BASE CHANGE AND IWASAWA MAIN CONJECTURES FOR GL_2

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Abstract. Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N, p an odd prime of good ordinary reduction such that $E[p]$ is an irreducible $G_{\mathbb{Q}}$-module, and K an imaginary quadratic field with all primes dividing Np split. We prove Iwasawa Main Conjectures for the $\mathbb{Z}_p$-cyclotomic and $\mathbb{Z}_p$-anticyclotomic deformations of E over $\mathbb{Q}$ and K respectively, dispensing with any of the ramification hypotheses on $E[p]$ in previous works.

The strategy employs base change and the two-variable zeta element associated to E over K, via which the sought after main conjectures are deduced from Wan's divisibility towards a three-variable main conjecture for E over a quartic CM field containing K.

As an application, we prove cases of the two-variable main conjecture for E over K. The aforementioned one-variable main conjectures imply the p-part of the conjectural Birch and Swinnerton-Dyer formula for E if $\mathrm{ord}_{s=1} L(E,s) \leq 1$. They are also an ingredient in the proof of Kolyvagin's conjecture and its cyclotomic variant in our joint work with Grossi [BCGS23].

Contents

1. Introduction

Let E be an elliptic curve defined over $\mathbb{Q}$, p an odd prime of good ordinary reduction for E, and K an imaginary quadratic field. In this paper we study Iwasawa theory of E along the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and the anticyclotomic $\mathbb{Z}_p$-extension of K, proving corresponding Iwasawa Main Conjectures (see Theorems 1.1.2, 1.2.2 and 1.2.3).

1.1. **Cyclotomic Main Conjecture.** Let $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Put $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, and let $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$ be the cyclotomic Iwasawa algebra.

We consider the classical Selmer group $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty) = \varinjlim_n \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_n)$, where $\mathbb{Q}_n$ is the subfield of $\mathbb{Q}_\infty$ with $[\mathbb{Q}_n : \mathbb{Q}] = p^n$. Its Pontryagin dual

$$\mathfrak{X}_{\mathrm{ord}}(E/\mathbb{Q}_\infty) := \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$$

is a finitely generated Λ-module. Let $\mathcal{L}_p(E/\mathbb{Q}) \in \Lambda \otimes \mathbb{Q}_p$ be the p-adic L-function attached to E by Mazur–Swinnerton-Dyer [MSD74]. In [Maz72], Mazur conjectured the following.

Conjecture 1.1.1 (Mazur's Main Conjecture)**.** *The Λ-module $\mathfrak{X}_{\mathrm{ord}}(E/\mathbb{Q}_\infty)$ is torsion, with*

$$\mathrm{ch}_\Lambda\left(\mathfrak{X}_{\mathrm{ord}}(E/\mathbb{Q}_\infty)\right) = (\mathcal{L}_p(E/\mathbb{Q}))$$

as ideals in Λ.

Note that implicit in Conjecture 1.1.1 is the integrality statement $\mathcal{L}_p(E/\mathbb{Q}) \in \Lambda$; this is most well-understood under the assumption that p is odd and

$$E[p] \text{ is an irreducible } G_{\mathbb{Q}}\text{-module} \tag{$\mathrm{irr}_{\mathbb{Q}}$}$$

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(see [GV00, Prop. 3.1]) where $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the absolute Galois group of $\mathbb{Q}$. (We similarly use G_L to denote the absolute Galois of a number field L.) Let T be the p-adic Tate module of E.

In [Kat04], Kato proved the Λ-torsionness of $\mathfrak{X}_{\mathrm{ord}}(E/\mathbb{Q}_\infty)$ and the inclusion $p^c \cdot \mathcal{L}_p(E/\mathbb{Q}) \in \mathrm{ch}_\Lambda(\mathfrak{X}_{\mathrm{ord}}(E/\mathbb{Q}_\infty))$ for some $c \geq 0$, with $c = 0$ when T has "large" image. These results are consequences of his seminal construction of an Euler system for T. Let N be the conductor of E. Assuming further that

$$\text{there exists a prime } q \| N \text{ such that } E[p] \text{ is ramified at } q, \tag{mult}$$

the converse divisibility, and hence Conjecture 1.1.1 was proved by Skinner–Urban [SU14]. Their work employs Eisenstein congruences on the unitary group $\mathrm{GU}(2,2)$ over imaginary quadratic fields.

Our main result towards Conjecture 1.1.1 removes the hypothesis (mult):

Theorem 1.1.2. *Let E be an elliptic curve defined over $\mathbb{Q}$ and p a prime of good ordinary reduction for E.*

(a) *If $p > 3$ satisfies ($\mathrm{irr}_{\mathbb{Q}}$), then $\mathfrak{X}_{\mathrm{ord}}(E/\mathbb{Q}_\infty)$ is Λ-torsion, with*
$$\mathrm{ch}_{\Lambda_{\mathbb{Q}}}(\mathfrak{X}_{\mathrm{ord}}(E/\mathbb{Q}_\infty)) = \big(\mathcal{L}_p(E/\mathbb{Q})\big)$$
in $\Lambda \otimes \mathbb{Q}_p$.

(b) *If in addition*
$$\text{there exists an element } \sigma \in G_{\mathbb{Q}(\mu_{p^\infty})} \text{ such that } T/(\sigma - 1)T \simeq \mathbb{Z}_p, \tag{im}$$
then the equality holds in Λ, and hence Conjecture 1.1.1 holds.

Remark 1.1.3.

(i) For non-CM curves the condition (im) holds for all sufficiently large primes p by Serre's open image theorem [Ser72]. In fact, it is expected that $p \geq 37$ suffices.

(ii) The only prior result towards Conjecture 1.1.1 without assuming the hypothesis (mult) is due to Wan [Wan15], based on Eisenstein congruences on the unitary group $\mathrm{GU}(2,2)$ over CM fields. However, it is conditional on a p-integral comparison of certain automorphic periods, which still remains open. Our proof of Theorem 1.1.2 relies on a main result of [Wan15] but sidesteps the period comparison.

1.2. **Anticyclotomic Main Conjectures.** Assume that the discriminant $D_K < 0$ satifies

$$D_K \text{ is odd and } D_K \neq -3. \tag{disc}$$

Moreover, assume that K satisfies the *Heegner hypothesis*, namely

$$\text{every prime } \ell | N \text{ splits in } K, \tag{Heeg}$$

and that

$$p = v\bar{v} \text{ splits in } K \tag{spl}$$

for v the prime of K above p induced by an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, which we fix throughout.

Let K_∞^-/K be the anticyclotomic $\mathbb{Z}_p$-extension, $\Gamma_K^- = \mathrm{Gal}(K_\infty^-/K)$, and $\Lambda_K^- = \mathbb{Z}_p[\![\Gamma_K^-]\!]$ the anticyclotomic Iwasawa algebra. In view of (Heeg) and the p-ordinarity hypothesis, the Kummer images of Heegner points of p-power conductor give rise to a Λ_K^--adic class

$$\kappa_1^{\mathrm{Heeg}} \in \mathrm{H}^1_{\mathcal{F}_\Lambda}(K, \mathbf{T}).$$

Here $\mathbf{T} = \varprojlim_n \mathrm{Ind}_{K_n^-/K}(T)$, with K_n^- the subfield of K_∞^- with $[K_n^- : K] = p^n$, and $\mathrm{H}^1_{\mathcal{F}_\Lambda}(K, \mathbf{T}) \subset \mathrm{H}^1(K, \mathbf{T})$ is the compact ordinary Selmer group[1] interpolating the classical Selmer groups $\varprojlim_m \mathrm{Sel}_{p^m}(E/K_n^-)$ as n varies. Let $\mathfrak{X}_{\mathrm{ord}}(E/K_\infty^-)$ be the Pontryagin dual of $\mathrm{Sel}_{p^\infty}(E/K_\infty^-) = \varinjlim_n \mathrm{Sel}_{p^\infty}(E/K_n^-)$.

