace{2cm} \text{Monotonicity} \hspace{2cm} $\| \Sigma' \setminus B(r, r, \Sigma'') \| \geq C \cdot r^2$.

\hspace{2cm} $\Rightarrow \hspace{2cm} \int_{r} (\Sigma' \setminus B(r, r, \Sigma'')) \hspace{2cm} \text{in HP}$.

\hspace{2cm} \text{\textbullet $\Rightarrow$}.

3. \hspace{2cm} \text{let} \hspace{2cm} $\nu_B = \text{unit normal of } \Sigma''$, \hspace{2cm} $\nu \in \Sigma''$. 

Claim. \( \lim_{r \to \frac{\pi}{2}} v(2z) = 2(\bar{z}) \) \\
if not \n\exists B(r, x) \\
\n\Sigma'' \cap B(r, x) \neq \emptyset \\
\n\vdash \exists r \in \mathbb{R} \quad (\Sigma'' \cap B(r, x)) = \text{Hyp} \cap B(0,1) \quad \text{by} \quad g \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
Step 3: Removable of Singularity

$x \in \text{sp}(W)$

\[ \Sigma \Lambda \Phi_n(x, k, \rho) = \left( \sum \limits_{i=1}^{m_i} \rho_{i}(\rho) \right) \Sigma_p \]

extend \( \rho \to 0 \)

\[ \Sigma \Lambda \Phi_p(x) \{ \gamma \} = \left( \sum \limits_{i=1}^{m_i} \gamma_{i} \right) \Sigma_i \]

\( \Sigma_i \text{ start in } \Phi_p(x), \quad TV(x, \Sigma_i) \text{ planes of } m_i \geq 1 \)

\( \Rightarrow \Sigma_i \text{ graph near } x \)

\( \Rightarrow x \text{ removable} \)
6. Construction of Comparison Surfaces

Thm. \( \{ \Sigma^j \} \): \( \Sigma^j \rightarrow \) starting manifold \( V \)

1. \( \forall r \geq m \rightarrow R^+ \): set \( \Sigma^j \) \( \neq \) -a.m. in \( C^{\infty} \text{ for } \theta_0 \in B(0,1) \).

2. \( \Sigma^j \cap \Omega \) smooth for \( j \) large.

\( \Rightarrow V \) smooth.

Prop. Such \( \text{"} V \text{"} \) has \text{"} good replacement property \text{"}.

Meek-Simon-Yau: \( \Sigma \subset M \): generalized surface. \( U \subset M \) open.

\( U \cap \Omega \). \( C^\infty \).

Problem. \( (\Sigma, T_\Sigma(\Omega)) \):

\[ \text{Prove } \text{H}^2(4(1,2)) \]

\( \forall \{ \Sigma^k \} = \{ T^{(1,2)} \} \) \( k \in B(0,1) \) - minimizing seq. for \( (\Sigma, T_\Sigma(\Omega)) \)

Then \( \Sigma^k \rightarrow V \) strongly \& \( \forall M \subset \subset \Omega \) stable min.

Problem. \( (\Sigma^i, T_{\Sigma^j}(\Omega^i)) \):

\[ m_j = \text{min } \text{H}^2(4(1, \Sigma^j)) \]

\( T_{\Sigma^j} \in \Omega^i \cap \Omega_{\text{NN}, x} \).

\[ T_{\Sigma^j}(\Omega^i) = \{ 4 \in T_{\Sigma^j}(\Omega^i), \text{H}^2(4(1, \Sigma^j)) \leq \text{H}^2(\Sigma^j) + \frac{1}{s^j} \} \]

Lemma. \( \{ \Sigma^j, k \} \) \( k \in \Omega \) minimizing seq. for \( T_{\Sigma^j}(\Omega^i)) \) \( \Rightarrow \Sigma^j \rightarrow V \) strongly

\& \( V^j \cap \Omega \) is stable min.

Prop. \( V^j \) as above. \( V^j \leftarrow \lim_{j \rightarrow \infty} V^j \). Then \( V^j \) is a replacement of \( V \) in \( \Omega \).
\[ V^* = V \text{ on } \delta \Omega^c \]
\[ \Rightarrow \quad V^* = V \text{ on } \delta \Omega^c \]

2. By construction,
\[ H^2(\Omega) - \frac{1}{2} \leq H^2(\Sigma^j, k) \leq H^2(\Omega) \]
\[ \Rightarrow \quad H^2(\Omega) - \frac{1}{2} \leq H^2(T^j, k) \leq H^2(\Omega) \]
\[ \Rightarrow \quad H^2(\Omega) - \frac{1}{2} \leq \| V^* \|_{H^1(\Omega)} = \| V \|_{H^1(\Omega)} \]

3. \( V^* \) stationary, \( V^* \) stationary in \( \Omega \setminus \delta \Omega^c \)

Only need to show \( V^* \) starting in \( \delta \Omega^c \) at \( \delta \Omega^c \leq C \Omega \) for some \( C > 0 \).

