KATO'S MAIN CONJECTURE FOR NONORDINARY MODULAR FORMS

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Abstract. We prove Kato's main conjecture for modular forms at nonordinary primes for weights in the Fontaine–Laffaille range. The key ingredients in the proof are a reformulation of the conjecture in terms of signed Selmer groups due to Lei–Loeffler–Zerbes, certain p-adic families of Rankin–Eisenstein classes arising from the work of Lei–Loeffler–Zerbes and Kings–Loeffler–Zerbes, and the lower bound divisibility in an Iwasawa–Greenberg main conjecture for Rankin–Selberg p-adic L-functions obtained in our earlier work [CLW22].

Contents

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1. Introduction

In [Kat93, §4], Kato formulated a vast generalisation of Iwasawa's main conjecture, conditional on a strong form of Beilinson's conjectures [Bei84]. For the motive attached to a newform $f \in S_k(\Gamma_0(N))$ of weight $k \geq 2$ and the cyclotomic extension $\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}$, the conjecture is unconditional, see [Kat04, Conj. 12.10].

The main result of this paper is a proof of Kato's main conjecture for newforms of weight $2 \leq k < p$ at nonordinary primes $p > 2$ under mild hypotheses.

1.1. Main results.

Let us prepare some notation for the precise statement of our main results. Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N))$ be a newform of (even) weight $k \geq 2$. Fix an odd prime p, and put

$$\widetilde{\Gamma} = \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \cong \Delta \times \Gamma,$$

where Γ is the Galois group of the cyclotomic $\mathbf{Z}_p$-extension $\mathbf{Q}_\infty/\mathbf{Q}$, and Δ is cyclic of order $p-1$. Let $F = \mathbf{Q}(\{a_n\}_n) \subset \mathbf{C}$ be the number field generated by the Fourier coefficients of f. Fix an embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, where $\overline{\mathbf{Q}}$ denotes the algebraic closure of $\mathbf{Q}$ inside $\mathbf{C}$, and let $L = F_{\mathfrak{P}}$ be the completion of F at the prime $\mathfrak{P}$ above p determined by ι_p. Denote by $\mathscr{O}$ the integer ring of F. Let $V_f \cong L^2$ be the p-adic Galois representation attached to f by Deligne [Del71], and for any $G_{\mathbf{Q}}$-stable $\mathscr{O}$-lattice $T_f \subset V_f$, let

$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T_f) := \varprojlim_r \mathrm{H}^1(\mathbf{Q}(\mu_{p^r}), T_f),$$
$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V_f) := \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T_f) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

be the Iwasawa cohomology groups for $\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}$. Let

$$\Lambda_{\mathscr{O}}(\widetilde{\Gamma}) = \mathscr{O}[\![\widetilde{\Gamma}]\!] \cong \mathscr{O}[\Delta][\![\Gamma]\!]$$

be the cyclotomic Iwasawa algebra. In his influential work [Kat04], Kato constructed a collection of cohomology classes $\mathbf{z}_\gamma^{(p)} \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V_f)$, depending L-linearly on $\gamma \in V_f$, whose image under the Bloch–Kato dual exponential map is related to the special values $L(f, \chi, r)$, for $1 \leq r \leq k-1$, of the complex L-function of f twisted by finite order characters $\chi : \widetilde{\Gamma} \to \mathbf{C}^\times$ (see [*op. cit.*, Thm. 12.5]).

For $\gamma \in T_f \subset V_f$, Kato showed that $\mathbf{z}_\gamma^{(p)}$ lands in $\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T_f)$ provided $\rho_f : G_{\mathbf{Q}} \to \mathrm{Aut}_L(V_f) \cong \mathrm{GL}_2(L)$ has "large image"; assume from now on that this is the case, let $\mathbf{z}_f^{\mathrm{int}} \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T_f)$ be the resulting integral normalisation of $\mathbf{z}_\gamma^{(p)}$ as in §2.4, and denote by $(\mathbf{z}_f^{\mathrm{int}})$ the $\Lambda_{\mathscr{O}}(\widetilde{\Gamma})$-span of $\mathbf{z}_f^{\mathrm{int}}$.

Let $T_f^\vee(1) = \mathrm{Hom}(T_f, \mu_{p^\infty})$ be the Cartier dual of T_f, and

$$\mathrm{Sel}_{\mathrm{str}}(f/\mathbf{Q}(\mu_{p^\infty})) := \ker\left\{\mathrm{H}^1(\mathbf{Q}(\mu_{p^\infty}), T_f^\vee(1)) \longrightarrow \prod_\eta \mathrm{H}^1(\mathbf{Q}(\mu_{p^\infty})_\eta, T_f^\vee(1))\right\}$$

be the p-primary *strict Selmer group*, where η runs over the places of $\mathbf{Q}(\mu_{p^\infty})$. In this setting, Kato formulated the following generalised Iwasawa main conjecture.

Conjecture 1.1 (Kato's main conjecture)**.** *Let $p > 2$ be a prime. For every character $\eta : \Delta \to \mathbf{Z}_p^\times$, we have the equality*

$$\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\left(\frac{\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T_f)^\eta}{(\mathbf{z}_f^{\mathrm{int}})^\eta}\right) = \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(\mathrm{Sel}_{\mathrm{str}}(f/\mathbf{Q}(\mu_{p^\infty}))^{\eta,\vee}\big)$$

as characteristic ideals of torsion $\Lambda_{\mathscr{O}}(\Gamma)$-modules, where $M^{\eta,\vee} = \mathrm{Hom}(M^\eta, \mathbf{Q}_p/\mathbf{Z}_p)$ denotes the Pontryagin dual of the the η-isotypic component of M.

When p is an odd prime of good ordinary reduction for f, i.e. $p \nmid N$ and $a_p \notin \mathfrak{P}$, Conjecture 1.1 is known to be equivalent to Greenberg's main conjecture [Gre89] for the p-adic L-function of Mazur–Tate–Teitelbaum associated to the p-adic unit root of

$$x^2 - a_p x + p^{k-1} = (x-\alpha)(x-\beta). \tag{1.1}$$

In this case, the conjectures are known under mild hypotheses by the work of several mathematicians, including Kato, Skinner–Urban, and Wan (see [Kat04, SU14, Wan15, BCS25]).

For the statement of our main result on Conjecture 1.1, let $\pi_f = \otimes_\ell \pi_{f,\ell}$ be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by f, and consider the following *root number condition*:

$$\text{there exists a prime } \ell \| N \text{ such that } \pi_\ell \cong \sigma_\ell \otimes \xi_\ell, \tag{rtn}$$

where σ_ℓ denotes the special representation of $\mathrm{GL}_2(\mathbf{Q}_\ell)$, and ξ_ℓ is the unique unramified character of $\mathbf{Q}_\ell^\times$ sending ℓ to $-\ell^{k/2-1}$. Let $\omega : \Delta \to \mathbf{Z}_p^\times$ be the Teichmüller character.

Theorem A. *Let $f \in S_k(\Gamma_0(N))$ be a newform and $p > 2$ be a prime of good nonordinary reduction. Suppose*

- $2 \leq k < p$;
- *N is squarefree;*
- *f is p-regular, i.e. $\alpha \neq \beta$;*
- *either* $\begin{cases} k = 2, \text{or} \\ \text{Condition (rtn) holds.} \end{cases}$

Then Conjecture 1.1 for $\eta = \omega^{k/2}$ holds.

As explained in more detail in Section 1.2 below, two key ingredients in the proof of Theorem A are certain Rankin–Eisenstein classes arising from the work of Lei–Loeffler–Zerbes [LLZ14] and Kings–Loeffler–Zerbes [KLZ20, KLZ17, LZ16], and the "lower bound" divisibility in an Iwasawa–Greenberg main conjecture for Rankin–Selberg p-adic L-functions obtained in our earlier work [CLW22].

1.2. **Strategy of the proof.** Our starting point for the proof of Theorem A is an equivalent form of Conjecture 1.1 discovered by Lei–Loeffler–Zerbes [LLZ10] building on integral p-adic Hodge theory, and more specifically, Berger's theory of Wach modules [Ber04]. To briefly explain this, recall that Perrin-Riou's p-adic regulator map [PR94] gives a way of attaching to Kato's $\mathbf{z}_f^{\mathrm{int}}$ an element

$$\mathscr{L}^{\mathrm{PR}}(\mathrm{res}_p(\mathbf{z}_f^{\mathrm{int}})) \in \mathcal{H}_L(\widetilde{\Gamma}) \otimes \mathbb{D}_{\mathrm{cris}}(V_f) \tag{1.2}$$

where $\mathcal{H}_L(\widetilde{\Gamma})$ is the algebra of tempered L-valued distributions on $\widetilde{\Gamma}$ and $\mathbb{D}_{\mathrm{cris}}(V_f)$ is the Dieudonné–Fontaine module associated to f. By virtue of Kato's explicit reciprocity law, pairing (1.2) against any vector $\nu \in \mathbb{D}_{\mathrm{cris}}(V_f^*(1))$ via the de Rham pairing on $\mathbb{D}_{\mathrm{cris}}(V_f) \times \mathbb{D}_{\mathrm{cris}}(V_f^*(1))$, where $V_f^* = \mathrm{Hom}_L(V_f, L)$ is the dual of V_f, gives an element $\mathscr{L}_\nu^{\mathrm{PR}}(\mathrm{res}_p(\mathbf{z}_f^{\mathrm{int}})) \in \mathcal{H}_L(\widetilde{\Gamma})$ whose values at certain characters χ of $\widetilde{\Gamma}$ is related to the critical values of $L(f,\chi,s)$. In particular, the p-adic L-functions $\mathscr{L}_{p,\xi}^{\mathrm{MTT}}$ of Mazur–Tate–Teitelbaum [MTT86] associated to any root $\xi \in \{\alpha, \beta\}$ of (1.1) with $v_p(\xi) < k-1$ are recovered in this manner. In the ordinary case, taking $\xi = \alpha$ to be the p-adic unit root of (1.1) and $\nu = \nu_\alpha$ a certain Frobenius eigenvector with eigenvalue α, one thus obtains an equality

$$\mathscr{L}_{\nu_\alpha}^{\mathrm{PR}}(\mathrm{res}_p(\mathbf{z}_f^{\mathrm{int}})) = \mathscr{L}_{p,\alpha}^{\mathrm{MTT}} \tag{1.3}$$

as elements in $\Lambda_{\mathscr{O}}(\widetilde{\Gamma})$ (cf. [Kat04, Thm. 16.6]). With the help of (1.3), Kato was able to deduce from his results on Conjecture 1.1, a proof of a divisibility in the Iwasawa main conjecture for these bounded p-adic L-functions. A proof of the opposite divisibility—and hence a full proof of Conjecture 1.1—in the ordinary case under mild hypotheses, was then obtained in the work of Skinner–Urban [SU14] on the Iwasawa main conjecture for certain Rankin–Selberg p-adic L-functions related to $\mathscr{L}_{p,\alpha}^{\mathrm{MTT}}$ using Eisenstein congruences on $\mathrm{GU}(2,2)$.