The formulation of a Main Conjecture in this setting is due to Perrin-Riou [PR87].

Conjecture 1.2.1 (Heegner point Main Conjecture)**.** *The Λ_K^--modules $\mathfrak{X}_{\mathrm{ord}}(E/K_\infty^-)$ and $\mathrm{H}^1_{\mathcal{F}_\Lambda}(K, \mathbf{T})$ have Λ_K^--rank one, and*
$$\mathrm{ch}_{\Lambda_K^-}\big(\mathfrak{X}_{\mathrm{ord}}(E/K_\infty^-)_{\mathrm{tor}}\big) = \mathrm{ch}_{\Lambda_K^-}\big(\mathrm{H}^1_{\mathcal{F}_\Lambda}(K, \mathbf{T})/(\boldsymbol{\kappa}_1^{\mathrm{Heeg}})\big)^2$$
as ideals in Λ_K^-.

[1] See e.g. [CGLS22, §4.1] for a review of the construction, whose notations we largely follow.

The first general results towards Perrin-Riou's Heegner point Main Conjecture are due to Bertolini [Ber95] and Howard [How04], relying on the Heegner point Kolyvagin system. These works established the rank statements in Conjecture 1.2.1, and the latter proved the divisibility "$\supseteq$" if

$$\text{(sur)} \qquad \bar{\rho}_E : G_{\mathbb{Q}} \to \mathrm{Aut}_{\mathbb{F}_p}(E[p]) \text{ is surjective.}$$

The first cases of the opposite divisibility, and hence of Conjecture 1.2.1 appeared in [BCK21], which builds on Wei Zhang's resolution of Kolyvagin's conjecture [Zha14b] and assumes the hypothesis (sur) in addition to certain ramification hypotheses on $E[p]$. Via level raising and rank lowering, Wei Zhang's work and [BCK21] rely on the results of Skinner–Urban [SU14], inheriting the hypotheses therein. Prior to [BCK21], Wan [Wan20] established the first cases of the Heegner point Main Conjecture under a generalized (non-classical) Heegner hypothesis. It employs Eisenstein congruences on the unitary group $\mathrm{GU}(3,1)$. In addition to (sur), it requires that N is square-free and a ramification hypotheses on $E[p]$.

Our different approach dispenses with any of the ramification hypotheses, leading to the following result.

Theorem 1.2.2. *Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N, p be a prime of good ordinary reduction for E, and K an imaginary quadratic field satisfying* (disc)*,* (Heeg)*, and* (spl)*.*

(a) *If $p > 3$ satisfies* ($\mathrm{irr}_{\mathbb{Q}}$)*, then both $\mathfrak{X}_{\mathrm{ord}}(E/K_\infty^-)$ and $\mathrm{H}^1_{\mathcal{F}_\Lambda}(K, \mathbf{T})$ have Λ_K^--rank one, and*

$$\mathrm{ch}_{\Lambda_K^-}\left(\mathfrak{X}_{\mathrm{ord}}(E/K_\infty^-)_{\mathrm{tor}}\right) = \mathrm{ch}_{\Lambda_K^-}\left(\mathrm{H}^1_{\mathcal{F}_\Lambda}(K, \mathbf{T})/(\boldsymbol{\kappa}_1^{\mathrm{Heeg}})\right)^2$$

in $\Lambda_K^- \otimes \mathbb{Q}_p$.

(b) *If further $p > 3$ satisfies* (sur)*, then the equality holds in Λ_K^- and hence Conjecture 1.2.1 holds.*

Such an equality has applications to the Birch and Swinnerton-Dyer conjecture. For example, it yields a p-converse to the Gross–Zagier and Kolyvagin theorem:

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E) = 1 \implies \mathrm{ord}_{s=1} L(E,s) = 1$$

(cf. [Ski20, Wan21, Cas17, BT20, BST21]). Note that the p-converse does not assume finiteness of $\text{Ш}(E)[p^\infty]$, in fact deduces it as a consequence.

In light of the Λ_K^--adic analogue of the p-adic Waldspurger formula of [BDP13] (see [CH18]), Conjecture 1.2.1 is equivalent to the prediction that the p-adic L-function $\mathcal{L}_p^{\mathrm{BDP}}(E/K) \in \Lambda_K^{-,\mathrm{ur}}$ constructed in *op. cit.* generates the characteristic ideal of the anticyclotomic Selmer group $\mathfrak{X}_{\mathrm{Gr}}(E/K_\infty^-)$ whose classes are locally trivial (resp. unrestricted) at the primes above $\overline{v}$ (resp. v). Here we put

$$\Lambda_K^{-,\mathrm{ur}} = \Lambda_K^- \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{ur}},$$

where $\mathbb{Z}_p^{\mathrm{ur}}$ denotes the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$. Hence, Theorem 1.2.2 also yields the following.

Theorem 1.2.3. *Let (E, p, K) be as in Theorem 1.2.2.*

(a) *If $p > 3$ satisfies* ($\mathrm{irr}_{\mathbb{Q}}$)*, then $\mathfrak{X}_{\mathrm{Gr}}(E/K_\infty^-)$ is Λ_K^--torsion, and*

$$\mathrm{ch}_{\Lambda_K^-}\left(\mathfrak{X}_{\mathrm{Gr}}(E/K_\infty^-)\right) = \left(\mathcal{L}_p^{\mathrm{BDP}}(E/K)\right)$$

in $\Lambda_K^{-,\mathrm{ur}} \otimes \mathbb{Q}_p$.

(b) *If further $p > 3$ satisfies* (sur)*, then the equality of characteristic ideals holds in $\Lambda_K^{-,\mathrm{ur}}$.*

1.3. **Application to the Birch and Swinnerton-Dyer formula.** A consequence of Theorems 1.1.2 and 1.2.3 is the following.

Corollary 1.3.1. *Let $E/\mathbb{Q}$ be a non-CM elliptic curve. Let $p > 3$ be a prime of good ordinary reduction such that* ($\mathrm{irr}_{\mathbb{Q}}$) *and* (im) *hold. If $\mathrm{ord}_{s=1} L(E,s) = r \in \{0,1\}$, then*

$$\left|\frac{L^{(r)}(E,1)}{\mathrm{Reg}(E)\cdot\Omega_E}\right|_p^{-1} = \left|\#\text{Ш}(E)\prod_{\ell\nmid\infty} c_\ell(E)\right|_p^{-1}$$

and hence the p-part of the conjectural BSD formula for E is true.