If not, \( \exists X \in A(\delta \Omega^c) \) \( \delta V^*(X) \leq -c < 0 \).

\[ \Rightarrow \quad \delta V^*(X) \leq -c \quad \text{for } j \gg 1 \]
\[ \Rightarrow \quad \delta \Sigma^j, k(X) \leq -c \quad \text{for } j \gg 1, k \gg 1 \]

Can deform the mass \( \| \Sigma^j, k \| \) down a fixed amount
\[ \Rightarrow \quad \exists \varepsilon^*, \quad \Sigma^j, k = \varphi^j(\cdot, \varepsilon^*) \]

\[ \Sigma^j, k(\varepsilon) = \varphi^j(\varepsilon \Sigma^j, k) \quad \text{for } \varepsilon \leq \varepsilon^* \]

\[ \| \Sigma^j, k(\varepsilon) \| = \| \Sigma^j, k \| - \frac{c}{j} \]

---

**Question.** Show \( V^* = \text{min} \text{ soln of } (\Sigma^j, T^j_0(\Omega)) \).

3. \( V^* \) in \( \delta \Omega^c \)

\[ \Sigma = \Sigma^j, \quad \Sigma^2 = \Sigma^j, k. \quad (\Sigma^2(\varepsilon^k) = mL(\Omega^c)) \quad V^* = V^j \]

\( V^j \) starting in \( \delta \Omega^c \).

**Lemma 2.** Fix \( \{ \Sigma^k \} \). \( \forall X \in \Omega \). \( \exists \varepsilon > 0 \). \( \forall k \gg 1 \)

\[ \text{C1. Any } \varphi \in T^j_0(\Omega^c) \text{ with } H^2(\varphi(1, \varepsilon^k)) \leq H^2(\varepsilon^k) \]
\[ \exists \varphi \in T^j_0(\Omega^c) \text{ with } \| \Sigma^j, k(\varepsilon^k) \| = \| \Sigma^j, k \| - c \quad \frac{1}{j} \]

'\( \varepsilon \) uniform for \( \Sigma^k \)' with \( \frac{1}{j} \frac{\varepsilon^k}{(\varepsilon^k)^2} \quad \frac{1}{(\varepsilon^k)^2} \)

\[ H^2(\Omega^c, \Sigma^k) \leq H^2(\Sigma^k) + \frac{1}{j} \]
Lemma 2 ⇒ Lemma 1

1. Will show that $V^*$ has "good replacement property" in $A_{v}$.

\[ A_{v} \in A_{v} \text{ with } \forall x \in A_{v}, \exists y \text{ such that } \overline{A_{v}} = A_{v} \cup \{x, y, z, w\} \subseteq B_{v}(x) \]

\[ \{ \Sigma^{k, l} \} \text{ is minimizing seq for } (\Sigma^{k, l}; \beta_{n}(A_{v})) \]

\[ \Sigma^{k, l} \rightarrow W^{k}, \quad W^{k} \in B_{v}(x) \text{ stable min} \]

\[ W^{k} \rightarrow W \quad W \in B_{v}(x) \text{ stable ...} \]

Want $W$ is a replacement of $V^*$ in $A_{v}$.

Clearly, $W = \beta_{v}(A_{v}; V^*) = V'$ by $\beta_{v}(\overline{A_{v}})$

\[ W = \beta_{v}(\Sigma^{k, l}; V^*) \text{ minimizing seq for } (\Sigma^{k, l}; \beta_{v}(A_{v})) \]

\[ \Sigma^{k, l} \text{ connected to } \Sigma_{k} \text{ by } \Sigma_{k}^{k, l} \in \beta_{v}(B_{v}(x)) \]

\[ \Rightarrow \quad \|V^1\|_{M} \leq H^2(\Sigma^{k, l}) \leq H^2(\Sigma_{k}) \]

\[ \Rightarrow \quad \|W\|_{M} = \|V^1\|_{M} \]

Want: $W$ has further replacement property

\[ W = \beta_{v}(\Sigma^{k, l}; V^*) \text{ minimizing seq for } (\Sigma^{k, l}; \beta_{v}(A_{v})) \]

\[ \Rightarrow \quad W \text{ has replacement in } A_{v} \in A_{v} \text{ uniformly by } (\Sigma^{k, l}) \]

in $A_{v} \in A_{v} \cap B_{v}(x)$. The same as $\Sigma$ uniform for $[\Sigma^{k, l}]$

\[ \text{with } \Sigma^{k, l} \in \Sigma \text{ and } B_{v}(x) \]

Proof of Lemma 2

Rough ideas: \( \{\Sigma^{k}\} \) minimizing seq for \((\Sigma, \beta_{n}(A_{v}))\).

\[ x \in A_{v}, \quad \text{find } \varepsilon \text{ such that every } y \in \beta_{n}(B_{v}) \]

\[ \rightarrow y \in \beta_{n}(B_{v}) \]
\[ \Rightarrow H(4(t + \Sigma)) \leq H(1 + \Sigma) + o(\varepsilon) + o\left(\left(1 - t^2/k\right)\right) \]

\text{can be made small}
§5. Index bound & Application

**Theorem (M^3, g).** closed, orientable, $R_g > 0$. Assume $M$ has no non-orientable closed surfaces ($\tau_1(M) = 0$).

Then a minimal $\Sigma^2$ of index $1$ achieving the connected.