For the proof of Theorem A in the nonordinary case, our approach builds on Kato's work and our earlier main result [CLW22] on the Iwasawa main conjecture for Rankin–Selberg p-adic L-functions using Eisenstein congruences on $\mathrm{GU}(3,1)$. However, compared to the ordinary case, two fundamental new difficulties arise:

(1) For any of the roots $\xi \in \{\alpha, \beta\}$ of (1.1), the image of the map
$$\mathscr{L}^{\mathrm{PR}}_{\nu_\xi} : \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), T_f) \longrightarrow \mathcal{H}_L(\widetilde{\Gamma})$$
is not contained in $\Lambda_{\mathscr{O}}(\widetilde{\Gamma})$.

(2) The Rankin–Selberg p-adic L-function
$$\mathcal{L}_{\overline{v}}(f/K)^\eta \tag{1.4}$$
studied in [CLW22] has very little relation with the p-adic distributions $\mathscr{L}^{\mathrm{MTT}}_{p,\xi}$ (in particular, their ranges of p-adic interpolation are disjoint).

To address (1), we begin by recalling that in the case $a_p = 0$ (so $\beta = -\alpha$), Lei [Lei11] showed that certain linear combinations of $\mathscr{L}^{\mathrm{PR}}_{\nu_\alpha}$ and $\mathscr{L}^{\mathrm{PR}}_{\nu_{-\alpha}}$ give rise to $\Lambda_{\mathscr{O}}(\widetilde{\Gamma})$-linear *(signed) Coleman maps*
$$\mathscr{C}^{\pm} : \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), T_f) \longrightarrow \Lambda_{\mathscr{O}}(\widetilde{\Gamma}) \tag{1.5}$$
recovering Kobayashi's construction in the case of rational elliptic curves [Kob03]; in particular, with the property that $\mathscr{C}^{\pm}(\mathrm{res}_p(\mathbf{z}_f^{\mathrm{int}}))$ recovers Pollack's signed p-adic L-functions [Pol03] attached to f. This allowed him, as originally done in [Kob03] for rational elliptic curves, to formulate (and prove results towards) signed main conjectures equivalent to Kato's main conjecture.

Abstracting this idea, we note that given a $\Lambda_{\mathscr{O}}(\widetilde{\Gamma})$-linear map $\mathscr{C}$ as in (1.5), one can define a dual Selmer group
$$\mathrm{Sel}_{\mathscr{C}}(\mathbf{Q}(\mu_{p^\infty}), T_f^\vee(1)) \subset \mathrm{H}^1(\mathbf{Q}(\mu_{p^\infty}), T_f^\vee(1)) \tag{1.6}$$
taking the orthogonal complement $\ker(\mathscr{C})^\perp \subset \mathrm{H}^1(\mathbf{Q}_p(\mu_{p^\infty}), T_f^\vee(1))$ under local Tate duality as the local condition cutting $\mathrm{Sel}_{\mathscr{C}}(\mathbf{Q}(\mu_{p^\infty}), T_f^\vee(1))$ at the prime above p. In favorable circumstances, Poitou–Tate duality then gives rise to an exact sequence
$$\begin{aligned}0 \longrightarrow \frac{\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T_f)}{(\mathbf{z}_f^{\mathrm{int}})} \longrightarrow &\frac{\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), T_f)}{(\mathrm{res}_p(\mathbf{z}_f^{\mathrm{int}}))} \\ &\longrightarrow \mathrm{Sel}_{\mathscr{C}}(\mathbf{Q}(\mu_{p^\infty}), T_f^\vee(1))^\vee \longrightarrow \mathrm{Sel}_{\mathrm{str}}(\mathbf{Q}(\mu_{p^\infty}), T_f^\vee(1))^\vee \longrightarrow 0,\end{aligned}$$
whereby Conjecture 1.1 amounts to the following statement.

Conjecture 1.2 ($\mathscr{C}$-main conjecture). *Suppose $\eta \in \Delta \to \mathbf{Z}_p^\times$ is such that*
$$\mathscr{C}^\eta : \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), T_f)^\eta \longrightarrow \Lambda_{\mathscr{O}}(\Gamma)$$
has $\Lambda_{\mathscr{O}}(\Gamma)$-torsion cokernel with characteristic ideal generated by $\xi_\eta \in \Lambda_{\mathscr{O}}(\Gamma)$, and
$$\mathscr{C}^\eta(\mathrm{res}_p(\mathbf{z}_f^{\mathrm{int},\eta})) \neq 0,$$
where $\mathbf{z}_f^{\mathrm{int},\eta}$ is the projection of $\mathbf{z}_f^{\mathrm{int}}$ to the η-isotytic component of $\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T_f)$. Then the Pontryagin dual $\mathrm{Sel}_{\mathscr{C}}(\mathbf{Q}(\mu_{p^\infty}), T_f^\vee(1))^{\eta,\vee}$ is $\Lambda_{\mathscr{O}}(\Gamma)$-torsion, with
$$(\xi_\eta) \cdot \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(\mathrm{Sel}_{\mathscr{C}}(\mathbf{Q}(\mu_{p^\infty}), T_f^\vee(1))^{\eta,\vee}\big) = \big(\mathscr{C}^\eta(\mathrm{res}_p(\mathbf{z}_f^{\mathrm{int},\eta}))\big).$$

Our proof of Theorem A is deduced from a proof of Conjecture 1.2 for one of the Coleman maps $\mathscr{C}^{i,\eta}_{\mathrm{LLZ}}$ ($i \in \{1,2\}$) constructed by Lei–Loeffler–Zerbes [LLZ10]. Bypassing the use of Honda theory of formal groups in Kobayashi's original construction of signed Coleman maps, the much more general (albeit less explicit) construction of $\mathscr{C}^{i,\eta}_{\mathrm{LLZ}}$ is based on an understanding of the $\Lambda_{\mathscr{O}}(\widetilde{\Gamma})$-module structure of Berger's Wach module $\mathbb{N}(T_f)$.

As for (2), to deduce from the divisibility in [CLW22] a corresponding divisibility in Conjecture 1.2 for $\mathscr{C}_{\rm LLZ}$, we use certain Beilinson–Flach classes (denoted $\mathcal{BF}_{\rm int}^{j,\mathbf{h}_v,\eta}$ in the body of the text)

$$\mathcal{BF}^{j,\eta} \in \mathrm{H}^1_{\rm Iw}(K_\infty, T_f(\eta)), \quad j \in \{1,2\} \tag{1.7}$$

attached to the base-change of f to an auxiliary imaginary quadratic field $K/\mathbf{Q}$ in which $p = v\overline{v}$ splits, where K_∞ denotes the $\mathbf{Z}_p^2$-extension of K. The construction of (1.7) is deduced from the works of Kings–Loeffler–Zerbes [KLZ17], Loeffler–Zerbes [LZ16], and Büyükboduk–Lei [BL21], together with the study in [BSTW24] of lattices in the big Galois representations attached to a certain Hida family with CM by K. Building on the explicit reciprocity laws satisfied by these classes, for $i \in \{1,2\}$ and $w \in \{v, \overline{v}\}$ we then construct two-variable signed Coleman and logarithm maps

$$\mathscr{C}_{i,w}^{\eta} : \prod_{\tilde{w}|w} \mathrm{H}^1_{\rm Iw}(K_{\infty,\tilde{w}}, T(\eta)) \longrightarrow \Lambda_{\mathscr{O}}(\Gamma_K), \qquad \mathscr{L}_{i,w}^{\eta} : \ker(\mathscr{C}_{i,w}) \longrightarrow \Lambda_{\mathscr{O}^{\rm ur}}(\Gamma_K), \tag{1.8}$$

where $\mathscr{O}^{\rm ur} = \mathscr{O}\hat{\otimes}_{\mathbf{Z}_p} W(\overline{\mathbf{F}_p})$, connecting:

- $\mathrm{res}_v(\mathcal{BF}^{i,\eta})$ (upon restriction to the cyclotomic $\mathbf{Z}_p$-extension of K) to the product
$$\mathscr{C}^{i,\eta}_{\rm LLZ}(\mathrm{res}_p(\mathbf{z}_f^{\mathrm{int},\eta})) \cdot \mathscr{C}^{i,\eta}_{\rm LLZ}(\mathrm{res}_p(\mathbf{z}_{f\otimes\epsilon_K}^{\mathrm{int},\eta})),$$
for $f \otimes \epsilon_K$ the twist of f by the quadratic character attached to K; and
- $\mathrm{res}_{\overline{v}}(\mathcal{BF}^{j,\eta})$ for $j \neq i$ to the p-adic L-function
$$\mathcal{L}_{\overline{v}}(f/K)^\eta,$$

respectively. From here, the desired divisibility transfer from [CLW22] follows easily by global duality. Together with Kato's opposite divisibility, we thus arrive at the proof of Theorem A.

1.3. **Applications.** As a consequence of Theorem A, we deduce cases of the p-part of the Tamagawa Number Conjecture [BK90, BF01] for modular forms in analytic rank zero.

Theorem B. *Let $f \in S_k(\Gamma_0(N))$ be a newform and $p > 2$ be a prime of good nonordinary reduction. Suppose*

- $2 \leq k < p$;
- *N is squarefree;*
- *f is p-*regular*, i.e. $\alpha \neq \beta$;*
- *either* $\begin{cases} k = 2, \text{or} \\ \text{Condition (rtn) holds.} \end{cases}$

If $L(f, k/2) \neq 0$ then the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, T_f^\vee(1-k/2))$ is finite, and

$$\left[\mathscr{O} \colon \frac{L(f,k/2)}{(-2\pi i)^{k/2}\cdot \Omega^{\pm}_{\omega_f}}\right] = \#\mathrm{Sel}(\mathbf{Q}, T_f^\vee(1-k/2)) \cdot \prod_{\ell | N} \mathrm{Tam}_\ell(T_f(k/2)).$$

where $\pm = (-1)^{k/2-1}$, $\Omega^{\pm}_{\omega_f} \in \mathbf{C}^\times$ is the period of Definition 2.11, and

$$\mathrm{Tam}_\ell(T_f(k/2)) = \left[\mathrm{H}^1_f(\mathbf{Q}_\ell, T_f(k/2)) : \mathrm{H}^1_{\rm unr}(\mathbf{Q}_\ell, T_f(k/2))\right]$$

is the Tamagawa factor of $T_f(k/2)$ at ℓ.

In particular, specialised to the case where f is of weight $k = 2$ with rational Fourier coefficients, Theorem B recovers the main result of [BSTW24] on the p-part of the Birch–Swinnerton-Dyer formula in analytic rank zero, and extends it to the case $a_p \neq 0$:

Corollary C. *Let $E/\mathbf{Q}$ be a semistable elliptic curve of conductor N, and $p>2$ be a prime of good nonordinary reduction for E. If $L(E,1)\neq 0$ then $E(\mathbf{Q})$ and $\text{Ш}(E/\mathbf{Q})[p^\infty]$ are both finite, and*

$$\mathrm{ord}_p\biggl(\frac{L(E,1)}{\Omega_E}\biggr)=\mathrm{ord}_p\biggl(\#\text{Ш}(E/\mathbf{Q})[p^\infty]\cdot\prod_{\ell\mid N}c_\ell(E)\biggr),$$

where $\Omega_E\in\mathbb{R}_{>0}$ is the positive Néron period. Hence the p-part of the Birch–Swinnerton-Dyer formula holds for E.