Proof. In the case $r = 0$, this follows from the equality of characteristic ideals in Theorem 1.1.2, the interpolation property of $\mathcal{L}_p(E/\mathbb{Q})$ at the trivial character, and the formula (up to a p-adic unit) in the control theorem [Gre99, Thm. 4.1] for the value of a characteristic power series for $\mathfrak{X}_{\rm ord}(E/\mathbb{Q}_\infty)$ at the trivial character.

Similarly, for a suitably chosen imaginary quadratic field K, the result in the case $r = 1$ follows from Theorem 1.2.3, the p-adic Waldspurger formula [BDP13] for the value of $\mathcal{L}_p^{\rm BDP}(E/K)$ at the trivial character, the anticyclotomic control theorem [JSW17, Thm. 3.3.1], and the $r = 0$ result for the K-quadratic twist of E. (See also [Cas24, §1] for a more detailed review of these arguments.) □

Remark 1.3.2. The condition (im) in Corollary 1.3.1 excludes only finitely many primes p (cf. Remark 1.1.3(i)). For an overview of prior results, the reader may refer to [BST21, BSTW23].

1.4. **On the two-variable Main Conjectures.** Our approach to the above theorems also gives a proof of the two-variable Iwasawa Main Conjectures for E/K under an additional hypothesis on $E[p]$.

For the precise statement, following a terminology from [Dia97] consider the set of "vexing primes" for $E[p]$:

$$\mathcal{V} := \bigl\{\ell \equiv -1 \,(\mathrm{mod}\, p) \ \mid \ \ \bar\rho_E|_{G_{\mathbb{Q}_\ell}} \text{ is irreducible and } \bar\rho_E|_{I_\ell} \text{ is reducible}\bigr\},$$

where $I_\ell \subset G_{\mathbb{Q}_\ell}$ are inertia and decomposition groups at ℓ, respectively.

Let K_∞/K denote the $\mathbb{Z}_p^2$-extension of K, and put $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ and $\Lambda_K = \mathbb{Z}_p[\![\Gamma_K]\!]$. Let $\mathfrak{X}_{\rm ord}(E/K_\infty)$ be the Pontryagin dual of the Selmer group $\mathrm{Sel}_{p^\infty}(E/K_\infty)$, and let $\mathcal{L}_p^{\rm PR}(E/K) \in \Lambda_K$ be the two-variable p-adic Rankin L-series constructed by Perrin-Riou [PR88] (normalized as in [CGS23, §1.2]).

Theorem 1.4.1. *Let (E, p, K) be as in Theorem 1.2.2. Assume that $\mathcal{V} = \emptyset$.*

(a) *If $p > 3$ satisfies (irr$_\mathbb{Q}$), then $\mathfrak{X}_{\rm ord}(E/K_\infty)$ is Λ_K-torsion, with*
$$\mathrm{ch}_{\Lambda_K}\bigl(\mathfrak{X}_{\rm ord}(E/K_\infty)\bigr) = \bigl(\mathcal{L}_p^{\rm PR}(E/K)\bigr)$$
in $\Lambda_K \otimes \mathbb{Q}_p$.

(b) *If further $p > 3$ satisfies (sur), then the equality of characteristic ideals holds in Λ_K.*

Remark 1.4.2. Since the global root number of E over K equals -1 (cf. (Heeg)), Theorem 1.4.1 complements the results on the two-variable Iwasawa Main Conjecture in [SU14]. On the other hand, some cases of the two-variable Main Conjecture when $\varepsilon(E/K) = -1$ for semistable elliptic curves E appeared in [CW22, Thm. A]. However, the latter excludes classical Heegner hypothesis and assumes a ramification hypothesis on $E[p]$.

1.5. **About the proofs.** The key new idea is to base change E to a quartic CM field M containing K for which the main result of [Wan15] towards a three-variable Main Conjecture applies, and utilize the two-variable zeta element associated to E over K as recently constructed in [BSTW23].

To begin, the main result of [Wan15] yields the divisibility

$$\bigl(\mathcal{L}_p^{\rm PR}(E/K) \cdot \mathcal{L}_p^{\rm PR}(E^F/K)\bigr) \supset \mathrm{ch}_{\Lambda_K}\bigl(\mathfrak{X}_{\rm ord}(E/K_\infty)\bigr) \cdot \mathrm{ch}_{\Lambda_K}\bigl(\mathfrak{X}_{\rm ord}(E^F/K_\infty)\bigr) \tag{1.1}$$

in $\Lambda_K \otimes \mathbb{Q}_p$, where E^F is the quadratic twist of E for the real subfield F contained in M. In view of the two-variable zeta elements of [BSTW23] and their explicit reciprocity laws, this translates into the divisibility

$$\bigl(\mathcal{L}_p^{\rm Gr}(E/K) \cdot \mathcal{L}_p^{\rm Gr}(E^F/K)\bigr) \supset \mathrm{ch}_{\Lambda_K}\bigl(\mathfrak{X}_{\rm Gr}(E/K_\infty)\bigr) \cdot \mathrm{ch}_{\Lambda_K}\bigl(\mathfrak{X}_{\rm Gr}(E^F/K_\infty)\bigr) \tag{1.2}$$

in $\Lambda_K^{\rm ur} \otimes \mathbb{Q}_p$, where $\mathcal{L}_p^{\rm Gr}(E^\cdot/K) \in \Lambda_K^{\rm ur} := \Lambda_K \hat\otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\rm ur}$ is a two-variable p-adic Rankin L-series specializing to $\mathcal{L}_p^{\rm BDP}(E/K)$ (up to a unit) under the natural projection $\Lambda_K^{\rm ur} \to \Lambda_K^{-,\rm ur}$, and $\mathfrak{X}_{\rm Gr}(E/K_\infty)$ is the counterpart of $\mathfrak{X}_{\rm Gr}(E/K_\infty^-)$ over K_∞/K.

In view of the vanishing of the Iwasawa μ-invariant of $\mathcal{L}_p^{\rm BDP}(E/K)$ proved in [Hsi14, Bur17] following ideas in [Hid10], the divisibilities (1.1) and (1.2) both hold integrally.

The proof of Theorem 1.1.2 then follows from (1.1) (for a suitably chosen K) by descending to the cyclotomic $\mathbb{Z}_p$-extension K_∞^+/K and appealing to Kato's work [Kat04]. Similarly, the proof of Theorem 1.2.3 (and hence of Theorem 1.2.2) follows from (1.2) by descending to K_∞^-/K and appealing to the Kolyvagin system bound developed in [CGLS22, CGS23] applied to the Heegner point Euler system. Without any restriction on $\mathcal{V}$, we thus arrive at the equality

$$\bigl(\mathcal{L}_p^{\rm PR}(E/K) \cdot \mathcal{L}_p^{\rm PR}(E^F/K)\bigr) = \mathrm{ch}_{\Lambda_K}\bigl(\mathfrak{X}_{\rm ord}(E/K_\infty)\bigr) \cdot \mathrm{ch}_{\Lambda_K}\bigl(\mathfrak{X}_{\rm ord}(E^F/K_\infty)\bigr) \tag{1.3}$$

(and likewise for (1.2)), and assuming $\mathcal{V} = \emptyset$ we separate the two factors, concluding the proof of Theorem 1.4.1.