**Lemma.** $\Sigma^1, \Sigma^2$, closed minimal in $(M, g)$, $R_g > 0$.

$\Rightarrow \Sigma^1 \cap \Sigma^2 \neq \emptyset$.

**Proof.** If not, there is a minimizing geodesic $\nu$ connecting $\Sigma^1 \to \Sigma^2$, meeting $\Sigma^1, \Sigma^2$ orthogonally.

--- by 1st variation.

- Given $X \in T_p\Sigma^1 \times T_q\nu$. parallel transport $X$ to $X(t)$ along $\nu$. to get $X(t)$. Then $X(t) \in T_{\nu(t)}\Sigma^2$.

- Let $\nu = \text{flow of } N(t)$ by $X(t)$.

- \[ \frac{d}{ds} \| N(t) \|^2 = \frac{d}{ds} \int_0^1 \left< N(t) \dot{N}(t) \right> dt \]

- $= -2 \frac{d}{ds} \int_0^1 \left< \frac{\partial}{\partial s} \nu(t) \nu(t) \left| \dot{N}(t) \right> dt \]

- $= 2 \int_0^1 \left< \frac{\partial}{\partial s} \nu(t) \frac{\partial}{\partial s} \nu(t) \right> dt$
\[ = 2 \int_0^1 \left< \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial \xi} \right> \left< \frac{\partial R}{\partial \xi}, \frac{\partial R}{\partial t} \right> dt + R\int_0^1 R^\prime (K, \phi) \frac{\partial R}{\partial t} dt \]

\[ = 2 \int_0^1 \frac{\partial}{\partial \xi} \left< \frac{\partial \phi}{\partial \xi}, \frac{\partial \phi}{\partial t} \right> - R^\prime (K, \phi) \frac{\partial R}{\partial t} dt \]

\[ \geq 0 \]

Let \( X = e^1 \ldots e^m \) o.n. basis for \( \Sigma \) and sum

\[ 2 \left( H^2 - 4^2 - \int_0^1 R^\prime (R, R) \frac{\partial R}{\partial t} dt \right) \geq 0 \]

\[ \text{in } (M^3, g) \text{ Rbg} \geq 0. \]

\[ \text{Lemma}^2: \text{ } \Sigma^2 \text{ closed (orientable)} \Rightarrow \Sigma \text{ separable} \]

i.e. \( M|\Sigma = \text{two connected component} \)

\[ \text{Proof}: \text{ If not, } [\Sigma^2] \text{ non-trivial} \]

\[ \Rightarrow \text{ can minimize area in } [\Sigma^2], \Rightarrow \text{ stable min } \Sigma_0 \]

\[ \text{as above.} \]

\[ \text{Prop: } \exists \Sigma^2 \text{ minimal, } \exists \{\Sigma_t\}_{t \in [0, \infty]} \text{ generalized family} \]

\[ \text{s.t. } 0 \Rightarrow (\Sigma_t = \{ f = t \}, \text{ for some Morse function } f: M \to \mathbb{R}) \]

\[ \Rightarrow \Sigma_0 = \Sigma \]

\[ \{\Sigma_t\}_{t \in (-\infty, \infty)} \text{ is a smooth foliation} \]

\[ \text{i.e. } \Sigma_t = \{ \exp_x \left( u(t, \nu(x)) \right), x \in \Sigma \} \]

\[ \nu \text{ unit normal of } \Sigma. \]

\[ \text{s. only if } t > 0. \]
\[ M | \Sigma = M_1 \cup M_2. \] 
Points to \( M_1 \).

\[ R_{ij} > 0. \] \( \Sigma \) unstable.

\[ L_1 U_i = \partial \Sigma U_i + (R_{ij} + \Lambda l^2) U_i. \]

\[ \lambda_1 \text{ first eigenvalue, } \quad L_1 U_i = -\lambda_1 U_i, \quad \lambda_1 > 0. \]

\[ \Sigma_+ = \{ \exp_x \left( t U_{1(x)} \nu_{1(x)} \right), \quad x \in \Sigma \} \quad t \in (-\varepsilon, \varepsilon). \]

- smooth foliation for \( \varepsilon \ll 1 \).

\[ \frac{\partial}{\partial t} ||\Sigma_+|| = - \int u_{1(x)} \nu (x) \cdot \nu = 0. \]

\[ \frac{\partial^2}{\partial t^2} ||\Sigma_+|| = - \int u_{1(x)} L_1 u_{1(x)} = \lambda_1 \int u_i^2 < 0. \]

\[ \Rightarrow \quad ||\Sigma_+|| < ||\Sigma||. \]

**Claim.** Can extend \( \{ \Sigma_+ \}_{t \in \varepsilon (-\varepsilon, \varepsilon)} \) to \( t \in [-1,1] \), satisfying above.