Remark 1.3. An earlier result towards the Birch–Swinnerton-Dyer conjecture for elliptic curves $E/\mathbf{Q}$ at nonordinary primes $p>2$ with $a_p\neq 0$ was obtained in [IS24, Thm. 5.3] conditional on the existence of signed Beilinson–Flach classes and reciprocity laws relating them to certain p-adic L-functions (see Conjecture 3.33 in *op. cit.*) akin to some of the results in §3 below.

Remark 1.4. It should be possible to deduce from Theorem B an extension of Corollary C to abelian varieties $A/\mathbf{Q}$ of GL_2-type (cf. [CÇSS18, Thm. C]).

We also note that by a method introduced by Wei Zhang in the ordinary case [Zha14], Theorem B also has applications to variants of Kolyvagin's conjecture [Kol91] for nonordinary primes and higher weight modular forms as in the work of N. Sweeting [Swe21] and E. Da Ronche [DR25]; and together with Theorem A, it should have applications to higher weight analogues of Kurihara's nonvanishing conjectures [Kur14] and their refinements, as studied in [BCGS26, Kim26, Kim25, CS26].

1.4. **Relation to prior work.** In [Wan14b], the third-named author outlined a proof of Kobayashi's signed main conjectures for elliptic curves $E/\mathbf{Q}$ at supersingular primes $p>2$ with $a_p=0$. At the time, the approach was based on a combination of the following three ingredients:

(1) The "upper bound" divisibility in the signed main conjectures proved by Kobayashi [Kob03];
(2) The "lower bound" divisibility in an Iwasawa–Greenberg main conjecture for certain Rankin–Selberg convolutions obtained in the companion paper [Wan14a] via Eisenstein congruences on $\mathrm{GU}(3,1)$;
(3) The construction of "signed Beilinson–Flach elements" attached to E and an auxiliary imaginary quadratic field K, deduced from the work of Kings–Loeffler–Zerbes [KLZ17] and Loeffler–Zerbes [LZ16] and their explicit reciprocity laws, allowing to transfer the divisibility in (2) to a corresponding divisibility in (1).

Since then, the approach to the lower bound divisibility in [Wan14a] (in particular, the foundational Hida theory for semi-ordinary forms) has been developed in [CLW22], while the construction of signed Beilinson–Flach elements for rational elliptic curves with $a_p=0$ has been established in [BSTW24] building on related work of Büyükboduk–Lei [BL21] and the detailed analysis of the geometry of the Coleman–Mazur eigencurve at certain weight one points by Betina–Dimitrov–Pozzi [BDP22].

Kobayashi's formulation of Iwasawa main conjectures in the supersingular setting, and his proof of an upper bound divisibility building on Kato's work, has been extended to other settings, including:

(1) Rational elliptic curves with good supersingular reduction at $p>2$ (allowing nonzero a_p for $p=3$) in work of Sprung [Spr12];
(2) Elliptic newforms $f\in S_k(\Gamma_0(N))$ of weight $k\geq 2$ with $a_p=0$ in work of Lei [Lei11];
(3) Elliptic newforms $f\in S_k(\Gamma_0(N))$ of weight $k\geq 2$ with nonordinary reduction at $p>2$ (with a_p not necessarily zero) in work Lei–Loeffler–Zerbes [LLZ10].

As mentioned above, to arrive at Theorem A we prove one of the signed main conjectures of Lei–Loeffler–Zerbes [LLZ10]; since these are known to specialise to conjectures equivalent to Lei's in the case $a_p=0$, and to Kobayashi's and Sprung's in the case of rational elliptic curves, Theorem A also yields a proof of the result anticipated in [Wan14b, Thm. 1.3][1] (not intended for publication) and its corresponding extension to the settings (1)–(3) above. Note however that the approach outlined in

[1]See also [BSTW24] for a recent detailed proof in this case building on [CLW22].

op. cit. (and extended by Sprung [IS24] to setting of rational elliptic curves at supersingular primes $p > 2$ as in [Spr12]) differs markedly from the approach carried out in this paper. In particular, our construction of the signed logarithm maps (1.8) and the proof of the associated explicit reciprocity law connecting $\mathrm{res}_{\overline{v}}(\mathcal{BF}^{j,\eta})$ to $\mathcal{L}_{\overline{v}}(f/K)^{\eta}$ is new (cf. §§3.5 and 5.4).

Finally, we note that it should be possible to adapt the methods in this paper to give a new proof of the main result of [SU14] towards the two-variable Iwasawa main conjecture for modular forms in the p-ordinary case (cf. [CW22, FW21]).

Remark 1.5. In [Wan16] (incorporated as part of [FW21]), the third-named author outlined a different proof of Theorem 1.1. Contrary to the approach via signed Iwasawa theory carried out in this paper, the approach in *op. cit.* directly builds on the *unbounded* Beilinson–Flach classes of Loeffler–Zerbes [LZ16] and a so-called "unramified Iwasawa theory" along an appropriately chosen line in a 2-variable p-adic deformation of T_f.

1.5. **Remarks on the hypotheses.** We give some comments on the need for the hypotheses in our Theorem A (and its applications).

(1) As already noted, it should be possible to extend Theorem A to good ordinary primes $p > 2$ by similar methods. (In some sense, this case is simpler, since there is no 'signed decomposition' to be taken, and one can directly use the explicit reciprocity laws from [KLZ17].)
(2) The weight restriction $k < p$ arises from our use of integral p-adic Hodge theory, in particular, the choice of a good basis of the Wach module $\mathbb{N}(T_f)$ using Fontaine–Laffaille theory. This condition (together with p-nonordinarity) implies irreducibility of the residual representation $\bar{\rho}_f|_{G_{\mathbf{Q}_p}}$, a condition that should be added in general. It might be possible to go beyond the Fontaine–Laffaille range by building on very recent advances in integral p-adic Hodge theory[2], a direction that we intend to explore in future work.
(3) The hypothesis that N be squarefree is carried over from [CLW22], and it should be removable with some more work.
(4) The p-regularity hypothesis on f is carried over from [LZ16], but it is also used in some of our arguments here. It might be possible to remove it building on a different construction of p-adic L-functions interpolating Rankin–Selberg L-values, as explained in [FW21, §4.7].
(5) The restriction to the $\eta = \omega^{k/2}$-isotypic component for the action of Δ arises from our use of anticyclotomic p-adic L-functions interpolating (twists of) the central L-values $L(f/K, k/2)$, and seems essential to our method.
(6) Finally, the last hypothesis in Theorem A could be significantly relaxed by a suitable application of the anticyclotomic $\mu = 0$ results of [CH18b] in higher weight (cf. §4.3).

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[2]We are grateful to Matthias Flach for informative remarks concerning this point.

2. Preliminaries

The goal of this section is to give a precise formulation of Conjectures 1.1 and 1.2 in our setting and explain their equivalence. Throughout we fix a prime $p > 2$, and complex and p-adic embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$.

2.1. **Modular forms.** Let $f = \sum_{n=1}^\infty a_n q^n \in S_k(\Gamma_0(N))$ be a newform of weight $k \geq 2$, level N with $p \nmid N$, and trivial nebentypus. Throughout we assume that

$$\text{(FL)} \qquad k < p.$$

Let $F = \mathbf{Q}(\{a_n\}_n)$ be the Hecke field of f, viewed inside $\overline{\mathbf{Q}}$ via ι_∞, and let L denote the completion of the image of F under ι_p. Let $\mathscr{O}$ be the ring of integers of L, and assume that f is *nonordinary* at p, meaning that

$$|\iota_p(a_p)| < 1.$$

Let V_f be the L-linear Galois representation associated to f by Deligne. We adopt the convention that the p-adic cyclotomic character $\varepsilon : G_{\mathbf{Q}_p} \to \mathbf{Z}_p^\times$ has Hodge–Tate weight $+1$, so the Hodge–Tate weights of $V_f|_{G_{\mathbf{Q}_p}}$ are 0 and $1-k$. Let T_f denote the canonical $G_{\mathbf{Q}}$-stable $\mathscr{O}$-lattice in V_f defined by Kato in [Kat04, §8.3].

2.2. **Iwasawa algebras.** Let $\widetilde{\Gamma}$ be the Galois group of the cyclotomic $\mathbf{Z}_p^\times$-extension $\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}$; this decomposes as a direct product

$$\widetilde{\Gamma} \cong \Delta \times \Gamma,$$

where Γ denotes the Galois group of the cyclotomic $\mathbf{Z}_p$-extension $\mathbf{Q}_\infty/\mathbf{Q}$, and $\Delta = \mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ is cyclic of order $p-1$. Put

$$\Lambda_{\mathscr{O}}(\widetilde{\Gamma}) := \mathscr{O}[\![\widetilde{\Gamma}]\!] \cong \mathscr{O}[\Delta][\![\Gamma]\!], \qquad \Lambda_{\mathscr{O}}(\Gamma) := \mathscr{O}[\![\Gamma]\!]$$

for the cyclotomic Iwasawa algebras. Upon the choice of a topological generator $\gamma \in \Gamma$, we shall often identify Λ with the formal power series ring $\mathscr{O}[\![X]\!]$ via $X = \gamma - 1$.

Let $\mathcal{H}_L \subset L[\Delta][\![X]\!]$ be the subring of power series convergent on the p-adic unit disk $|X| < 1$ and put

$$\mathcal{H}_L(\Gamma) = \{h(\gamma - 1) \mid h \in \mathcal{H}_L\};$$

this is identified with the ring of locally analytic L-valued distributions on Γ, and it naturally contains $\Lambda_{\mathscr{O}}(\Gamma)$ as the subring of $\mathscr{O}$-valued measures on Γ. Put also

$$\Lambda_L(\Gamma) := L \otimes_{\mathscr{O}} \Lambda_{\mathscr{O}}(\Gamma),$$

which is similarly identified with the subring of bounded elements in $\mathcal{H}_L(\Gamma)$, and

$$\Lambda_L(\widetilde{\Gamma}) := L[\Delta] \otimes_L \Lambda_L(\Gamma), \qquad \mathcal{H}_L(\widetilde{\Gamma}) := L[\Delta] \otimes_L \mathcal{H}_L(\Gamma).$$

2.3. **Signed Coleman maps.** In this section we recall the definition of signed Coleman maps originating from the work of Lei–Loeffler–Zerbes [LLZ10], where such maps are constructed from suitable bases of the Wach module of T_f exploiting its $\Lambda_{\mathscr{O}}(\widetilde{\Gamma})$-module structure through the Mellin transform.

