MATH 207A

Euler Systems and the BSD Conjecture

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Francesc Castella

http://www.math.ucla.edu/~castella/207A/index.html

Let E be an elliptic curve over $\mathbf{Q}$, and let

$$\varrho : G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}_{\mathbf{C}}(V_\varrho) \cong \mathrm{GL}_n(\mathbf{C})$$

be an n-dimensional Artin representation. The L-series $L(E, \varrho, s)$ of E twisted by ϱ is defined on the right half-plane $\Re(s) > 3/2$ by a convergent Euler product, and is expected to admit an analytic continuation to the whole complex plane with a functional equation relating its values at s and $2 - s$.

Let $H/\mathbf{Q}$ be a finite Galois extension through which ϱ factors. A natural Galois-equivariant refinement of the Birch and Swinnerton-Dyer conjecture predicts that

$$\mathrm{ord}_{s=1} L(E, \varrho, s) \stackrel{?}{=} \dim_{\mathbf{C}} E(H)^{\varrho}, \tag{*}$$

where $E(H)^{\varrho} := \mathrm{Hom}_{G_{\mathbf{Q}}}(V_\varrho, E(H) \otimes \mathbf{C})$ is the ϱ-isotypical component of the Mordell–Weil group of E. Much is known about (*) in the case of *analytic rank zero* (and $n \leq 2$). Most notably:

(1) When $n = 1$, i.e., when ϱ is a Dirichlet character, by the work of Kato;

(2) When ϱ is induced from a ring class character of an imaginary quadratic field, by the works of Gross–Zagier and Kolyvagin;

(3) When ϱ is induced from a ring class character of a *real* quadratic field, by the recent work of Darmon–Rotger [1];

(4) When ϱ is induced from a (not necessarily anticyclotomic) character of an imaginary quadratic field, by the recent work of Kings–Loeffler–Zerbes [2].

In this course, after some generalities on the method of Euler systems (following [3]), we will study the works mentioned in (3) and (4), which represent two genuinely new applications of this machinery and shed new light on the classical works mentioned in (1) and (2).

References

[1] H. Darmon and V. Rotger, *Diagonal cycles and Euler systems II: the Birch and Swinnerton-Dyer conjecture for Hasse–Weil–Artin L-series*, preprint (2014) available at http://www.math.mcgill.ca/darmon/pub/pub.html.

[2] G. Kings, D. Loeffler, and S. L. Zerbes, *Rankin–Selberg Euler systems and p-adic interpolation*, http://arxiv.org/abs/1405.3079.

[3] Karl Rubin, *Euler systems*, Annals of Mathematics Studies, vol. 147, Princeton University Press, Princeton, NJ, 2000, Hermann Weyl Lectures. The Institute for Advanced Study.