* Min-max Problem for nef-lc with \( \Delta. \quad (M, \omega_M) \)

**Def.** \( (M, \omega_M), \{ \Sigma_+ \}_{t \in \varepsilon (-\varepsilon, \varepsilon)} \) is called a "generalized family" if "as before" & 18: \( \Sigma_0 = \omega_M \).

\[ \Sigma_+ = \left\{ \exp_x \left\{ t U_{1(x)} \nu_{1(x)} \right\}, \quad x \in \Sigma \right\} \quad t \in (-\varepsilon, \varepsilon) \]

for \( t \in [0, \varepsilon) \).

**Def.** \( \mathcal{J}_S (M, \omega_M) = \} \psi \in C^\infty_{\omega_M} (G_{\omega_M} \times M \rightarrow M), \quad \psi_0 = \text{id}. \)

\[ \psi(t \cdot ) |_{\omega_M} = \text{id}. \quad \psi(t \cdot ) \in \text{Diff}. \]

\[ \text{Let } \Lambda = \text{saturated family under, } \mathcal{J}_S (M, \omega_M) \]

\[ \mathcal{W}(\Lambda) = \inf \left\{ \max_{\psi \in \mathcal{J}_S (M, \omega_M)} H^2 (\psi(t \Sigma_+)) \right\}. \]
Theorem. If $W > \lambda \Delta (M)$ and $\lambda > 0$, then $\overline{Z}$ minimize $\Sigma_{\infty}$ and $\overline{Z}$ converges to $\Sigma_{\infty}$ as $\lambda \to \infty$.

Lemma 3. A minimizing seq $\{\Sigma_t\} \ni \exists \delta > 0 \text{ s.t. } ||\Sigma_t|| > W - \delta.$

Claim. All is the same as before then!

Lemma 4: Fix $\{\Sigma_t\} \in \Sigma_{\infty}$ generalized family. For all $t$.

Then, $\exists t_0 \ni ||\Sigma_{t_0}|| = W - \delta.$

Proof of Lemma 4: $F_t = \text{flow of } \frac{\partial}{\partial t} \Sigma_t \text{ near } \Sigma_{t_0}.$
Let $M_0 = \{ x \in M : \text{dist}(x, x_0) > a \}$.

For $a > 0$. Let $M \setminus M_0 \approx (\sigma_{0.2a}) \times \Sigma$, $dr^2 + g_r$.

$$c = \text{sup} \left\{ \| \nabla_x \Sigma | x \Sigma | + t \in (0, 2a) \right\}, \quad H_{am_x} > 0 \quad \text{for } (2a, a).$$

$$x = k(r) \phi(r) \frac{2}{2r},$$

$$k(r) = \begin{cases} 0 & r < \frac{a}{2} \\ 1 & r < a \\ \phi(r) \leq -c \phi(r) & r > a, \end{cases}$$

$$\nu = \text{flow of } x.$$

$\Rightarrow \nu$ is $C^1$ on $M_{am_x}$.

$$\frac{\partial}{\partial t} | T \Sigma | = \int \Sigma \nabla_x (\phi(r) \frac{2}{2r}),$$

$$= \int \Sigma \phi(r) \frac{2}{2r} \frac{\partial}{\partial t} \left( \phi(r) \frac{2}{2r} \right),$$

$$= \int \left( \phi'(r) (1 - \| x e_i \|^2) \phi(r) \frac{2}{2r} \right).$$

$$\leq \int \left( \phi'(r) + c \phi \right) (1 - \| x e_i \|^2) \phi(r) \frac{2}{2r} H_{am_x},$$

$$\leq 0.$$
To be continued...

1°. Can find Morse function \( f : [-1, 1] \rightarrow \mathbb{R} \). \[ \Sigma_t = \{ f = \alpha \} \quad \alpha \epsilon \{(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \} \]

- by perturb dist \( f \) (\( \Sigma \))

2°. Consider \( (M_{\frac{\varepsilon}{2}}, \Sigma_{\frac{\varepsilon}{2}}) \)

\[ \exists \phi \in \Sigma_{\frac{\varepsilon}{2}} (M_{\frac{\varepsilon}{2}}, \Sigma_{\frac{\varepsilon}{2}}) \quad \Sigma_t \]

\[ \{ \Sigma_t' = \phi (\Sigma_t) \} \quad \text{satisfies} \quad \| \Sigma_t' \| \leq \| \Sigma_{\frac{\varepsilon}{2}} \| + \delta \quad \delta \ll 1 \]

if not. As \( H \Sigma_{\frac{\varepsilon}{2}} > 0 \). run min-max.

\[ \Rightarrow \Sigma_0 \quad \text{minimal} \quad \subset M_{\frac{\varepsilon}{2}} \quad \land \quad \Sigma = \phi \]

\[ \Rightarrow \text{let} \quad \Sigma_0' = \begin{cases} \Sigma_0' & t \epsilon \left( \frac{\varepsilon}{2}, 1 \right) \\ \Sigma_t & t \epsilon \left( -\frac{\varepsilon}{2}, 0 \frac{\varepsilon}{2} \right) \\ \Sigma_t' & t \epsilon \left( 1, -\frac{\varepsilon}{2} \right) \end{cases} \]

\[ \Rightarrow \{ \Sigma_t' \} \quad \text{satisfies Prop 1} \]