2.3.1. *Wach modules.* Put $\mathbb{A}^+_{\mathbf{Q}_p} = \mathscr{O}[\![\pi]\!]$ for the power series ring[3] over $\mathscr{O}$ in the variable π. The ring $\mathbb{A}^+_{\mathbf{Q}_p}$ is equipped with $\mathscr{O}$-linear actions of φ and $\widetilde{\Gamma}$ defined by

$$\varphi(\pi) = (1+\pi)^p - 1, \qquad \sigma(\pi) = (1+\pi)^{\varepsilon(\sigma)} - 1$$

[3] Note that this ring is most often denoted $\mathscr{O} \otimes_{\mathbf{Z}_p} \mathbb{A}^+_{\mathbf{Q}_p}$ with $\mathbb{A}^+_{\mathbf{Q}_p} = \mathbf{Z}_p[\![\pi]\!]$, but since our coefficient ring will be fixed to be $\mathscr{O}$, the extension of scalars will be omitted for the ease of notation.

for $\sigma \in \widetilde{\Gamma}$. Put $\mathbb{B}^+_{\mathbf{Q}_p} = \mathbb{A}^+_{\mathbf{Q}_p}[1/p] \subset L[\![\pi]\!]$, and let $\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p}$ be the subring of $L[\![\pi]\!]$ of formal power series convergent in the p-adic unit disk $|\pi| < 1$. The actions of φ and $\widetilde{\Gamma}$ extend to these larger rings, and φ has a left inverse ψ satisfying

$$(\varphi \circ \psi)(g)(\pi) = \frac{1}{p} \sum_{\zeta \in \mu_p} g(\zeta(1+\pi) - 1).$$

One easily checks that $\psi(1+\pi) = 0$, and that the action of $\widetilde{\Gamma}$ on $1+\pi$ extends to isomorphisms

$$\Lambda_{\mathscr{O}}(\widetilde{\Gamma}) \cong (\mathbb{A}^+_{\mathbf{Q}_p})^{\psi=0}, \qquad \Lambda_L(\widetilde{\Gamma}) \cong (\mathbb{B}^+_{\mathbf{Q}_p})^{\psi=0}, \qquad \mathcal{H}_L(\widetilde{\Gamma}) \cong (\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0} \tag{2.1}$$

of $\Lambda_{\mathscr{O}}(\widetilde{\Gamma})$-, $\Lambda_L(\widetilde{\Gamma})$-, and $\mathcal{H}_L(\widetilde{\Gamma})$-modules, respectively, all of which will be denoted by $\mathfrak{M}$ and referred to as the *Mellin transform*.

Let $\mathbb{N}(T_f) \subset \mathbb{N}(V_f)$ be the Wach modules attached to the $G_{\mathbf{Q}_p}$-representations $T_f \subset V_f$ by [Ber04]. These are free rank 2 modules over $\mathbb{A}^+_{\mathbf{Q}_p}$ and $\mathbb{B}^+_{\mathbf{Q}_p}$, respectively, with compatible actions of φ and $\widetilde{\Gamma}$. By Corollary III.4.5 in *op. cit.* there is an isomorphism of filtered φ-modules

$$\mathbb{N}(V_f)/\pi\mathbb{N}(V_f) \cong \mathbb{D}_{\mathrm{cris}}(V_f), \tag{2.2}$$

and the image of $\mathbb{N}(T_f)$ under this isomorphism defines an $\mathscr{O}$-lattice $\mathbb{D}_{\mathrm{cris}}(T_f) \subset \mathbb{D}_{\mathrm{cris}}(V_f)$.

Lemma 2.1. *There exists an $\mathscr{O}$-basis (v_1, v_2) of $\mathbb{D}_{\mathrm{cris}}(T_f(k-1))$ with*

(i) *v_1 an $\mathscr{O}$-module generaotor of $\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(T_f(k-1)) \subset \mathbb{D}_{\mathrm{cris}}(T_f(k-1))$;*
(ii) *$v_2 = \varphi(v_1)$.*

In particular, the matrix A_φ of φ with respect to such basis (v_1, v_2) is given by

$$A_\varphi = \begin{pmatrix} 0 & -1/p^{k-1} \\ 1 & a_p/p^{k-1} \end{pmatrix}.$$

Proof. The existence of bases (v_1, v_2) as claimed under the Fontaine–Laffaille condition (FL) is shown in [LLZ17, Lem. 3.1]. Note that the proof uses the fact that $\mathbb{D}_{\mathrm{cris}}(T_f(k-1))$ is then a *strongly divisible* lattice of $\mathbb{D}_{\mathrm{cris}}(V_f(k-1))$ in the sense of [FL82, §7.7]. The form of A_φ then follows from the relation $\varphi^2 = a_p p^{1-k}\varphi - p^{1-k}$. ☐

Note that by definition we have an isomorphism

$$\mathbb{N}(T_f(k-1))/\pi\mathbb{N}(T_f(k-1)) \cong \mathbb{D}_{\mathrm{cris}}(T_f(k-1)) \tag{2.3}$$

coming from (2.2).

Lemma 2.2. *There exists an $\mathscr{O}$-basis (v_1, v_2) of $\mathbb{D}_{\mathrm{cris}}(T_f(k-1))$ as in Lemma 2.1 and an $\mathbb{A}^+_{\mathbf{Q}_p}$-basis (n_1, n_2) of $\mathbb{N}(T_f(k-1))$ lifting (v_1, v_2) under (2.3) and satisfying the following conditions:*

(i) *The matrix P_φ of φ acting on $\mathbb{N}(T_f(k-1))$ with respect to the basis (n_1, n_2) satisfies*

$$P_\varphi \equiv P_0 \pmod{\pi}, \quad \textit{where} \quad P_0 = \begin{pmatrix} 0 & -1/q^{k-1} \\ \delta^{k-1} & a_p/q^{k-1} \end{pmatrix}$$

with $q = \varphi(\pi)/\pi$ and $\delta = p/(q - \pi^{p-1})$;
(ii) *$((1+\pi)\varphi(n_1), (1+\pi)\varphi(n_2))$ is a $\Lambda_{\mathscr{O}}(\widetilde{\Gamma})$-basis of $(\varphi^*\mathbb{N}(T_f(k-1)))^{\psi=0}$.*

Proof. In weight $k \geq 3$, this is shown in [LLZ17, Thm. 3.3], in which case one can even take $P_\varphi = P_0$ in part (i). In general, by the proof of [Ber04, Prop. V.2.3] we may choose bases (v_1, v_2) and (n_1, n_2) for which (i) holds with $P_\varphi = P_0$; then by [LLZ10, Thm. 3.5] we may replace (n_1, n_2) by an $\mathbb{A}^+_{\mathbf{Q}_p}$-basis $(n_1', n_2') \equiv (n_1, n_2) \pmod{\pi}$ for which (ii) also holds. ☐

By [Ber04, Prop. III.2.1] we have an inclusion of $\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p}$-modules

$$\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p}\otimes_{\mathbb{A}^+_{\mathbf{Q}_p}}\mathbb{N}(T_f(k-1))\subset\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p}\otimes_{\mathscr{O}}\mathbb{D}_{\mathrm{cris}}(T_f(k-1)). \tag{2.4}$$

In particular, for any bases (v_1,v_2) and (n_1,n_2) as in Lemma 2.2, which we fix from now on, there is a change of basis matrix $M\in M_{2\times2}(\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p})$ such that

$$(n_1\ ,\ n_2)=(v_1\ ,\ v_2)\ M. \tag{2.5}$$

In particular, $M\equiv I_2\pmod{\pi}$, where I_2 denotes the 2×2 identity matrix.

Definition 2.3. The *logarithm matrix* $M_{\log}\in M_{2\times2}(\mathcal{H}_L(\widetilde{\Gamma}))$ associated to the bases (v_1,v_2), (n_1,n_2) is given by

$$M_{\log}:=\mathfrak{M}^{-1}((1+\pi)A_\varphi\cdot\varphi(M)),$$

where M is the change of matrix (2.5) and $\mathfrak{M}$ is the Mellin transform (2.1).

Remark 2.4. One can check that in fact $M_{\log}$ is defined over $\mathcal{H}_L(\Gamma)$ (see [LLZ11, §3.A]).

Remark 2.5. Since $1\mapsto1+\pi$ under the Mellin transform, replacing (n_1,n_2) by another basis (n_1',n_2') of $\mathbb{N}(T_f(k-1))$ with $n_i'\equiv n_i\ (\mathrm{mod}\,\pi)$ leaves the resulting logarithm matrix $M_{\log}$ unchanged.

2.3.2. *Signed Coleman maps.* For any finite dimensional $\mathbf{Q}_p$-vector space V equipped with a continuous action of $G_{\mathbf{Q}_p}$ put

$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),T):=\varprojlim_n\mathrm{H}^1(\mathbf{Q}_p(\mu_{p^n}),T),$$
$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),V):=\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),T)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p,$$

where $T\subset V$ is any $G_{\mathbf{Q}_p}$-stable lattice. Put

$$V=V_f(k-1),\qquad T=T_f(k-1).$$

Note that for any $h\geq1$ the $G_{\mathbf{Q}_p}$-representation $V^*(1):=\mathrm{Hom}_L(V,L(1))$ satisfies $\mathrm{Fil}^{-h}\mathbb{D}_{\mathrm{cris}}(V^*(1))=\mathbb{D}_{\mathrm{cris}}(V^*(1))$ (indeed, in our conventions $V^*(1)$ has Hodge–Tate weights $\{2-k,1\}$). Hence by Perrin-Riou's work [PR94] for any integer $h\geq1$ there is a big exponential map

$$\Omega_{V^*(1),h}:(\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}\otimes_{\mathbf{Q}_p}\mathbb{D}_{\mathrm{cris}}(V^*(1))\longrightarrow\mathcal{H}_L(\widetilde{\Gamma})\otimes_{\Lambda_L(\widetilde{\Gamma})}\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),V^*(1)),$$

where $(\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ is being identified with $\mathcal{H}_L(\widetilde{\Gamma})$ via $\mathfrak{M}$, characterised by an interpolation property described in [*op. cit.*, §3.2.3]. (The construction of $\Omega_{V^*(1),h}$ depends on the choice of a compatible system $(\zeta_{p^n})_{n\geq1}$ of p-power roots of unity, which we fix once and for all and omit from the notation.) Composing with the $\mathcal{H}_L(\widetilde{\Gamma})$-linear extension of the Perrin-Riou pairing

$$\langle\,,\,\rangle_V:\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),V)\times\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),V^*(1))\longrightarrow\Lambda_L(\widetilde{\Gamma})$$

given by $\langle(x_n)_n,(y_n)_n\rangle_V=\varprojlim_n\sum_\tau\langle x_n,y_n^\tau\rangle_{\mathrm{Tate}}\cdot\tau$, where τ runs over the elements in $\mathrm{Gal}(\mathbf{Q}_p(\mu_{p^n})/\mathbf{Q}_p)$ and $\langle\,,\,\rangle_{\mathrm{Tate}}:\mathrm{H}^1(\mathbf{Q}_p(\mu_{p^n}),T)\times\mathrm{H}^1(\mathbf{Q}_p(\mu_{p^n}),T^*(1))\to\mathscr{O}$ is the Tate local duality pairing, for every $\Xi\in(\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}\otimes_{\mathbf{Q}_p}\mathbb{D}_{\mathrm{cris}}(V^*(1))$ we obtain a map

$$\begin{aligned}\mathscr{L}_{V,\Xi}:\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),V)&\longrightarrow\mathcal{H}_L(\widetilde{\Gamma})\\ \mathbf{z}&\longmapsto\langle\mathbf{z},\Omega_{V^*(1),1}(\Xi)\rangle_V.\end{aligned}$$