Remark 1.5.1. An Euler system for E/K extending[2] the construction in [BSTW23] would give rise to a divisibility

$$\mathrm{ch}_{\Lambda_K}\big(\mathfrak{X}_{\mathrm{ord}}(E/K_\infty)\big) \supset \big(\mathcal{L}_p^{\mathrm{PR}}(E/K)\big),$$

possibly after inverting p. With this divisibility in hand, the proof of Theorem 1.4.1 follows from (1.3) without the additional hypothesis that $\mathcal{V}=\emptyset$.

Remark 1.5.2. In the main text we prove Theorems 1.1.2, 1.2.2, 1.2.3, and 1.4.1 for any weight two elliptic newform with good ordinary reduction at p.

We conclude this Introduction by noting that when $p>3$ satisfies ($\mathrm{irr}_{\mathbb{Q}}$), Theorems 1.1.2 and 1.2.3 are one of the key ingredients[3] in the proof of Kolyvagin's conjecture and its analogue for Kato's Euler systems in the joint work [BCGS23] of the authors with Grossi. When $p>3$ satisfies (sur), Theorems 1.1.2 and 1.2.3 are also used in [BCGS23] to prove the refinement of Kolyvagin's conjecture and its cyclotomic analog formulated by W. Zhang [Zha14a] and C.-H. Kim [Kim22], respectively.

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2. Main Conjecture over CM fields

In this section we briefly recall the formulation of the Iwasawa Main Conjecture over CM fields M/F at the base of the proof of our main results.

2.1. **Setting.** Let F be a totally real field of degree $d=[F:\mathbb{Q}]$.

Let $g\in S_2(\Gamma_0(\mathfrak{n}))$ be a Hilbert modular newform over F of parallel weight 2. Let p be a prime with

$$p\nmid D_F, \tag{ur}$$

where D_F denotes the discriminant of $F/\mathbb{Q}$. For a prime λ of the Hecke field F_g over p, let $\rho_g:G_F\to\mathrm{GL}_2(F_{g,\lambda})$ be the associated Galois representation and $V_g:=V_{g,\lambda}$ the underlying $F_{g,\lambda}$-vector space. Let $T_g\subset V_g$ be a G_F-stable $\mathcal{O}:=\mathcal{O}_{F_{g,\lambda}}$-lattice and,

$$\bar\rho_g:G_F\to\mathrm{GL}_2(\bar{\mathbb{F}}_p)$$

the corresponding residual representation. Suppose that g is ordinary at each prime w of F over p, which we abbreviate as p being a prime of ordinary reduction for g. Let $0\subset\mathrm{Fil}_w^+(V_g)\subset V_g$ be the associated filtration of $F_{g,\lambda}[G_{F_w}]$-modules and put $\mathrm{Fil}_w^+(T_g)=T_g\cap\mathrm{Fil}_w^+(V_g)$.

Let M be a CM quadratic extension of F such that

$$\text{any prime of } F \text{ above } p \text{ splits in } M. \tag{spl_F}$$

Denote by Γ_M^- (resp. Γ_M^+) the Galois group of the anticyclotomic $\mathbb{Z}_p^d$-extension M_∞/M (resp. cyclotomic $\mathbb{Z}_p$-extension M_∞^+/M). Let $M_\infty=M_\infty^-M_\infty^+$ be the compositum, and put $\Gamma_M=\mathrm{Gal}(M_\infty/M)$ and $\Lambda_M=\mathcal{O}[\![\Gamma_M]\!]$.

Selmer groups. We consider the $\mathcal{O}[G_M]$-module

$$\mathcal{M}_g=T_g\otimes_{\mathbb{Z}_p}\Lambda_M^\vee,$$

where $\Lambda_M^\vee=\mathrm{Hom}_{\mathrm{cts}}(\Lambda_M,\mathbb{Q}_p/\mathbb{Z}_p)$ denotes the Pontryagin dual, and G_M acts on Λ_M and $\Lambda_M^\vee$ via $\Psi:G_M\twoheadrightarrow\Gamma_M\hookrightarrow\Lambda_M^\times$ and Ψ^{-1}, respectively. For every prime w of M above p, put $\mathcal{M}_{g,w}^+=\mathrm{Fil}_w^+(T_g)\otimes_{\mathbb{Z}_p}\Lambda_M^\vee$. Let Σ be a finite set of places of M containing the primes above $\mathfrak{n}p\infty$, let M^Σ be the maximal extension of M unramified outside Σ, and define the *ordinary Selmer group* by

$$\mathrm{H}^1_{\mathcal{F}_\Lambda}(M,\mathcal{M}_g):=\ker\Big\{\mathrm{H}^1(M^\Sigma/M,\mathcal{M}_g)\to\prod_{w\in\Sigma,w\nmid p}\mathrm{H}^1(M_w,\mathcal{M}_g)\times\prod_{w\mid p}\mathrm{H}^1(I_w,\mathcal{M}_g/\mathcal{M}_{g,w}^+)\Big\}.$$

[2] The construction of such an Euler system will complement the one in [LLZ15], where the authors need to twist by a non-Eisenstein and p-distinguished Hecke character.

[3] A similar input when ($\mathrm{irr}_{\mathbb{Q}}$) is not satisfied is provided by the main results of [CGS23].

We put $\mathfrak{X}_{\mathrm{ord}}(g/M_\infty) = \mathrm{H}^1_{\mathcal{F}_\Lambda}(M, \mathcal{M}_g)^\vee$ to denote the Pontryagin dual, and for any subextension M' of M_∞/M let $\mathfrak{X}_{\mathrm{ord}}(g/M')$ be the analogously defined Selmer group with $\mathrm{Gal}(M'/M)$ in place of Γ_M.

We shall also consider the *Greenberg Selmer group*

$$\mathrm{H}^1_{\mathcal{F}_{\mathrm{Gr}}}(M, \mathcal{M}_g) := \ker \left\{ \mathrm{H}^1(M^\Sigma/M, \mathcal{M}_g) \to \prod_{w \in \Sigma, w \nmid p} \mathrm{H}^1(M_w, \mathcal{M}_g) \times \prod_{w \mid \overline{v}} \mathrm{H}^1(I_w, \mathcal{M}_g) \right\}$$

and its Pontryagin dual $\mathfrak{X}_{\mathrm{Gr}}(g/M_\infty)$, as well as their variants for any M' as above.

p-adic L-functions. Assume that the prime p is odd and unramified in F. Let M/F be a CM quadratic field extension satisfying (spl_F) and such that

(Δ) M is not contained in H_F, and any prime ramified in $F/\mathbb{Q}$ splits in M.

Here H_F denotes the Hilbert class field of F. Let

$$\mathcal{L}_p(g/M) \in \Lambda_M \otimes \mathbb{Q}_p$$

be the associated $(d+1)$-variable p-adic L-function as in [Wan15, §7.3], which interpolates the central L-values $L^{\mathrm{alg}}(g/M \otimes \chi, 1)$ as χ varies over finite order characters of Γ_M (cf. [Wan15, Thm. 82(i)]). If an underlying Hecke algebra is Gorenstein, then [Wan15, Thm. 82(ii)] shows the inclusion $\mathcal{L}_p(g/M) \in \Lambda_M$.