Proof of Thm:

\( \Sigma \) c\textsuperscript{o} minimal \( \Rightarrow \{ \Sigma_t \} \quad \text{satisfying Prop 1} \)

smallest

\( \Lambda = \text{saturated family containing all} \{ \{ \Sigma_t \} \} \)

Run-min-max in \( \Lambda \) \( \Rightarrow V = \text{Im} \Sigma_t \)
\[ \mathcal{M}(\nu) = \frac{1}{2} \sum_{i=1}^{n} m_i \| z_i \| \leq \sup_{\| z \|} \inf_{2 \min} \| z \| \]

\[ \| z \| \leq \inf_{2 \min} \| z \| \]

Claim: \( \Sigma_i \) is Brelux 1.

If not. \( \forall u(x) \perp \Sigma_i u(x) \)

\[ \Sigma_{s+t} = \text{Deformation of } \Sigma_t \]

uncenter \( \overline{X} = \overline{X(t), \overline{X(s)}} \)

(extend to nbhd of \( \Sigma \))

\[ \frac{d}{ds} |\Sigma_{s+t}| = 0 \]

\[ \frac{d^2}{ds^2} |\Sigma_{s+t}| = -\int \nabla \mathcal{L} \cdot \nabla \mathcal{L} < 0. \]

\[ \| \Sigma_{s+t} \| < \| \Sigma_t \| \]

\[ \exists \varepsilon \to 0 \left\{ \Sigma_{s+t} \right\} \in \mathcal{A} \]
6. \( (M^3,g) \), Ricz > 0. No non-orientable surface. \( (S^3,g) \).

\( Bg \geq 6 \) then the min-max surface has area \( \leq 4 \).

\( = \) only if \( (M^3,g) = (S^3,g_0) \).

**Recall:** Given \( \Sigma \) min \( \rightarrow \{ \Sigma + \epsilon E_{11} \} \), \( \Sigma_0 = \Sigma \).

\( \Lambda = \) smallest saturated set of \( \{ \Sigma' \} \).

\( W(M, \Lambda, g) = ||\Sigma_0||, \quad \Sigma_0 = \min \) of index = 1.

- If \( \Lambda^h = \) smallest saturated set of \( \{ \Sigma' \} \) with genus \( h \)

\( \Rightarrow W(M, \Lambda^h, g) = ||\Sigma_0||, \quad \Sigma_0 = \text{genus } = h. \)

**Lemma** (Estimate of index one surface)

\( (M^3,g) \), 3-mfld. \( Bg \geq k_0 > 0 \). \( \Sigma^2 \rightarrow (M^3,g) \).

Then \( \int_S (Ric_{(S^3)} + 1) \, du \leq 8 \pi \left( \left\lfloor \frac{g(1)}{2} \right\rfloor + 1 \right) \).

and \( k_0 \leq 24 \pi + 16 \pi \left( \frac{g(2)}{2} - \left\lfloor \frac{g(2)}{2} \right\rfloor \right) \).

**Proof:** By complex analysis. 7 branch covering \( \Phi : \mathbb{C} \rightarrow S^2 \) conformal.

- Let \( u_1 > 0 \). - first eigenfunction of \( L_2 \). \& \( \int_S u_1 \, du = 1 \).

- Let \( u = \Phi^* (u_1 \text{Ricz} \, du) \). - probability measure

- \( \exists \Phi \in S^2 \rightarrow S^2 \) conformal. st.

\( \int_{S^2} \Phi \, du = 0 \).

- Let \( \Phi = \Phi \circ \Phi \).
\[ \int_{\Sigma} \phi \cdot (\Omega_{1}, \text{dr}) \, d\mu_{\Sigma} = 0. \]

\[ \frac{3}{2} \Rightarrow \int_{\Sigma} (\text{Ric}(\nabla \phi) + 11\phi) \, d\mu \leq \int_{\Sigma} 10\phi \, d\mu = 2 \text{area}(\phi(\Sigma)) \]

\[ = 8\pi \, \text{deg}(\phi) \]

\[ \text{deg}(\phi) \leq \left( \frac{g_{12}}{2} + 1 \right) + 1 \]

\[ k_{0} |\Sigma| \leq \int_{\Sigma} R_{g} \, d\mu = \int_{\Sigma} 2R_{2} (\nabla \phi) + 11\phi \, d\mu + 2k. \]

\[ R_{g} = 2R_{11,12} + 2R_{11,13} + 2 \sqrt{R_{23}}, \quad = 2 \text{Ric}(\nabla \phi) + 2 \left( R^{\Sigma} - h_{22} h_{33} + h_{2} h_{3} \right) \]

\[ \geq 1 \, \text{area}(\nabla \phi)^{0} \]

\[ \leq 2 \int_{\Sigma} (\text{Ric}(\nabla \phi) + 11\phi) \, d\mu + 2 \int_{\Sigma} k \]

\[ \leq 2 \cdot 8\pi \left( \frac{g_{12}}{2} + 1 \right) + 4\pi (2 - 2g_{12}) \]

\[ = 24\pi + 16\pi \left( \frac{g_{12}}{2} - \frac{g_{12}}{2} \right) \]