In particular, letting (v_1^*,v_2^*) denote the basis of $\mathbb{D}_{\mathrm{cris}}(T^*(1))$ dual to (v_1,v_2) under the natural pairing $[\,,\,]_{\mathrm{dR}}:\mathbb{D}_{\mathrm{cris}}(T)\times\mathbb{D}_{\mathrm{cris}}(T^*(1))\to\mathscr{O}$, a key result of Berger [Ber03, Thm. II.13] (see also [LLZ10, Cor. 3.24]) is the description of the resulting Perrin-Riou's *big logarithm map*[4]

$$\mathscr{L}_V:=(v_1\ ,\ v_2)\begin{pmatrix}\mathscr{L}_{V,(1+\pi)\otimes v_1^*}\\ \mathscr{L}_{V,(1+\pi)\otimes v_2^*}\end{pmatrix}:\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),V)\longrightarrow\mathbb{D}_{\mathrm{cris}}(V)\otimes_{\mathbf{Q}_p}\mathcal{H}_L(\widetilde{\Gamma}) \tag{2.6}$$

[4]Or *big p-adic regulator map.*

as the composition

$$\begin{aligned}\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),V) \xrightarrow{\cong} \mathbb{N}(V)^{\psi=1} &\xrightarrow{1-\varphi} (\varphi^*\mathbb{N}(V))^{\psi=0} \\ &\hookrightarrow \mathbb{D}_{\mathrm{cris}}(V)\otimes_{\mathbf{Q}_p}(\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0} \\ &\xrightarrow{\mathfrak{M}^{-1}} \mathbb{D}_{\mathrm{cris}}(V)\otimes_{\mathbf{Q}_p}\mathcal{H}_L(\widetilde{\Gamma}),\end{aligned} \tag{2.7}$$

where the first arrow is given by Fontaine's isomorphism (see [CC99, §II.1] for a reference), as refined by Berger [Ber03] in terms of Wach modules (see also [LLZ17, Thm. 2.6]), and the second arrow is the composite embedding

$$(\varphi^*\mathbb{N}(V))^{\psi=0} \xrightarrow{\mathrm{id}\otimes 1} (\varphi^*\mathbb{N}(V))^{\psi=0}\otimes_{\Lambda_L(\widetilde{\Gamma})}\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p} \cong (\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0} \subset \mathbb{D}_{\mathrm{cris}}(V)\otimes_{\mathbf{Q}_p}(\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$$

explained in [LLZ11, Prop. 2.11], where $\varphi^*\mathbb{N}_{\mathrm{rig}}(V)$ denotes the $\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p}$-span of $\varphi(\mathbb{N}_{\mathrm{rig}}(V))$ for

$$\mathbb{N}_{\mathrm{rig}}(V) := \mathbb{N}(V)\otimes_{\mathbb{B}^+_{\mathbf{Q}_p}}\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p}.$$

The first two maps in (2.7) admit integral versions, with T in place of V. As introduced in [LLZ10], this leads to the following decomposition of $\mathscr{L}_V$ is terms of *signed Coleman maps* $\widetilde{\mathscr{C}_i}$ in the spirit of Kobayashi [Kob03].

Proposition 2.6. *Let* (n_1,n_2) *be a* $\mathbb{A}^+_{\mathbf{Q}_p}$*-basis of* $\mathbb{N}(T)$ *as in Lemma 2.2, and for* $i\in\{1,2\}$ *define*

$$\widetilde{\mathscr{C}_i} : \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),T) \longrightarrow \Lambda_{\mathscr{O}}(\widetilde{\Gamma})$$

by the composition

$$\widetilde{\mathscr{C}_i} : \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),T) \xrightarrow{\cong} \mathbb{N}(T)^{\psi=1} \xrightarrow{1-\varphi} (\varphi^*\mathbb{N}(T))^{\psi=0} \xrightarrow{\mathrm{pr}_i\circ\mathfrak{J}} \Lambda_{\mathscr{O}}(\widetilde{\Gamma}),$$

where the last arrow takes the i*-th coordinate with respect to the basis* $((1+\pi)\varphi(n_1),(1+\pi)\varphi(n_2))$ *of* $(\varphi^*\mathbb{N}(T))^{\psi=0}$*. Then*

$$\mathscr{L}_V = (v_1\ ,\ v_2)\cdot M_{\log}\cdot\begin{pmatrix}\widetilde{\mathscr{C}_1}\\ \widetilde{\mathscr{C}_2}\end{pmatrix}$$

where $v_i = n_i \bmod \pi \in \mathbb{D}_{\mathrm{cris}}(T)$*.*

Proof. This is clear from the description of $\mathscr{L}_V$ in (2.7) and the definition of $M_{\log}$. □

Remark 2.7. Let α and β be the roots of $x^2 - a_px + p^{k-1}$, where a_p is the p-th Fourier coefficient of $f\in S_k(\Gamma_0(N))$, and suppose the p-regularity hypothesis $\alpha\neq\beta$. (As recalled in the Introduction, this is known to hold for $k=2$ by [CE98].) Then

$$v_\alpha := \frac{1}{\alpha-\beta}(\alpha v_1 - p^{k-1}v_2), \qquad v_\beta := \frac{1}{\alpha-\beta}(-\beta v_1 + p^{k-1}v_2)$$

are φ-eigenvectors in $\mathbb{D}_{\mathrm{cris}}(T)$ with eigenvalues $1/\alpha$ and $1/\beta$, respectively, and for $\xi\in\{\alpha,\beta\}$ letting

$$\mathscr{L}_{V,(1+\pi)\otimes v_\xi^*} : \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),V) \longrightarrow \mathcal{H}_L(\widetilde{\Gamma})$$

denote the coordinate of $\mathscr{L}_V$ with respect to v_ξ, so (v_α^*,v_β^*) is the basis of $\mathbb{D}_{\mathrm{cris}}(V^*(1))$ dual to (v_α,v_β), the decomposition in Proposition 2.6 can be rewritten as

$$\begin{pmatrix}\mathscr{L}_{V,(1+\pi)\otimes v_\alpha^*}\\ \mathscr{L}_{V,(1+\pi)\otimes v_\beta^*}\end{pmatrix} = Q^{-1}_{\alpha,\beta}M_{\log}\cdot\begin{pmatrix}\widetilde{\mathscr{C}_1}\\ \widetilde{\mathscr{C}_2}\end{pmatrix}, \tag{2.8}$$

where $Q_{\alpha,\beta} = \frac{1}{\alpha-\beta}\left(\begin{smallmatrix}\alpha & -\beta\\ -p^{k-1} & p^{k-1}\end{smallmatrix}\right)$.

2.3.3. *Images of the signed Coleman maps.* For every character $\eta:\Delta\to\mathbf{Z}_p^\times$, applying the projector $e_\eta=\frac{1}{p-1}\sum_{\delta\in\Delta}\eta^{-1}(\sigma)\sigma\in\mathbf{Z}_p[\Delta]$ to the η-isotypic component we get $\Lambda_{\mathscr{O}}(\Gamma)$-module maps

$$e_\eta\widetilde{\mathscr{C}_i}:\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),T)^\eta\longrightarrow\Lambda_{\mathscr{O}}(\widetilde{\Gamma})^\eta\cong\Lambda_{\mathscr{O}}(\Gamma).\tag{2.9}$$

For $m\geq 1$, put

$$\delta_m:=\prod_{j=0}^{m-1}(u^{-j}\gamma-1)\in\Lambda_{\mathscr{O}}(\Gamma),$$

where $\gamma\in\Gamma$ and $u\in 1+p\mathbf{Z}_p$ are topological generators.

Proposition 2.8. *Let $i\in\{1,2\}$. For every character $\eta:\Delta\to\mathbf{Z}_p^\times$, there exists a factor $\xi_{\eta,i}$ of δ_{k-1} such that*

$$\mathrm{Image}(e_\eta\widetilde{\mathscr{C}_i})\subset\xi_{\eta,i}\Lambda_{\mathscr{O}}(\Gamma)$$

with finite index.

Proof. See Corollary 5.3 and Theorem 5.10 in [LLZ11]. □

Remark 2.9. As observed in [BL21, Rem. 2.15], we have $\xi_{\eta,1}=1$ for all η, while $\xi_{\eta,2}\neq 1$ unless $k=2$ and $\eta=\mathbb{1}$ is the trivial character of Δ.

2.4. **Kato's Euler system.** Let $V_F(f)$ and $S(f)$ be the subspaces of de Rham and Betti cohomology attached to the eigenform $f\in S_k(\Gamma_0(N))$ as in [Kat04, §6.3]; these are F-vector spaces of dimension 1 and 2, respectively. For any commutative ring extension A of F, we put $V_A(f)=V_F(f)\otimes_F A$ and $S_A(f)=S(f)\otimes_F A$. Then one has canonical isomorphisms

$$V_L(f)\cong V_f,\qquad S_L(f)\cong\mathrm{Fil}^1\mathbb{D}_{\mathrm{cris}}(V_f),\tag{2.10}$$

where we recall that $L=F_{\mathfrak{P}}$ is the completion of F at the prime $\mathfrak{P}$ above p induced by ι_p.

Let $V_{\mathbf{C}}(f)^\pm$ denote the $\pm$-eigenspace of $V_{\mathbf{C}}(f)$ under the action of complex conjugation. For every $\gamma\in V_F(f)$ with $\gamma^\pm\neq 0\in V_{\mathbf{C}}(f)^\pm$ and an F-basis $\omega\in S(f)$, define $\Omega^\pm_{\gamma,\omega}\in\mathbf{C}^\times$ by

$$\mathrm{per}_f(\omega)=\Omega^+_{\gamma,\omega}\cdot\gamma^++\Omega^-_{\gamma,\omega}\cdot\gamma^-,$$

where $\mathrm{per}_f:S(f)\to V_{\mathbf{C}}(f)$ is the period map of [Kat04, §6.3].

2.4.1. *Kato's explicit reciprocity law.* Let $q\in\mathbf{Z}$. Following [Kat04, §12.2], for V a finite dimensional $\mathbf{Q}_p$-vector space equipped with a continuous $G_{\mathbf{Q}}$-action unramified outside a finite set of primes and $T\subset V$ any $G_{\mathbf{Q}}$-stable lattice, we put

$$\mathrm{H}^q_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),T):=\varprojlim_n\mathrm{H}^q(\mathbf{Z}[\mu_{p^n},1/p],j_*T),$$
$$\mathrm{H}^q_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),V):=\mathrm{H}^q_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),T)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p,$$

where $j:\mathrm{Spec}(\mathbf{Q}(\mu_{p^n}))\to\mathrm{Spec}(\mathbf{Z}[\mu_{p^n},1/p])$ is the natural map. These are zero for $q\neq 1,2$, and the latter is independent of the choice of T.

