MULTIPlicitTY ONE Conjecture IN MIN-MAX THEORY

XIN ZHOU

The min-max theory is a powerful tool to find minimal surfaces, which are the mathematical models for soap films. Motivated by Yau’s conjecture on minimal surfaces [15], Marques and Neves proposed a program to establish the Morse theory for the Area functional [6, 7, 8], in which they explored the notion of “volume spectrum” introduced by Gromov in 1980s [4]. One of their goals is to understand the key feature of the min-max theory, that is, the Morse index. The long-standing challenge of min-max theory, especially for Marques-Neves’s program, was the “Multiplicty One Conjecture” [8, 1.2]. The conjecture said that minimal hypersurfaces produced by the min-max theory are always two-sided and have multiplicity one a generic scenario. This conjecture is a natural nonlinear analog of a famous result by Uhlenbeck [12] for the linear “Laplacian spectrum” in 1960s. This conjecture was proved by the author in [16].

Now we start to state the precise result. Let \((M^{n+1}, g)\) be a closed orientable Riemannian manifold of dimension \(3 \leq (n + 1) \leq 7\). In [1], Almgren proved that the space of mod-2 cycles \(Z_n(M, \mathbb{Z}_2)\) is weakly homotopic the Eilenberg-MacLane space \(K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty\); (see also [8] for a simpler proof). Later, Gromov [4], Guth [5], Marques-Neves [7] introduced the notion of volume spectrum as a nonlinear version of spectrum for the area functional in \(Z_n(M, \mathbb{Z}_2)\). In particular, the volume spectrum is a non-decreasing sequence of positive numbers

\[
0 < \omega_1(M, g) \leq \cdots \leq \omega_k(M, g) \leq \cdots \to +\infty,
\]

which is uniquely determined by the metric \(g\) in a given closed manifold \(M\).

By adapting the celebrated min-max theory developed by Almgren [2], Pitts [9] (for \(3 \leq (n + 1) \leq 6\), and Schoen-Simon [11] (for \(n + 1 = 7\)), Marques-Neves [7] proved that each \(\omega_k(M, g)\) is associated with an integral varifold \(V_k\) whose support is a disjoint collection of smooth, connected, closed, embedded, minimal hypersurfaces \(\{\Sigma^k_{i_1}, \cdots, \Sigma^k_{i_{l_k}}\}\), such that

\[
(0.1) \quad \omega_k(M, g) = \sum_{i=1}^{l_k} m^k_i \cdot \text{Area}(\Sigma^k_i),
\]

where \(\{m^k_1, \cdots, m^k_{l_k}\} \subset \mathbb{N}\) is a set of positive integers, usually called multiplicities.

Our main theorem states that if a component \(\Sigma^k_i\) is not weakly stable, then \(\Sigma^k_i\) has to be two-sided and the associated integer multiplicity is identically equal to one, i.e. \(m^k_i = 1\). Note that a closed minimal hypersurface \(\Sigma\) is said to be weakly stable if it has a 0 as the lowest eigenvalue for the second variation of area; (when \(\Sigma\) is one-sided, one has to pass to its two-sided double cover).

**Theorem 0.1.** Given a closed manifold \((M^{n+1}, g)\) of dimension \(3 \leq (n + 1) \leq 7\), denote \(\{\Sigma^k_i : k \in \mathbb{N}, i = 1, \cdots, l_k\}\) as the min-max minimal hypersurfaces associated with volume spectrum. Then every connected component of \(\{\Sigma^k_i : k \in \mathbb{N}, i = 1, \cdots, l_k\}\) which is not weakly stable is two-sided and has multiplicity one. That is, if \(\Sigma^k_i\) is not weakly stable, \(k \in \mathbb{N}, 1 \leq i \leq l_k\), then \(\Sigma^k_i\) is two-sided.
and \(m_i^k = 1\), and
\[
\sum_{i=1}^{l_k} \text{index}(\Sigma_i^k) \leq k.
\]

**Remark 0.2.** Theorem 0.1 is an equivalent formulation of the Multiplicity One Conjecture of Marques-Neves \([8, 1.2]\) proved by the author in \([16, \text{Theorem A}]\). Indeed, \([16, \text{Theorem A}]\) asserts that for a bumpy metric \(g\), all connected components of \(\{\Sigma_k^i : k \in \mathbb{N}, i = 1, \ldots, l_k\}\) are two-sided and have multiplicity one. Theorem 0.1 directly implies \([16, \text{Theorem A}]\), as weakly stable minimal hypersurfaces are degenerate and hence do not exist in a bumpy metric. Now we argue that \([16, \text{Theorem A}]\) implies Theorem 0.1. A metric \(g\) is called bumpy if every closed immersed minimal hypersurface is non-degenerate. White proved that the set of bumpy metrics is generic in Baire sense \([13, 14]\). For an arbitrary metric \(g\), we can take a sequence of bumpy metrics \(\{g_j\}_{j \in \mathbb{N}}\) such that \(g_j \to g\) smoothly.

Now fix \(k \in \mathbb{N}\); for each \(g_j\), the associated min-max minimal hypersurfaces \(V_{k,j}\) are all two-sided and have multiplicity one by \([16, \text{Theorem A}]\). By the compactness theorem \([10, \text{Theorem A.6}]\), \(V_k\) converges up to a subsequence to a limit integral varifold \(V\), such that the support \(\text{spt}(V)\) of \(V\) is smooth embedded minimal hypersurfaces. Now using \([10, \text{Theorem A.6}]\) again, if a connected component of \(\text{spt}(V)\) either has multiplicity greater than one or is one-sided, it (or its two-sided double cover when one-sided) has to carry a positive Jacobi field for the second variation of area, and hence it is weakly stable.

**Remark 0.3.** Recently, Chodosh-Mantoulidis \([3]\) proved this conjecture in dimension three \((n+1) = 3\) for the Allen-Cahn setting; they also proved that the total index is exactly \(k\) for their \(k\)-min-max solutions when \((n+1) = 3\). After our results were posted, Marques-Neves finished their program and also proved the same optimal index estimates for \(3 \leq (n+1) \leq 7\) \([8, \text{Addendum}]\).

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA SANTA BARBARA, SANTA BARBARA, CA 93106, USA

*Email address:* zhou@math.ucsb.edu