Ricci flow:

\[ \frac{\partial g_{ij}}{\partial t} = -2 \text{Ric}(g_{ij}), \]

Lemma:

\[ + \rightarrow W(M, \mathcal{L}^{h}, g_{12}), \text{ Lipschitz} \]

Fix $\phi \in [0, T)$. \]

\[ C = \sup_{t \in [0, T]} \| \text{Ric}(g_{ij}) \|, \quad t \in [0, t_{0}] \]

\[ e^{2c(t_{1} - t_{2})} g(t_{1}) \leq g(t_{2}) \leq e^{2c(t_{2} - t_{1})} g(t_{1}), \quad t_{1}, t_{2} \in [0, t_{0}] \]
\[ A \leq 0. \quad \exists \{ \Sigma_t \} \subset \{ \Sigma \} \quad \forall t \in [0, T]. \]

\[ \sup_{\sigma \in \mathcal{H}^{1,1} \setminus \mathcal{H}^1} H^2(\Sigma_t) \leq W(M, \Lambda^h g(t)) + \delta. \]

\[ W(M, \Lambda^h g(t)) \leq \sup_{\sigma \in \mathcal{H}^{1,1} \setminus \mathcal{H}^1} H^2(\Sigma_t) \leq e^{\chi(t-t_0)} \sup_{\sigma \in \mathcal{H}^{1,1} \setminus \mathcal{H}^1} H^2(\Sigma_t). \]

\[ \lim_{t \to 0} \]

No: non-orientable surfaces.

Prop: \[ M^3 \text{, } h - \text{Heegaard genus. } (M, g(t)) \quad \text{Reg.} \geq 0. \]

Then, \[ W(M, \Lambda^h g(t)) \leq W(M, \Lambda^h g_0) - (16t - 8\pi \frac{h}{2})^2. \]

\[ \text{If not, } \exists \tau \in (0, T). \]

\[ W(M, \Lambda^h g(t)) < W(M, \Lambda^h g_0) - (16\tau - 8\tau \frac{h}{2})^2. \]

\[ \exists \tau > 0. \quad W(M, \Lambda^h g(t)) < W(M, \Lambda^h g_0) - (16\tau - 8\tau \frac{h}{2})^2. \]

Let \[ t' = \inf \{ t \in (0, T) : W(M, \Lambda^h g(t)) < W(M, \Lambda^h g_0) - (16\tau - 8\tau \frac{h}{2})^2 \} \]

\[ t' \in [0, 2). \quad \& \]

\[ W(M, \Lambda^h g(t')) - W(M, \Lambda^h g(t)) \leq -(16\tau - 8\tau \frac{h}{2})^2 (t' - t) + \epsilon. \]

Calculate \[ \lim_{t \to 0} W(M, \Lambda^h g(t)). \]

Let \[ \{ \Sigma_t \} \subset \{ \Sigma \} \text{ be the optimal family w.r.t. } (\Lambda^h g(t)). \]
\[ \| \Sigma_{0} \|_{d_4} = \mathcal{W}(\mathcal{M}, \mathcal{L}^h, \mathcal{G}(\mathbf{u})^3). \]

\[ \int_{\mathcal{Q}} \frac{\partial}{\partial t} \left( \| \Sigma_{0} \|_{d_4} \right) = \int_{\mathcal{Q}} \frac{\partial}{\partial t} \sqrt{\det \left( \mathcal{G}(\mathbf{u})^3 \right)} \, d\mathbf{x}. \]

\[ = \int_{\mathcal{Q}} \mathcal{G}(\mathbf{u})^3 \frac{\partial}{\partial t} \sqrt{\det \left( \mathcal{G}(\mathbf{u})^3 \right)} \, d\mathbf{x}. \]

\[ = -\int_{\mathcal{Q}} \mathcal{G}(\mathbf{u})^3 \frac{\partial}{\partial t} \mathcal{R} \, d\mathbf{x}. \]

\[ = -\int_{\mathcal{Q}} \mathcal{G}(\mathbf{u})^3 \frac{\partial}{\partial t} \left( R - \mathcal{R} \mathcal{U}^3 \mathcal{U}^3 \right) \, d\mathbf{x}. \]

\[ = \int_{\mathcal{Q}} \left( \mathcal{R} \frac{\partial}{\partial t} \mathcal{U}^3 \mathcal{U}^3 \right) \, d\mathbf{x}. \]

\[ = -\int_{\mathcal{Q}} \left( \mathcal{R} \frac{\partial}{\partial t} \mathcal{U}^3 \mathcal{U}^3 \right) \, d\mathbf{x}. \]

\[ \geq -8\pi \left( \left( \frac{h+1}{2} \right) + \frac{1}{2} \right) \]

\[ = -16\pi - 8\pi \left( \left[ \frac{h+1}{2} \right] - h \right). \]

\[ = -16\pi + 8\pi \left[ \frac{h}{2} \right]. \]

\[ \Rightarrow \| \Sigma_{0} \|_{d_4} \leq \mathcal{W}(\mathcal{M}, \mathcal{L}^h, \mathcal{G}(\mathbf{u})^3) \cdot (t' - t). \]

\[ \mathcal{W}(\mathcal{M}, \mathcal{L}^h, \mathcal{G}(\mathbf{u})^3) \cdot (t' - t). \]
Theorem. \( (M^3, g), \ Rg > 0 \). No non-orientable surfaces.