By [Kat04, Thm. 12.5], there is an L-linear map

$$V_f\longrightarrow\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),V_f):\ \gamma\longmapsto\mathbf{z}_\gamma^{(p)}$$

with the property that for $0\leq j\leq k-2$, if $\gamma\in V_F(f)\subset V_f$ under the identification (2.10), then the image of $\mathbf{z}_\gamma^{(p)}$ by the composition

$$\begin{aligned}\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),V_f)\xrightarrow{\otimes((\zeta_{p^n})_{n\geq0})^{\otimes(k-1-j)}}\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),V(-j))&\xrightarrow{\mathrm{pr}_{\mathbf{Q}}}\mathrm{H}^1(\mathbf{Q},V(-j))\\&\xrightarrow{\mathrm{res}_p}\mathrm{H}^1(\mathbf{Q}_p,V(-j))\\&\xrightarrow{\exp^*}\mathrm{Fil}^0\mathbb{D}_{\mathrm{cris}}(V_f^*(-j))=\mathrm{Fil}^1\mathbb{D}_{\mathrm{cris}}(V_f)\end{aligned}$$

is contained in $S(f) \subset \mathrm{Fil}^1\mathbb{D}_{\mathrm{cirs}}(V_f)$ under the identification (2.10), and satisfies

$$\mathrm{per}_f^{\pm}(\text{image of } \mathbf{z}_{\gamma}^{(p)}) = \frac{L_{\{p\}}(f, j+1)}{(-2\pi i)^{j+1}} \cdot \gamma^{\pm}, \tag{2.11}$$

where $L_{\{p\}}(f,s)$ is the L-function on f with the Euler factor at p removed, the sign is $\pm = (-1)^j$, and $\mathrm{per}_f^{\pm} : S(f) \to V_{\mathbf{C}}(f)^{\pm}$ is the composition of per_f with the projection to the $\pm$-eigenspace.

2.4.2. *Integral normalisations.* Let $\mathcal{O} = \mathcal{O}_{F,\mathfrak{P}}$ denote the localization of the ring of integers of F at the prime $\mathfrak{P}$ above p determined by ι_p.

Taking cohomology with coefficients in $\mathcal{O}$ in the definition of $V_F(f)$ defines a lattice $T_{\mathcal{O}}(f) \subset V_F(f)$ contained in T_f. In the following we put

$$T_{\mathcal{O}}(f)^{\pm} = T_{\mathcal{O}}(f) \cap V_{\mathbf{C}}(f)^{\pm}$$

and fix $\mathcal{O}$-module generators $\gamma_f^{\pm} \in T_{\mathcal{O}}(f)^{\pm}$.

On the other hand, adapting the notation in [KLZ17, §2.8] let $M_{\mathcal{O}}(f)$ denote the maximal subspace of

$$\mathrm{H}^1_c(Y_1(N)(\mathbf{C}), \mathrm{Sym}^{k-2}\mathscr{H}_{\mathbf{Z}}^{\vee}) \otimes_{\mathbf{Z}} \mathcal{O}$$

on which the Hecke operators T_n act as multiplication by a_n, and put

$$M_{\mathcal{O}}(f)^{\pm} := M_{\mathcal{O}}(f) \cap \left(\mathrm{H}^1_c(Y_1(N)(\mathbf{C}), \mathrm{Sym}^{k-2}\mathscr{H}_{\mathbf{Z}}^{\vee}) \otimes_{\mathbf{Z}} \mathbf{C}\right)^{\pm},$$

and fix $\mathcal{O}$-module generators $\delta_f^{\pm} \in M_{\mathcal{O}}(f)^{\pm}$.

From the definitions, we have natural inclusions $M_{\mathcal{O}}(f)^{\pm} \hookrightarrow T_{\mathcal{O}}(f)^{\pm}$ which become isomorphisms after extending scalars to F.

Lemma 2.10. *We can write*

$$\delta_f^{\pm} = c^{\pm} \cdot \gamma_f^{\pm}$$

with $c^{\pm} = c_f \in \mathcal{O}$ a generator of the congruence ideal for f.

Proof. Let $\kappa_L = \mathscr{O}/\mathfrak{m}_L$ be the residue field of $\mathscr{O}$. By a result of Fontaine (see e.g. [Edi92, Thm. 2.6]), the residual representation $\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa_L)$ attached to f is irreducible (even after restriction to $G_{\mathbf{Q}_p}$), and so the localisation of the Hecke algebra at the corresponding maximal ideal is known to be Gorenstein (see [FJ95, Thm. 2.1]). Thus the argument in [BSTW24, Lem. 2.5] applies. □

Following [Mak26, §2.1] we consider the following p-normalisation of periods.

Definition 2.11. For any nonzero $\omega \in S(f)$, the *p-normalised periods* of ω are the complex numbers $\Omega_{\omega}^{\pm} \in \mathbf{C}^{\times}$ defined by

$$\mathrm{per}_f(\omega) = \Omega_{\omega}^{+} \cdot \delta_{\mathcal{O}}^{+} + \Omega_{\omega}^{-} \cdot \delta_{\mathcal{O}}^{-}.$$

In other words, $\Omega_{\omega}^{\pm} = \Omega_{\delta_{\mathcal{O}},\omega}^{\pm}$ where $\delta_{\mathcal{O}} := \delta_{\mathcal{O}}^{+} + \delta_{\mathcal{O}}^{-}$.

On the other hand, we consider the following p-integral normalisation of Kato's class:

$$\mathbf{z}_f^{\mathrm{int}} := \mathbf{z}_{\gamma_{\mathcal{O}}}^{(p)} \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^{\infty}}), V_f),$$

where $\gamma_{\mathcal{O}} := \gamma_{\mathcal{O}}^{+} + \gamma_{\mathcal{O}}^{-} \in T_{\mathcal{O}}(f) \subset T_f$.

Proposition 2.12. *Then class $\mathbf{z}_f^{\mathrm{int}}$ is contained in $\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^{\infty}}), T_f) \subset \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^{\infty}}), V_f)$.*

Proof. As noted in the proof of Lemma 2.10, under our running assumptions the residual representation $\bar{\rho}_f$ is irreducible, so the result thus follows from (the proof of) [Kat04, Thm. 12.5(4)]. □

Definition 2.13. For $\xi \in \{\alpha, \beta\}$ and $i \in \{1, 2\}$ put

$$L_\xi(f) := \mathscr{L}_{V,(1+\pi)\otimes v_\xi^*}(\mathrm{res}_p(\mathbf{z}^{\mathrm{int}}_{f,k-1})), \qquad L_i(f) := \widetilde{\mathscr{C}}_i(\mathrm{res}_p(\mathbf{z}^{\mathrm{int}}_{f,k-1})),$$

where $\mathbf{z}^{\mathrm{int}}_{f,k-1}$ denotes the image of $\mathbf{z}^{\mathrm{int}}_f$ under the twisting map

$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V_f) \xrightarrow{\otimes((\zeta_{p^n})_{n\geq 0})^{\otimes(k-1)}} \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V).$$

Note that these are elements in $\mathcal{H}_L(\widetilde{\Gamma})$ and $\Lambda_{\mathscr{O}}(\widetilde{\Gamma})$, respectively.

2.4.3. *Interpolation formulas.* Fix a topological generator $\gamma \in \Gamma$, and for ζ a primitive p^t-th root of unity, let $\psi_\zeta : G_{\mathbf{Q}} \twoheadrightarrow \Gamma \to \overline{\mathbf{Q}}_p^\times$ be the character determined by $\psi_\zeta(\gamma) = \zeta$. By abuse of notation, we also let ψ_ζ the resulting Dirichlet character $(\mathbf{Z}/p^{t+1}\mathbf{Z})^\times \to \overline{\mathbf{Q}}_p^\times$. Let

$$\varepsilon : \widetilde{\Gamma} \longrightarrow \mathbf{Z}_p^\times, \qquad \langle\varepsilon\rangle = \varepsilon\omega^{-1} : \Gamma \longrightarrow 1 + p\mathbf{Z}_p,$$

be the p-adic cyclotomic character, where $\omega : \Delta \to \mathbf{Z}_p^\times$ is the mod p cyclotomic character.

Definition 2.14. For every character $\eta : \Delta \to \mathbf{Z}_p^\times$, let

$$L_\xi(f)^\eta \in \mathcal{H}_L(\Gamma), \qquad (\text{resp. } L_i(f)^\eta \in \Lambda_{\mathscr{O}}(\Gamma))$$

denote the images of $L_\xi(f)$ and $L_i(f)$ under the projections$\mathcal{H}_L(\widetilde{\Gamma}) \to \mathcal{H}_L(\Gamma)$ (resp. $\Lambda_{\mathscr{O}}(\widetilde{\Gamma}) \to \Lambda_{\mathscr{O}}(\Gamma)$) defined by e_η.

As in [Kat04, §14.22], the image of the differential $\omega_f \in S(f)$ attached to f (see e.g. [KLZ20, §6.1]) under the Faltings–Tsuji comparison isomorphism (2.10) gives an $\mathscr{O}$-basis of $\mathrm{Fil}^1\mathbb{D}_{\mathrm{cris}}(T_f)$. Thus in the following we shall assume that the basis (v_1, v_2) from Lemma 2.2 is such that $v_1 = \omega_f$.

Theorem 2.15. *Let $\xi \in \{\alpha, \beta\}$ and $\eta : \Delta \to \mathbf{Z}_p^\times$ be a character. Then $L_\xi(f)^\eta$ satisfies the following interpolation property: For all characters $\chi = \langle\varepsilon\rangle^j\psi_\zeta$ of Γ with $0 \leq j \leq k-2$ and ζ a primitive p^t-th root of unity for some $t \geq 0$, we have*

$$\phi_\chi(L_\xi(f)^\eta) = e_{p,\xi,\eta}(f,\chi) \cdot \frac{j! \cdot L(f \otimes \psi_\zeta^{-1}\eta\omega^j, j+1)}{(-2\pi i)^{j+1} \cdot \mathfrak{g}(\psi_\zeta^{-1}\eta\omega^j) \cdot \Omega^\pm_{\omega_f}},$$

where $\mathfrak{g}(\psi_\zeta^{-1}\eta\omega^j)$ is the Gauss sum, the sign is $\pm = (-1)^j$, $\Omega^\pm_{\omega_f}$ is the p-normalised period in Definition 2.11, and

$$e_{p,\xi,\eta}(f,\chi) = \frac{p^{t'(j+1)}}{\xi^{t'}} \cdot \left(1 - \frac{\psi_\zeta^{-1}\eta\omega^j(p)p^{k-2-j}}{\xi}\right)\left(1 - \frac{\psi_\zeta\eta^{-1}\omega^{-j}(p)p^j}{\xi}\right)$$

with $t' = 0$ or $t+1$ according to whether or not $\eta = \omega^{-j}$ and $t = 0$.