2.2. **Iwasawa Main Conjecture.**

Conjecture 2.2.1. *Let $g \in S_2(\Gamma_0(\mathfrak{n}))$ be a Hilbert modular newform over a totally real field F and p an odd prime unramified in F and good ordinary for g. Let M/F be a CM quadratic extension satisfying* (spl_F) *and such that*

(irr_M) *$\bar{\rho}_g$ is irreducible as G_M-representation.*

Then $\mathfrak{X}_{\mathrm{ord}}(g/M_\infty)$ is Λ_M-torsion, with

$$\mathrm{ch}_{\Lambda_M}\big(\mathfrak{X}_{\mathrm{ord}}(g/M_\infty)\big) = \big(\mathcal{L}_p(g/M)\big).$$

Remark 2.2.2. Without conditions (ur), and (irr_M), the conjecture is still expected to hold, with the equality of characteristic ideals being possibly up to tensoring with $\mathbb{Q}_p$.

3. Main Conjectures over quartic CM fields

We describe a consequence of a result [Wan15] towards Conjecture 2.2.1 which will be central to the proofs of main results.

3.1. **A hypothesis.** The results of [Wan15] are conditional on the following hypothesis.

Hypothesis 3.1.1.

- (H1) $\bar{\rho}_g|_{G_{F(\zeta_p)}}$ is absolutely irreducible, and for $p = 5$ the following case is excluded: the projective image $\bar{G}$ of $\bar{\rho}_g|_{G_F}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_p)$ and the mod p cyclotomic character factors through $G_F \to \bar{G}^{\mathrm{ab}} \simeq \mathbb{Z}/2\mathbb{Z}$ (in particular $[F(\zeta_5) : F] = 2$).
- (H2) There is a minimal modular lifting of $\bar{\rho}_g|_{G_F}$ (cf. [Fuj06, Def. 6.11]).
- (H3) For any finite place v of F, if $\bar{\rho}_g|_{G_{F_v}}$ is absolutely irreducible and $\bar{\rho}_g|_{I_v}$ is absolutely reducible, then $q_v \not\equiv -1 \pmod{p}$.

Here $I_v \subset G_{F_v}$ are inertia and decomposition groups at v, respectively, and q_v denotes the size of the residue field of v.

Remark 3.1.2.

- (i) Hypothesis (H1) implies that a certain Hecke algebra is Gorenstein (cf. [Fuj06, Thm. 11.1]), hence one has $\mathcal{L}_p(g/M) \in \Lambda_M$. (Note that here $\bar{\rho}_g$ is automatically p-distinguished in the sense of *loc. cit.*.)
- (ii) Under (H3), the exceptional case 0_E in [Fuj06, p. 16] does not occur, and hence the results of [Fuj06] apply to the setting of [Wan15, §7–9] (cf. [Fuj06, p. 57]).
- (iii) The case excluded by (H1) does not occur for g corresponding to elliptic curves (cf. [Fuj06, Prop. 9.8]).

3.2. The result.

Theorem 3.2.1 (Wan). *Let $g \in S_2(\Gamma_0(N))$ be an elliptic newform, and let $p > 3$ be a prime of good ordinary reduction for g. Let F be a real quadratic field with $(pN, D_F) = 1$, and let g_F denote the base-change of g to F. Let M/F be a CM quadratic extension satisfying (spl_F), (Δ), and $(N\mathcal{O}_F, D_{M/F}) = (1)$. Write*

$$N\mathcal{O}_F = \mathfrak{n}^+\mathfrak{n}^-,$$

with $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) divisible only by primes which are split (resp. inert) in M/F. Suppose that:

(i) *$\bar\rho_{g_F} = \bar\rho_g|_{G_F}$ satisfies (irr_M).*
(ii) *Hypothesis 3.1.1 holds.*
(iii) *$\mathfrak{n}^-$ is the square-free product of an even number of primes.*
(iv) *$\bar\rho_{g_F}$ is ramified at every prime dividing $\mathfrak{n}^-$.*

Then we have the divisibility

$$\big(\mathcal{L}_p(g_F/M)\big) \supset \mathrm{ch}_{\Lambda_M}\big(\mathfrak{X}_{\mathrm{ord}}(g/M_\infty)\big)$$

in Λ_M.

Proof. By [Wan15, Thm. 3], we have the divisibility

$$\big(\mathcal{L}_p(\mathbf{g}_F/M)\big) \supset \mathrm{ch}_{\mathbb{I}[\![\Gamma_M]\!]}\big(\mathfrak{X}_{\mathrm{ord}}(\mathbf{g}/M_\infty)\big)$$

in $\mathbb{I}[\![\Gamma_M]\!]$, where $\mathbf{g}_F$ denotes the parallel weight Hida family passing through g_F (cf. [Hid88], [Hid89]) and $\mathbb{I}$ is its coefficient ring.

Since $\mathfrak{X}_{\mathrm{ord}}(\mathbf{g}/M_\infty)$ specializes to $\mathfrak{X}_{\mathrm{ord}}(g/M_\infty)$ under the map induced by the specialization $\mathbb{I} \to \mathcal{O}$ corresponding to the p-ordinary stabilization of g_F (cf. [SU14, (3.5)]), and $\mathcal{L}_p(g_F/M)$ is defined as an analogous specialization of $\mathcal{L}_p(\mathbf{g}_F/M)$, the assertion follows. □

4. Main Conjectures over imaginary quadratic fields

We collect some known results in the direction of Conjecture 2.2.1 (and some variants) in the case $F = \mathbb{Q}$. Although some of these results are known under weaker hypotheses, here we shall assume that

(irr_K) $\qquad \bar\rho_g$ is irreducible as G_K-representation,

where $K = M$ is imaginary quadratic in this section.

We refer the reader to §§1.2 and 1.4 of [CGS23] for a review of the construction and interpolation property of the two-variable p-adic L-functions $\mathcal{L}_p^{\mathrm{PR}}(g/K) \in \Lambda_K$ and $\mathcal{L}_p^{\mathrm{Gr}}(g/K) \in \Lambda_K^{\mathrm{ur}}$ appearing below.

4.1. Two-variable Main Conjectures.

Conjecture 4.1.1. *Let $g \in S_2(\Gamma_0(N))$ be a newform and $p > 2$ a prime of good ordinary reduction for g. Let K be an imaginary quadratic field satisfying (irr_K). Then $\mathfrak{X}_{\mathrm{ord}}(g/K_\infty)$ is Λ_K-torsion, with*

$$\big(\mathcal{L}_p^{\mathrm{PR}}(g/K)\big) = \mathrm{ch}_{\Lambda_K}\big(\mathfrak{X}_{\mathrm{ord}}(g/K_\infty)\big).$$

Note that it follows from the comparison of p-adic L-functions in [Wan15, Prop. 84] that Conjecture 4.1.1 is nothing but Conjecture 2.2.1 in the current setting.

Conjecture 4.1.2. *Let $g \in S_2(\Gamma_0(N))$ be a newform and $p > 2$ a prime of good reduction for g. Let K be an imaginary quadratic field satisfying (spl). Then $\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty)$ is Λ_K-torsion, with*

$$\big(\mathcal{L}_p^{\mathrm{Gr}}(g/K)\big) = \mathrm{ch}_{\Lambda_K}\big(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty)\big)$$

as ideals in Λ_K^{ur}.