- Heegaard genus. \( \chi \geq 6 \).

\[
\Rightarrow \quad W(M, \Lambda^1 g) \leq 4\chi - 2\chi \left[ \frac{h}{2} \right] \leq 4\chi.
\]

\( h = 1 \) only if \( (M^3, g) = S^3 \).

Proof. Let \( (g(t)) \in \{a, T\} \) be the maximal soln of Ricci flow.

\[
W(M, \Lambda^1 g(t)) = W(M, \Lambda^1 g) - (16\chi - 8\chi \left[ \frac{h}{2} \right]) t.
\]

\( \lim_{t \to T} W(M, \Lambda^1 g(t)) = 0. \)

By Lemma, \( \min_M R_{g(t)}, W(M, \Lambda^1 g(t)) \leq 24\chi + 16\chi \left| \frac{\chi}{2} - \frac{\chi}{2} \right| \)

As \( \lim_{t \to T} \min_M R_{g(t)} = 0. \)

\[
\Rightarrow \quad 0.
\]

\( \Rightarrow \quad W(M, \Lambda^1 g) \leq (16\chi - 8\chi \left[ \frac{h}{2} \right]) T. \)

2. \( \frac{2}{3} R_{g(t)} = \Delta R_{g(t)} + \frac{2}{3} R_{g(t)}^2 + \| R_{g(t)} \|^2. \)

Maximum Principle \( \Rightarrow \min_M R_{g(t)} = \frac{3 \min_M R_{g(t)}}{3 - 2 \min_M R_{g(t)} (t - t_i)} \)

\( t \in (0, T), \quad t \in (t_i, T) \)

\( t_i = 0, \quad Rg \geq 6. \)

\( \Rightarrow \quad \min_M R_{g(t)} \geq \frac{18}{3 - 12t} = \frac{6}{1 - 4t} \). \( \Rightarrow T \leq \frac{1}{4}. \)
\[ \Rightarrow \quad \omega(M, \Lambda_{\frac{h}{2}}, g) \leq 4\pi - 2\pi \left[ \frac{h}{2} \right]. \]

\[ \Rightarrow \quad r = \frac{h}{4} \quad \Rightarrow \quad R_{g_{1/4}} \equiv 0 \quad \Rightarrow \quad \text{Einstein} \quad & h > 0, \quad \text{w} \quad (3 - 2 \min_{M} \kappa_{2}(t-t_1)) > 0, \quad \text{for} \quad t_1 < t < \frac{h}{4} \]

\[ \Rightarrow \quad \min_{M} \kappa_{2}(t) \leq \frac{3}{2(\frac{h}{4} - t_1)} = \frac{6}{1 - 4t_1} \]

To show \( M = S^3 \), only need to show \( h > 0 \).

(Frankel: \( R_{g} > 0 \), \( \pi_{1}(S^3) \rightarrow \pi_{1}(M) \) surjective).

If \( h = 1 \), \( M \) hence sphere or \( S^3 \times S^1 \).

\[ \Rightarrow \quad \text{at minimal } T^2 \text{ of area } \frac{4\pi}{\rho} < 4\pi. \]

\[ \Rightarrow \quad \text{minimal } T^2 \text{ has area } = 4\pi. \]
1. Introduction to the Almgren-Pitts theory.

Currents.

$U \subseteq \mathbb{R}^n$ open, $\mathcal{D}^k(U)$ = $\mathcal{C}_0$-supported $k$-forms in $U$.

Definition 1: A $k$-current $T$ in $U$ is a linear functional on $\mathcal{D}^k(U)$.

Definition 2: $\mathcal{D}^k(U)$ = $(\mathcal{D}^k(U))^\ast$ = space of $k$-currents in $U$.

$\mathcal{D}^k(U)$ = weak-topology.

$T \in \mathcal{D}^k(U), \quad \sigma T \in \mathcal{D}^{k-1}(U)$

$\sigma T(w) = T(\delta w), \quad w \in \mathcal{D}^{k-1}(U)$.

$\forall \omega \in \mathcal{D}^k(U)$

$\overline{M}w(T) = \sup_{|\omega| \leq 1, \sup\{m(w) \leq u\}} T(w)$.

$S \subseteq U$ countably $k$-rectifiable set in $U$. $\partial_\sigma S$ $\mathcal{H}^k$-measurable.

$e_1, \ldots, e_n$ an. basis for $T \times S$. $\delta(x) = e_1 \ldots e_n$.