Proof. This is a reformulation of [Kat04, Thm. 16.6], and follows from Kato's explicit reciprocity law (2.11) (and its variant for twists by finite order characters of $\widetilde{\Gamma}$) and the interpolation property of $\mathscr{L}_V$. Indeed, for $\xi \in \{\alpha, \beta\}$ Kato considers the differentials $\eta_\xi \in \mathbb{D}_{\mathrm{cris}}(V_f)$ attached to ω_f characterised by the properties $\varphi(\eta_\xi) = \xi \cdot \eta_\xi$ and $[\eta_\xi, \omega_f]_{\mathrm{dR}} = 1$, and shows that the image

$$\eta_{\xi,1} := \eta_\xi \otimes t^{-1}e_1 \in \mathbb{D}_{\mathrm{cris}}(V_f) \otimes \mathbb{D}_{\mathrm{dR}}(\mathbf{Q}_p(1)) \cong \mathbb{D}_{\mathrm{cris}}(V^*(1))$$

(where $t \in \mathbb{B}_{\mathrm{dR}}$ and the basis $e_1 \in \mathbf{Q}_p(1)$ are the ones defined by our chosen $(\zeta_{p^n})_{n\geq 0}$) is such that

$$\mathscr{L}_{V,(1+\pi)\otimes\eta_{\xi,1}}(\mathrm{res}_p(\mathbf{z}^{\mathrm{int}}_{f,k-1})) \in \mathcal{H}_L(\widetilde{\Gamma})$$

has the stated interpolation property after applying e_η. Since a direct calculation shows that

$$\eta_{\xi,1} = v^*_{\xi'} \tag{2.12}$$

where ξ' is the element in $\{\alpha, \beta\}$ different from ξ, the result follows. ∎

Note that from (2.8) we have the decomposition

$$\begin{pmatrix} L_\alpha(f) \\ L_\beta(f) \end{pmatrix} = Q_{\alpha,\beta}^{-1} M_{\log} \cdot \begin{pmatrix} L_1(f) \\ L_2(f) \end{pmatrix} \in \mathcal{H}_L(\widetilde{\Gamma})^{\oplus 2}. \tag{2.13}$$

Proposition 2.16. *For every character $\eta : \Delta \to \mathbf{Z}_p^\times$, the projection $L_i(f)^\eta \in \Lambda_{\mathscr{O}}(\Gamma)$ is nonzero for some $i \in \{1,2\}$.*

Proof. By the interpolation property of Theorem 2.15 and the nonvanishing result of [Roh88] (or just [Shi76, Prop. 2] for $k > 2$), the left-hand side of (2.13) has nonzero image under e_η. Since $Q_{\alpha,\beta}^{-1} M_{\log}$ has nonzero determinant (see [LLZ11, Cor. 3.2]), the result follows. ☐

Remark 2.17. Contrary to the case $a_p = 0$ studied in [Pol03, Cor. 5.11], due to the inexplicit nature of $M_{\log}$ in general, the interpolation property of Theorem 2.15 does not translate into a similar property of $L_i(f)^\eta$ in a way allowing us to show that $L_i(f)^\eta \neq 0$ for *both* $i \in \{1,2\}$ from the aforementioned nonvanishing results (cf. [LLZ10, Cor. 3.29]).

2.5. **Main conjectures.** We recall the formulation of signed main conjectures from [LLZ10, LLZ11] and their relation with Kato's main conjecture.

2.5.1. *Formulations.* For $i \in \{1,2\}$, define the *i-th submodule* of $\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), T)$ by

$$\mathrm{H}^1_{\mathrm{Iw},i}(\mathbf{Q}_p(\mu_{p^\infty}), T) := \ker(\widetilde{\mathscr{C}_i}),$$

where $\widetilde{\mathscr{C}_i}$ is the signed Coleman map of Proposition 2.6. Let $V^*(1) = \mathrm{Hom}_L(V, L(1))$ be the Kummer dual of V, put $A_f(1) = V_f(1)/T_f(1)$, and using the identification[5]

$$V^*(1) = V_f(k-1)^*(1) \cong V_f(1). \tag{2.14}$$

let $\mathrm{H}^1_i(\mathbf{Q}_p(\mu_{p^\infty}), A_f(1))$ be the orthogonal complement of $\mathrm{H}^1_{\mathrm{Iw},i}(\mathbf{Q}_p(\mu_{p^\infty}), T)$ under local Tate duality

$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), T) \times \mathrm{H}^1(\mathbf{Q}_p(\mu_{p^\infty}), A_f(1)) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p.$$

Definition 2.18. For $i \in \{1,2\}$ put

$$\mathrm{Sel}_i(f/\mathbf{Q}(\mu_{p^\infty})) := \ker\left\{\mathrm{H}^1(\mathbf{Q}(\mu_{p^\infty}), A_f(1)) \longrightarrow \frac{\mathrm{H}^1(\mathbf{Q}_p(\mu_{p^\infty}), A_f(1))}{\mathrm{H}^1_i(\mathbf{Q}_p(\mu_{p^\infty}), A_f(1))} \times \prod_{w\nmid p} \mathrm{H}^1(\mathbf{Q}(\mu_{p^\infty})_w, A_f(1))\right\};$$

and define the *strict Selmer group* by

$$\mathrm{Sel}_{\mathrm{str}}(f/\mathbf{Q}(\mu_{p^\infty})) := \ker\left\{\mathrm{Sel}_i(f/\mathbf{Q}(\mu_{p^\infty})) \longrightarrow \mathrm{H}^1(\mathbf{Q}_p(\mu_{p^\infty}), A_f(1))\right\}$$

(which is of course independent of i).

Let $X_i(f/\mathbf{Q}(\mu_{p^\infty})) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}_i(f/\mathbf{Q}(\mu_{p^\infty})), \mathbf{Q}_p/\mathbf{Z}_p)$ be the Pontryagin dual of $\mathrm{Sel}_i(f/\mathbf{Q}(\mu_{p^\infty}))$ and similarly put $X_{\mathrm{str}}(f/\mathbf{Q}(\mu_{p^\infty})) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}_{\mathrm{str}}(f/\mathbf{Q}(\mu_{p^\infty})), \mathbf{Q}_p/\mathbf{Z}_p)$.

2.5.2. *Kato's main conjecture.* As we explain below, the following is a reformulation of Kato's main conjecture (cf. [Kat04, Conj. 12.10]).

Conjecture 2.19 (Kato's main conjecture)**.** *For every character $\eta : \Delta \to \mathbf{Z}_p^\times$ we have*

$$\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\left(\frac{\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T)^\eta}{(\mathbf{z}^{\mathrm{int}}_{f,k-1})^\eta}\right) = \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X_{\mathrm{str}}(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\big)$$

as characteristic ideals of torsion $\Lambda_{\mathscr{O}}(\Gamma)$-modules.

[5]Recall that in general $V_f(k-1)^* \cong V_{\bar f}$ where $\bar f$ is the form obtained by complex conjugating the Fourier coefficients of f; however $\bar f = f$ for f with trivial nebentypus.

Remark 2.20. Let S be a set of primes containing ∞ and the primes dividing Np, and let $\mathbf{Q}^S$ denote the maximal extension of $\mathbf{Q}$ unramified outside S. As shown in [Kur02, Lemma 4.3] (see also [Lei11, §6.2]), Poitou–Tate duality gives

$$X_{\rm str}(f/\mathbf{Q}(\mu_{p^\infty}))\cong\ker\biggl\{\mathrm{H}^2_{\mathrm{Iw},S}(\mathbf{Q}(\mu_{p^\infty}),T)\longrightarrow\prod_{\ell\in S}\prod_{w\mid\ell}\mathrm{H}^2_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty})_w,T)\biggr\}, \tag{2.15}$$

where $\mathrm{H}^2_{\mathrm{Iw},S}(\mathbf{Q}(\mu_{p^\infty}),T)=\varprojlim_n\mathrm{H}^2(\mathrm{Gal}(\mathbf{Q}^S/\mathbf{Q}(\mu_{p^n})),T)$. On the other hand, from the localisation sequence in étale cohomology we have

$$\mathrm{H}^2_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),T)\cong\ker\biggl\{\mathrm{H}^2_{\mathrm{Iw},S}(\mathbf{Q}(\mu_{p^\infty}),T)\longrightarrow\prod_{\ell\in S\smallsetminus\{p\}}\prod_{w\mid\ell}\mathrm{H}^2_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty})_w,T)\biggr\} \tag{2.16}$$

(see [Kur02, §6]). Since by [Lei11, Lem. 4.4][6] we have $\mathrm{H}^0(\mathbf{Q}_p(\mu_{p^\infty}),A_f(1))=0$, comparing (2.15) and (2.16) and using local Tate duality it follows that

$$X_{\rm str}(f/\mathbf{Q}(\mu_{p^\infty}))\cong\mathrm{H}^2_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),T).$$

Thus Conjecture 2.19 (for $\eta:\Delta\to\mathbf{Z}_p^\times$) is equivalent to Conjecture 1.1 (for $\eta\omega^{k-1}$) in the Introduction.

2.5.3. *Signed main conjectures.* The next recall the formulation of signed Iwasawa main conjectures due to Lei–Loeffler–Zerbes (cf. [LLZ11, Conj. 1.1]).

Conjecture 2.21 (Lei–Loeffler–Zerbes signed main conjectures)**.** *Let $i\in\{1,2\}$. For every character $\eta:\Delta\to\mathbf{Z}_p^\times$ we have*

$$\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\bigl(X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\bigr)=\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\biggl(\frac{\mathrm{Image}(\widetilde{\mathscr{C}_i})^\eta}{(L_i(f)^\eta)}\biggr);$$

equivalently, we have

$$(\xi_{\eta,i})\cdot\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\bigl(X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\bigr)=\bigl(L_i(f)^\eta\bigr),$$

where $\xi_{\eta,i}$ is as in Proposition 2.8.

2.5.4. *Equivalence.* Our results on Kato's main conjecture for f will build on the following reformulation discovered by Lei–Loeffler–Zerbes [LLZ10].

Proposition 2.22. *Let $i\in\{1,2\}$. If $\eta:\Delta\to\mathbf{Z}_p^\times$ is such that $L_i(f)^\eta\neq0$, then the following are equivalent:*

(i) *$X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta$ is $\Lambda_{\mathscr{O}}(\Gamma)$-torsion with*
$$(\xi_{\eta,i})\cdot\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\bigl(X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\bigr)\subset\bigl(L_i(f)^\eta\bigr);$$
(ii) *$X_{\rm str}(f/\mathbf{Q}(\mu_{p^\infty}))^\eta$ is $\Lambda_{\mathscr{O}}(\Gamma)$-torsion with*
$$\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\bigl(X_{\rm str}(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\bigr)\subset\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\biggl(\frac{\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),T)^\eta}{(\mathbf{z}^{\rm int}_{f,k-1})^\eta}\biggr);$$

and the same holds for the respective opposite divisibilities. In particular, Conjecture 2.21 for $L_i(f)^\eta$ and Kato's main Conjecture 2.19 for η are equivalent.