The central result of [BSTW23] is the existence of two-variable zeta element for E over K (see also [KLZ17]), which leads to the following useful equivalence between the preceding two-variable main conjectures of different nature.

Theorem 4.1.3. *Let $g \in S_2(\Gamma_0(N))$ be an elliptic newform, and $p \nmid 2N$ an ordinary prime for g. Let K be an imaginary quadratic field satisfying (spl), $(D_K, N) = 1$, and (irr_K). Then the following are equivalent:*

(i) *$\mathfrak{X}_{\mathrm{ord}}(g/K_\infty)$ is Λ_K-torsion, with*

$$\big(\mathcal{L}_p^{\mathrm{PR}}(g/K)\big) \supset \mathrm{ch}_{\Lambda_K}\big(\mathfrak{X}_{\mathrm{ord}}(g/K_\infty)\big) \quad \text{in } \Lambda_K \otimes \mathbb{Q}_p.$$

(ii) $\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty)$ *is* Λ_K*-torsion, with*

$$\big(\mathcal{L}_p^{\mathrm{Gr}}(g/K)\big) \supset \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty)) \quad \text{in } \Lambda_K^{\mathrm{ur}} \otimes \mathbb{Q}_p.$$

The same conclusion holds for the opposite divisibilities, and before inverting p. In particular, Conjecture 4.1.1 and Conjecture 4.1.2 are equivalent.

Proof. This is shown in [BSTW23, §9.3.2] (cf. [CGS23, Prop. 3.2.1] or [Cas24, §3.3]) building on a pair of four-term exact sequences coming from Poitou–Tate duality. □

Taking the direct sum of two pairs of four-term exact sequences as in the proof of Theorem 4.1.3, one deduces the following.

Corollary 4.1.4. *Let $g \in S_2(\Gamma_0(N))$ and $g' \in S_2(\Gamma_0(N'))$ be newforms, $p \nmid 2NN'$ an ordinary prime for both g and g', and K an imaginary quadratic field satisfying* (spl) *and $(D_K, NN') = 1$, and such that* (irr_K) *holds for both g and g'. Then the following are equivalent:*

(i) $\big(\mathcal{L}_p^{\mathrm{PR}}(g/K)\cdot\mathcal{L}_p^{\mathrm{PR}}(g'/K)\big) \supset \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{ord}}(g/K_\infty))\cdot \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{ord}}(g'/K_\infty))$.

(ii) $\big(\mathcal{L}_p^{\mathrm{Gr}}(g/K)\cdot\mathcal{L}_p^{\mathrm{Gr}}(g'/K)\big) \supset \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty))\cdot \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g'/K_\infty))$.

Moreover, the same holds for the opposite divisibility.

4.2. **Anticyclotomic Main Conjectures.** A refinement of Kolyvagin's methods (in the style of [How04] and [Nek07]) developed in [CGLS22, CGS23] yields the first part of the following result.

Theorem 4.2.1. *Let $g \in S_2(\Gamma_0(N))$ be an elliptic newform and p an odd prime of good ordinary reduction for g. Let K be an imaginary quadratic field satisfying* (disc), (Heeg), *and* (irr_K). *Then the following hold:*

(a) *Both $\mathfrak{X}_{\mathrm{ord}}(g/K_\infty^-)$ and $\mathrm{H}^1_{\mathcal{F}_\Lambda}(K, T_g\otimes\Lambda_K^-)$ have Λ_K^--rank one, and*

$$\mathrm{ch}_{\Lambda_K^-}\big(\mathfrak{X}_{\mathrm{ord}}(g/K_\infty^-)_{\mathrm{tor}}\big) \supset \mathrm{ch}_{\Lambda_K^-}\big(\mathrm{H}^1_{\mathcal{F}_\Lambda}(K, T_g\otimes\Lambda_K^-)/(\boldsymbol{\kappa}_1^{\mathrm{Heeg}})\big)^2 \quad \text{in } \Lambda_K^-\otimes\mathbb{Q}_p.$$

(b) *If K also satisfies* (spl), *then $\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty^-)$ is Λ_K^--torsion, and*

$$\mathrm{ch}_{\Lambda_K^-}(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty^-))\Lambda_K^{-,\mathrm{ur}} \supset (\mathcal{L}_p^{\mathrm{BDP}}(g/K)) \quad \text{in } \Lambda_K^{-,\mathrm{ur}}\otimes\mathbb{Q}_p.$$

Moreover, if (sur) *holds, then both divisibilities hold integrally.*

Proof. Part (a) is contained in [CGS23, Thm. 5.5.2], and part (b) then follows from [BCK21, Thm. 5.2]. Under (sur), part (a) follows from [How04, Thm. B], and part (b) again from [BCK21, Thm. 5.2]. □

The following vanishing of μ-invariant result based on Hida's ideas [Hid10] will also play an important role in our arguments.

Proposition 4.2.2. *Let $g \in S_2(\Gamma_0(N))$ be an elliptic newform with good reduction at $p > 2$, and suppose K is an imaginary quadratic field satisfying* (disc), (Heeg), (spl), *and* (irr_K). *Then*

$$\mu(\mathcal{L}_p^{\mathrm{Gr}}(g/K)) = \mu(\mathcal{L}_p^{\mathrm{BDP}}(g/K)) = 0.$$

Proof. By [Hsi14, Thm. B], $\mathcal{L}_p^{\mathrm{BDP}}(g/K)$ has vanishing μ-invariant. Since a direct comparison of the interpolation properties shows that the projection of $\mathcal{L}_p^{\mathrm{Gr}}(g/K)$ to $\Lambda_K^{-,\mathrm{ur}}$ generates the same ideal as $\mathcal{L}_p^{\mathrm{BDP}}(g/K)$ (see [CGS23, Prop. 1.4.5]), the result follows. □

5. Base change

Let $g \in S_2(\Gamma_0(N))$ be an elliptic newform, and $p > 3$ a good ordinary prime for g such that ($\mathrm{irr}_{\mathbb{Q}}$) holds.

5.1. **Base change.** Let M/F be a CM quadratic extension of a real quadratic field F for the form $M = FK$ with K an imaginary quadratic field.

For the $\mathbb{Z}_p^2$-extension $\tilde{K}_\infty := FK_\infty$ of M, put $\tilde{\Lambda}_K := \mathcal{O}[\![\mathrm{Gal}(\tilde{K}_\infty/M)]\!]$ and let

$$\pi_K : \Lambda_M \to \tilde{\Lambda}_K \simeq \Lambda_K$$

denote the map arising from the projection $\mathrm{Gal}(M_\infty/M) \twoheadrightarrow \mathrm{Gal}(\tilde{K}_\infty/M)$.

Lemma 5.1.1. *We have the divisibility*
$$\pi_K\big(\mathrm{ch}_{\Lambda_M}(\mathfrak{X}_{\mathrm{ord}}(g/M_\infty))\big) \supset \mathrm{ch}_{\Lambda_K}\big(\mathfrak{X}_{\mathrm{ord}}(g/K_\infty)\big)\cdot \mathrm{ch}_{\Lambda_K}\big(\mathfrak{X}_{\mathrm{ord}}(g^F/K_\infty)\big)$$
in Λ_K, where g^F is the twist of g by the quadratic character corresponding to $F/\mathbb{Q}$.