$k$-current $T = T(S, \theta, \delta)$

$T(S, \theta, \delta)(w) = \int_S \langle w, \delta \delta \rangle \Theta(x, dx)^k$.

rectifiable $k$-current.

$\mathcal{I}^k(U) = \text{space of rectifiable } k\text{-currents}$

$\mathcal{Z}^k(U) = \{ T \in \mathcal{I}^k(U), \delta T = 0 \}$

$k$-cycles

$M \subseteq \mathbb{R}^n$.

$\mathcal{I}^k(M), \mathcal{Z}^k(M)$ = $k$-currents or $k$-cycles supported on $M$.
Mass-norm:
\[ M_{w}(T_1, T_2) = \inf_{W \subset U} \left\{ M_{w}(T_1 - T_2) : T_1, T_2 \in I_{d}(U) \right\} \]

Flat-norm:
\[ f_{w}(T_1, T_2) = \inf_{R \in I_{d}(U), \, \lambda \in I_{b}(U)} \left\{ M_{w}(R) + M_{w}(\lambda) : T_1 - T_2 = R + \lambda \right\} \]

2. Blungsness's Setting:

**Definition:** Cell complex of \( I = (0, 1) \)

1. \( I = (0, 1) \)
   \[ J_0 = \{ [0, 1] \} \]

2. \( I(1. j) = \{ \left\{ \frac{j}{3^n}, \frac{j+1}{3^n} \right\}, \left\{ \frac{1}{3^n} \right\} \} \)
   \[ I(1. j)_{p} = p \text{- cells}, \quad p = 0 \text{ or } 1 \]
   \[ I_0(1. j) = \{ [0, 1], \{ 1/3 \} \} \]

3. \( \alpha \): 1-cell of \( I(1. j) \)
   \[ \alpha(1) = \text{crb complex of } I(1. j) \]
   \[ \alpha_{0} = 2 \text{- 0-cells} \]

4. \( \alpha : I(1. j) \to I(1. j) \)
   \[ \alpha(b) = \{ b \} \to \{ b \} \]

5. \( d : I(1. j) \times I(1. j) \to \mathbb{Z}^+ \)
   \[ d(x, y) = 3^j | x - y | \]

6. \( n_{1}(1. j) : I(1. i) \to I(1. j) \)
   \[ d(1, 2, n_{1})(x, y) = \inf \left\{ d(x, y) : y \in I(1. j) \right\} \]

**Given:** \( \phi : I(1. j) \to \mathbb{Z}_{n}(M^{2n+1}) \)

\[ f(\phi) = \sup \left\{ \lim \frac{\phi(x) - \phi(y)}{d(x, y)} : x, y \in I(1. j) \right\} \]

**finess:**
\[ \phi : \mathbb{I} \to (\mathbb{Z}, (M^n)^+) \]  

1. \( \phi(0) = \phi(1) = 0 \)

\[ \phi : \mathbb{I} \to (\mathbb{Z}, (M^n)^+) \]  

for \( i = 1 \) or \( 2 \).

\( \phi \) is \( 1 \)-homotopic to \( \phi' \) with fineness \( \delta \).

\[ f : k_3 \in \mathbb{N} \quad \phi' = \mathbb{I} (1, k_3) \quad \phi : \mathbb{I} (1, k_3) \times \mathbb{I} (1, k_3) \to \mathbb{Z}, (M^n)^+ \]

\[ f(4) \leq \delta \]

\[ 4 (i-1, x) = \phi' \cap (h_3, k_i) = x \]

\[ 4 (\mathbb{I} (1, k_3) \times \mathbb{I} (1, k_3)) = 0 \]

**Def.** (1, \( M \)) homotopy seq. \( \{ \phi_i \} \)

\[ \phi_i : \mathbb{I} (1, k_i) \to (\mathbb{Z}, (M^n)^+, \{ \delta \}) \]

\( \phi_i \sim \phi_{i+1} \) with fineness \( \delta_i \to 0 \)

\[ \sup \{ \delta_i \} < \infty \]

**Def.** \( S_i = \{ \phi_i \} \quad S_2 = \{ \phi_2 \} \) is homotopic.

\[ f : \quad \phi_2 \sim \phi_1 \] with fineness \( \delta_i \to 0 \)

**Def.** \( \pi_1 (\mathbb{Z}, (M^n)^+, \{ \phi \}) = \{ [S] : S \) (1-M) homotopy seq) \)

**Thm.** (extension). \( H_{+1} (M^n) \cong \pi_1 (\mathbb{Z}, (M^n)^+, \{ \phi \}) \)

**Def.** Given \( T \in \pi_1 (\mathbb{Z}, (M^n)^+, \{ \phi \}) \)

\[ L(T) = \sup \left\{ \liminf_{x \to \infty} \max_{x \in \text{Dom}(\phi)} \right\} \]
Then, given \( \Pi \sim [W] \)

\( \exists \) closed mind hypersurf \( \Sigma^n \)

\[ |\Sigma^n| = L(\Pi) \]

\[ \Sigma = \lim_{j \to \infty} \| P_j \cdot \phi_j \| \]