Proof. It follows readily from Shapiro's lemma and [SU14, Prop. 3.6 and Prop. 3.7] that the restriction map $\mathrm{H}^1(M, T_g\otimes\tilde{\Lambda}_K^\vee)\to \mathrm{H}^1(M,\mathcal{M}_g)[I_K]$, where $I_K=\ker(\pi_K)$, induces Λ_K-module isomorphisms
$$\begin{aligned}\mathrm{H}^1_{\mathcal{F}_\Lambda}(M,\mathcal{M}_g)[I_K] &\simeq \mathrm{H}^1_{\mathcal{F}_\Lambda}(M, T_g\otimes\tilde{\Lambda}_K^\vee)\\ &\simeq \mathrm{H}^1_{\mathcal{F}_\Lambda}(M, T_g\otimes\Lambda_K^\vee)\oplus \mathrm{H}^1_{\mathcal{F}_\Lambda}(M, T_g\otimes\eta_F\otimes\Lambda_K^\vee).\end{aligned}$$
Here η_F is the non-trivial character of $\mathrm{Gal}(F/\mathbb{Q})\simeq\mathrm{Gal}(M/K)$ and so
$$T_g\otimes\eta_F\simeq T_{g^F}.$$
Since by [SU14, Cor. 3.8] we have the divisibility
$$\pi_K\big(\mathrm{ch}_{\Lambda_M}(\mathfrak{X}_{\mathrm{ord}}(g/M_\infty))\big)\supset \mathrm{ch}_{\Lambda_K}\big(\mathfrak{X}_{\mathrm{ord}}(g/M_\infty)/I_K\big)$$
in $\Lambda_M/I_K\simeq\Lambda_K$, the proof concludes. ☐

Lemma 5.1.2. *Let the setting be as in Lemma 5.1.1. Then we have*
$$\big(\pi_K(\mathcal{L}_p(g_F/M))\big) = \big(\mathcal{L}_p^{\mathrm{PR}}(g/K)\cdot\mathcal{L}_p^{\mathrm{PR}}(g^F/K)\big)$$
in $\Lambda_K\otimes\mathbb{Q}_p$.

Proof. This just follows from a direct comparison of the interpolation properties (cf. [Wan21, Prop. 84]). ☐

5.2. **Proofs of main results.** We can now complete the proof of the results stated in §1. We begin with the following key proposition.

Proposition 5.2.1. *Let $g\in S_2(\Gamma_0(N))$ be an elliptic newform and $p\geq 5$ a prime of good ordinary reduction for g. Let K be an imaginary quadratic field satisfying* (disc)*,* (Heeg)*,* (spl)*, and* (irr_K)*. Suppose F is a real quadratic field of discriminant D_F satisfying the following hypotheses:*

(i) *p is inert in F.*
(ii) *D_F is odd and every prime dividing D_F splits in K.*
(iii) *Every prime $\ell|N$ is inert in F if $\ell\equiv -1 \pmod p$, and is split in F otherwise.*
(iv) (irr_M) *holds for $M=FK$.*
(v) *$\bar\rho_g|_{G_{F(\zeta_p)}}$ is irreducible.*
(vi) *If $p=5$, then $F\neq\mathbb{Q}(\zeta_5)^+$.*

Then we have the divisibilities
$$\big(\mathcal{L}_p^{\mathrm{PR}}(g/K)\cdot\mathcal{L}_p^{\mathrm{PR}}(g^F/K)\big)\supset \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{ord}}(g/K_\infty))\cdot\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{ord}}(g^F/K_\infty))\quad \textit{in } \Lambda_K,$$
and
$$\big(\mathcal{L}_p^{\mathrm{Gr}}(g/K)\cdot\mathcal{L}_p^{\mathrm{Gr}}(g^F/K)\big)\supset \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty))\cdot\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g^F/K_\infty))\quad \textit{in } \Lambda_K^{\mathrm{ur}}.$$

Proof. Note that the CM quadratic extension M/F and the residual representation $\bar\rho_g|_{G_F}$ satisfy the hypotheses of Theorem 3.2.1. Indeed, the conditions $(pN, D_F)=1$ and $(N\mathcal{O}_F, D_{M/F})=1$ are clear, and hypotheses (i)–(iii) and (Heeg) imply (spl_F) and (Δ). Further (ii) implies $\mathfrak{n}^-=\mathcal{O}_F$ and so hypotheses (iii) and (iv) of Theorem 3.2.1 are vacuous. Hypothesis 3.1.1 is readily verified: (H1) follows by (v) and (vi), (H2) follows as in [Wan21, Thm. 103] (indeed, by our choice of F, the base change to F of a minimal modular lifting of $\bar\rho_g$ gives a minimal modular lifting of $\bar\rho_{g_F}$), and (H3) also follows by condition (iii) on F.

Now Theorem 3.2.1 leads to the divisibility
$$\big(\pi_K(\mathcal{L}_p(g_F/M))\big)\supset \pi_K\big(\mathrm{ch}_{\Lambda_M}(\mathfrak{X}_{\mathrm{ord}}(g/M_\infty))\big)$$
in Λ_K, and so by Lemmas 5.1.1 and 5.1.2, we have
$$\big(\mathcal{L}_p^{\mathrm{PR}}(g/K)\cdot\mathcal{L}_p^{\mathrm{PR}}(g^F/K)\big)\supset \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{ord}}(g/K))\cdot\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{ord}}(g^F/K))$$
in $\Lambda_K\otimes\mathbb{Q}_p$. In turn Proposition 4.1.3 (and its proof) implies
$$\big(\mathcal{L}_p^{\mathrm{Gr}}(g/K)\cdot\mathcal{L}_p^{\mathrm{Gr}}(g^F/K)\big)\supset \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g/K))\cdot\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g^F/K))\tag{5.1}$$

in $\Lambda_K^{\mathrm{ur}} \otimes \mathbb{Q}_p$. Since $\mu(\mathcal{L}_p^{\mathrm{Gr}}(g/K) \cdot \mathcal{L}_p^{\mathrm{Gr}}(g^F/K)) = 0$ by Proposition 4.2.2, the divisibility (5.1) holds integrally in Λ_K^{ur}. By again appealing to Proposition 4.1.3, the proof concludes. ☐

Remark 5.2.2. In the case $p = 5$, if g corresponds to an elliptic curve $E/\mathbb{Q}$, then the condition $F \neq \mathbb{Q}(\zeta_5)^+$ is inessential (cf. Remark 3.1.2(iii)).

The existence of F satisfying the conditions in Proposition 5.2.1 is easily verified:

Lemma 5.2.3. *Let $g \in S_2(\Gamma_0(N))$ be an elliptic newform and $p \geq 5$ a prime of good ordinary reduction for g such that $\bar{\rho}_g$ is irreducible as $G_{\mathbb{Q}}$-representation. Then there exist an imaginary quadratic field K satisfying* (disc)*,* (Heeg)*,* (spl)*, and* (irr_K)*, and a real quadratic field F satisfying* (i)–(vi) *in Proposition 5.2.1.*

Proof. Since $p \nmid N$ by hypothesis, (i) and (iii) are independent splitting conditions which hold for a positive proportion of real quadratic fields F; fix one such F with odd discriminant D_F. Similarly, (spl), (Heeg), and (ii) hold for a positive proportion of imaginary quadratic fields K, and we can fix one such K satisfying these conditions in addition (disc) and (irr_K).

In view of (irr_K), if FK is not a subfield of the splitting field of $\rho_g|_{G_K}$, then (irr_M) holds for $M = FK$. For such F, if $\bar{\rho}_g|_{G_{F(\zeta_p)}}$ is reducible, then $\bar{\rho}_g|_{G_F}$ is induced by an index 2 subgroup G_L of G_F which contains $G_{F(\zeta_p)}$; but this forces $p = 5$ and $F = \mathbb{Q}(\zeta_5)^+$, so the result follows. ☐

Proof of Theorem 1.1.2. Pick an imaginary quadratic field K and a real quadratic field F as in Lemma 5.2.3.

Then by Proposition 5.2.1,

$$\big(\mathcal{L}_p^{\mathrm{PR}}(g/K) \cdot \mathcal{L}_p^{\mathrm{PR}}(g^F/K)\big) \supset \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{ord}}(g/K_\infty)) \cdot \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{ord}}(g^F/K_\infty)). \tag{5.2}$$

By [CGS23, Prop. 1.2.4] and Propositions 3.6 and 3.9 in [SU14], taking the image under the maps induced by the projection $\pi_+ : \Gamma_K \twoheadrightarrow \Gamma_K^+$, from (5.2) we get the divisibilities

$$\begin{aligned}
&\big(\mathcal{L}_p(g/\mathbb{Q}) \cdot \mathcal{L}_p(g^K/\mathbb{Q}) \cdot \mathcal{L}_p(g^F/\mathbb{Q}) \cdot \mathcal{L}_p(g^{FK}/\mathbb{Q})\big)\\
&\qquad \supset \pi_+\big(\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{ord}}(g/K_\infty)) \cdot \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{ord}}(g^F/K_\infty))\big)\\
&\qquad \supset \mathrm{ch}_{\Lambda}(\mathfrak{X}_{\mathrm{ord}}(g/\mathbb{Q}_\infty)) \cdot \mathrm{ch}_{\Lambda}(\mathfrak{X}_{\mathrm{ord}}(g^K/\mathbb{Q}_\infty)) \cdot \mathrm{ch}_{\Lambda}(\mathfrak{X}_{\mathrm{ord}}(g^F/\mathbb{Q}_\infty)) \cdot \mathrm{ch}_{\Lambda}(\mathfrak{X}_{\mathrm{ord}}(g^{FK}/\mathbb{Q}_\infty))
\end{aligned} \tag{5.3}$$

in $\Lambda_K^+ \simeq \Lambda$.

On the other hand, by [Kat04, Thm. 17.4] we have the divisibility

$$\big(\mathcal{L}_p(g/\mathbb{Q})\big) \subset \mathrm{ch}_{\Lambda}(\mathfrak{X}_{\mathrm{ord}}(g/\mathbb{Q}_\infty)) \tag{5.4}$$

in Λ if hypothesis (im) holds, and in $\Lambda \otimes \mathbb{Q}_p$ otherwise. Similar divisibilities hold for the twists g^K, g^F, and g^{FK}. Noting that a proper divisibility in (5.4) would contradict (5.3), the proof concludes. ☐

Proof of Theorem 1.2.2 and Theorem 1.2.3. Pick a real quadratic field F as in Lemma 5.2.3.

Then by Theorem 5.2.1,

$$\big(\mathcal{L}_p^{\mathrm{Gr}}(g/K) \cdot \mathcal{L}_p^{\mathrm{Gr}}(g^F/K)\big) \supset \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty)) \cdot \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g^F/K_\infty)). \tag{5.5}$$

By [CGS23, Prop. 1.4.5] and [JSW17, Cor. 3.4.2], taking the image under the maps induced by the projection $\pi_- : \Gamma_K \twoheadrightarrow \Gamma_K^-$, from (5.5) we get the divisibilities

$$\begin{aligned}
\big(\mathcal{L}_p^{\mathrm{BDP}}(g/K) \cdot \mathcal{L}_p^{\mathrm{BDP}}(g^F/K)\big) &\supset \pi_-\big(\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty)) \cdot \mathrm{ch}_{\Lambda_K^-}(\mathfrak{X}_{\mathrm{Gr}}(g^F/K_\infty))\big)\\
&\supset \mathrm{ch}_{\Lambda_K^-}(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty^-)) \cdot \mathrm{ch}_{\Lambda_K^-}(\mathfrak{X}_{\mathrm{Gr}}(g^F/K_\infty^-))
\end{aligned}$$

in $\Lambda_K^{-,\mathrm{ur}}$. Together with Theorem 4.2.1, this concludes the proof. ☐

Proof of Theorem 1.4.1. Pick an imaginary quadratic field K and two different real quadratic fields F, F' as in Lemma 5.2.3.

For each of the pairs (K, F), (K, F'), the equalities of characteristic ideals of Theorem 1.2.3, the divisibility (5.5), and the non-vanishing of the p-adic L-functions $\mathcal{L}_p^{\mathrm{BDP}}(g^{\cdot}/K)$ for $\cdot \in \{\emptyset, F, F'\}$ yields (by an application of [SU14, Lem. 3.2]) the equalities

$$\begin{aligned}
\big(\mathcal{L}_p^{\mathrm{Gr}}(g/K) \cdot \mathcal{L}_p^{\mathrm{Gr}}(g^F/K)\big) &= \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty)) \cdot \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g^F/K_\infty)),\\
\big(\mathcal{L}_p^{\mathrm{Gr}}(g/K) \cdot \mathcal{L}_p^{\mathrm{Gr}}(g^{F'}/K)\big) &= \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty)) \cdot \mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g^{F'}/K_\infty)),
\end{aligned}$$

and therefore

$$\big(\mathcal{L}_p^{\mathrm{Gr}}(g/K)^2\cdot\mathcal{L}_p^{\mathrm{Gr}}(g^F/K)\cdot\mathcal{L}_p^{\mathrm{Gr}}(g^{F'}/K)\big)=\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty))^2\cdot\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g^F/K_\infty))\cdot\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g^{F'}/K_\infty)).$$

When $\mathcal{V}=\emptyset$, condition (H3) in Hypothesis 3.1.1 is vacuous, and therefore the argument in the proof of Theorem 1.2.3 applies for any real quadratic F_0 satisfying conditions (i)-(ii) and (iv)-(vi) in Lemma 5.2.3, but not necessarily (iii). Thus taking F_0 to be the third real quadratic field in the compositum FF', as above we obtain the equality

$$\big(\mathcal{L}_p^{\mathrm{Gr}}(g^F/K)\cdot\mathcal{L}_p^{\mathrm{Gr}}(g^{F'}/K)\big)=\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g^F/K_\infty))\cdot\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g^{F'}/K_\infty)).$$

The combination of the last two equalities immediately gives $(\mathcal{L}_p^{\mathrm{Gr}}(g/K))=\mathrm{ch}_{\Lambda_K}(\mathfrak{X}_{\mathrm{Gr}}(g/K_\infty))$, which together with Proposition 4.1.3 yields the result. ☐

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