Proof. This is essentially shown in [LLZ10, Thm. 6.5] (cf. [Kob03, §7]), but we provide the details for the convenience of the reader. As noted in the proof of Proposition 2.12, under our assumptions the residual representation $\bar\rho_f$ is irreducible, and so by [Kat04, Thm. 12.4] we know that $\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),T)$ is a free $\Lambda_{\mathscr{O}}(\widetilde{\Gamma})$-module of rank 1. From Poitou–Tate duality we have the exact sequence

$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),T)\longrightarrow\frac{\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),T)}{\mathrm{H}^1_{\mathrm{Iw},i}(\mathbf{Q}_p(\mu_{p^\infty}),T)}\longrightarrow X_i(f/\mathbf{Q}(\mu_{p^\infty}))\longrightarrow X_{\rm str}(f/\mathbf{Q}(\mu_{p^\infty}))\longrightarrow0, \tag{2.17}$$

[6] Whose proof, building on [BLZ04, Prop. 4.1.4], does not require $a_p=0$.

and by definition and the hypothesis on η, the composition of the left-most map in (2.17) with $\widetilde{\mathscr{C}_i}$ is a nonzero map $\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T) \to \Lambda_{\mathscr{O}}(\widetilde{\Gamma})$ taking $\mathbf{z}^{\mathrm{int}}_{f,k-1}$ to $L_i(f)$. Thus the latter map is an injection, and quotienting out by the image of $\mathbf{z}^{\mathrm{int}}_{f,k-1}$ and taking η-isotypic components, (2.17) yields the short exact sequence

$$0 \longrightarrow \frac{\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T)^\eta}{(\mathbf{z}^{\mathrm{int}}_{f,k-1})^\eta} \longrightarrow \frac{\mathrm{Image}(\widetilde{\mathscr{C}_i})^\eta}{(L_i(f)^\eta)} \longrightarrow X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta \longrightarrow X_{\mathrm{str}}(f/\mathbf{Q}(\mu_{p^\infty}))^\eta \longrightarrow 0.$$

Using Proposition 2.8 and multiplicativity of characteristic ideals, this yields the result. $\square$

3. Beilinson–Flach elements

Let $K/\mathbf{Q}$ be an imaginary quadratic field of discriminant $-D_K < 0$ such that

$$\text{(spl)} \qquad p = v\overline{v} \quad \text{splits in } K.$$

For every prime w of K above p, let K^w_∞ denote the $\mathbf{Z}_p$-extension unramified of K outside w, and put $\Gamma^w = \mathrm{Gal}(K^w_\infty/K)$. Let $\mathrm{rec}_K : \mathbb{A}_K^\times \to G_K^{\mathrm{ab}}$ be the geometrically normalised reciprocity map of class field theory, and let $\mathrm{rec}_w : K_w^\times \to G^{\mathrm{ab}}_{K_w}$ denote the restriction of rec_K to $K_w^\times$. Then $\mathrm{rec}_w(1+p\mathbf{Z}_p)$ injects into Γ^w (here we identified $1+p\mathbf{Z}_p$ with the group of 1-units in $\mathcal{O}_{K_w}$), and so

$$[\Gamma^w : \mathrm{rec}_w(1+p\mathbf{Z}_p)] = p^{h_p}$$

for some $h_p \geq 0$ (with $h_p = 0$ if the class number h_K of K is coprime to p).

As usual, upon the choice of a topological generator $\gamma_w \in \Gamma^w$, we shall often identify $\Lambda_{\mathscr{O}}(\Gamma^w)$ with the one-variable power series ring $\mathscr{O}[\![T_w]\!]$ via $\gamma_w \mapsto 1 + T_w$.

3.1. **Canonical CM family.** Following [BSTW24, §4.1.6], define $\mathbf{h}_v \in \Lambda_{\mathbf{Z}_p}(\Gamma^v)[\![q]\!]$ by

$$\mathbf{h}_v(q) = \sum_{n=1}^\infty \mathbf{b}_n q^n, \qquad \mathbf{b}_n = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K, \mathbf{N}(\mathfrak{a})=n \\ v\nmid\mathfrak{a}}} \Theta_v(x_{\mathfrak{a}}),$$

with $\Theta_v : \mathbb{A}_K^\times \to \Gamma^v$ the composition of $\mathrm{rec}_K : \mathbb{A}_K^\times \to G_K^{\mathrm{ab}}$ with the projection $G_K^{\mathrm{ab}} \to \Gamma^v$, and where for each ideal $\mathfrak{a} \subset \mathcal{O}_K$, we let $x_{\mathfrak{a}} = (x_{\mathfrak{a},w})_w$ be any finite idèle with $\mathrm{ord}_w(x_{\mathfrak{a},w}) = \mathrm{ord}_w(\mathfrak{a})$ for all finite places w of K with $\mathrm{ord}_w(\mathfrak{a}) \neq 0$ and $x_{\mathfrak{a},w} = 1$ for all other places w.

The q-series $\mathbf{h}_v(q)$ is a *CM Hida family* of tame level D_K and tame character given by the quadratic Dirichlet character ϵ_K corresponding to K, in the sense that if $\phi : \Lambda_{\mathbf{Z}_p}(\Gamma^v) \to \mathbf{Q}_p$ is the $\mathbf{Z}_p$-algebra homomorphism defined by $\phi(\gamma_v^{h_p}) = (1+p)^m$ for some $m \geq 0$, the specialisation

$$\phi(\mathbf{h}_v)(q) := \sum_{n=1}^\infty \phi(\mathbf{b}_n) q^n$$

is the q-expansion of an ordinary p-stabilized newform $\phi(\mathbf{h}_v) \in S_{m+1}(\Gamma_0(D_K p), \epsilon_K\omega^{-m})$ with complex multiplication by K.

3.1.1. *Galois representations attached to CM eigenforms.* Let

$$\psi : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbf{C}^\times$$

be an algebraic Hecke character. We say that ψ has *infinity type* $(a,b) \in \mathbf{Z}^2$ is $\psi(z) = z^a \overline{z}^b$ for all $z \in (K \otimes_{\mathbf{Q}} \mathbf{R})^\times \cong \mathbf{C}^\times$, where the identification is via ι_∞. The restriction of ψ to the finite idèles $\mathbb{A}^\times_{K,\mathrm{f}}$ takes values in $\overline{\mathbf{Q}}^\times$. The *p-adic avatar* of ψ is the character $\hat{\psi} : G_K \to \overline{\mathbf{Q}}_p^\times$ defined by

$$\hat{\psi}(\sigma|_{K^{\mathrm{ab}}}) = \iota_p \iota_\infty^{-1}(\psi(x)) x_v^a x_{\overline{v}}^b,$$

where $x \in \mathbb{A}_{K,\mathrm{f}}^\times$ is such that $\mathrm{rec}_K(x) = \sigma|_{K^{\mathrm{ab}}}$. Thus, letting $\mathfrak{c}_\psi$ denote the conductor of ψ, we have

$$\hat{\psi}(\mathrm{Frob}_w) = \psi(w)$$

for all finite primes w of K with $w \nmid p\mathfrak{c}_\psi$, where Frob_w denotes a geometric Frobenius at w.

Suppose the infinity type of ψ is of the form $(1-k, 0)$ for some $k \geq 1$, and let $g = \theta_\psi$ be the theta series of weight k associated to ψ. Let L be the finite extension of $\mathbf{Q}_p$ generated by the image under ι_p of the field generated over $\mathbb{Q}$ by $\iota_\infty^{-1}(\psi(\mathbb{A}_{K,\mathrm{f}}^\times))$. Then the L-valued Galois representation V_g associated to g by Deligne [Del71] (see also in [Kat04, §8.3]) is such that

$$V_g \cong \mathrm{Ind}_K^{\mathbf{Q}}(\hat{\psi}).$$

3.1.2. *Galois representation attached to* $\mathbf{h}_v$. Put $\Lambda_D = \mathbf{Z}_p[\![\mathbf{Z}_p^\times]\!]$. For integers $M > 0$ prime to p, let

$$\mathfrak{h}_{Mp^\infty}^{\mathrm{ord}} := \varprojlim_r e'_{\mathrm{ord}} \mathfrak{h}_{\Gamma_1(Mp^r)}, \qquad \mathfrak{H}_{Mp^\infty}^{\mathrm{ord}} := \varprojlim_r e'_{\mathrm{ord}} \mathfrak{H}_{\Gamma_1(Mp^r)},$$

be Hida's anti-ordinary Hecke algebras acting on Λ_D-adic modular forms and cusp forms, respectively (see [KLZ17, §7.4]), and note that $\mathfrak{H}_{Mp^\infty}^{\mathrm{ord}}$ surjects onto $\mathfrak{h}_{Mp^\infty}^{\mathrm{ord}}$ by restriction. Put

$$\begin{aligned} H^1_{\mathrm{ord}}(Mp^\infty) &:= \varprojlim_r e'_{\mathrm{ord}} \mathrm{H}^1_{\mathrm{\acute{e}t}}(X_1(Mp^r)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)), \\ \widetilde{H}^1_{\mathrm{ord}}(Mp^\infty) &:= \varprojlim_r e'_{\mathrm{ord}} \mathrm{H}^1_{\mathrm{\acute{e}t}}(Y_1(Mp^r)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)), \end{aligned} \tag{3.1}$$

equipped with their natural $\mathfrak{h}_{Mp^\infty}^{\mathrm{ord}}[G_{\mathbf{Q}}]$- and $\mathfrak{H}_{Mp^\infty}^{\mathrm{ord}}[G_{\mathbf{Q}}]$-module structures, respectively.

Let $\varphi_v : \mathfrak{h}_{D_K p^\infty}^{\mathrm{ord}} \to \Lambda_{\mathbf{Z}_p}(\Gamma^v)$ be the Λ_D-algebra homomorphism corresponding to $\mathbf{h}_v$, where $\Lambda_{\mathbf{Z}_p}(\Gamma^v)$ is viewed as a Λ_D-module via the $\mathbf{Z}_p$-algebra homomorphism $\Lambda_D \to \Lambda_{\mathbf{Z}_p}(\Gamma^v)$ determined by $[\langle \varepsilon(\tilde{\gamma}) \rangle] \mapsto \varepsilon(\tilde{\gamma})^{-1}\gamma_v$ for all $\tilde{\gamma} \in G_K$ via the cyclotomic character $\varepsilon : G_K \to \mathbf{Z}_p^\times$, and put

$$\begin{aligned} \mathbb{T}_{\mathbf{h}_v} &:= H^1_{\mathrm{ord}}(D_K p^\infty) \otimes_{\mathfrak{h}_{D_K p^\infty}^{\mathrm{ord}}} \Lambda_{\mathbf{Z}_p}(\Gamma^v), \\ \widetilde{\mathbb{T}}_{\mathbf{h}_v} &:= \widetilde{H}^1_{\mathrm{ord}}(D_K p^\infty) \otimes_{\mathfrak{H}_{D_K p^\infty}^{\mathrm{ord}}} \Lambda_{\mathbf{Z}_p}(\Gamma^v), \end{aligned} \tag{3.2}$$

where the tensor products are with respect to φ_v and its composition with the projection $\mathfrak{H}_{D_K p^\infty}^{\mathrm{ord}} \to \mathfrak{h}_{D_K p^\infty}^{\mathrm{ord}}$, respectively.

Let $\Psi^v : G_K \to \Lambda_{\mathbf{Z}_p}(\Gamma^v)^\times$ denote the canonical projection $G_K \to \Gamma^v$ composed with the inclusion $\Gamma^v \hookrightarrow \Lambda_{\mathbf{Z}_p}(\Gamma^v)^\times$, and write $\Lambda_{\mathbf{Z}_p}(\Gamma^v)^\iota$ for the free $\Lambda_{\mathbf{Z}_p}(\Gamma^v)$-module of rank one on which G_K acts via the inverse of Ψ^v.

Theorem 3.1. *Let* $\mathbb{T}_{\mathbf{h}_v}^\square$ *denote the quotient of* $\mathbb{T}_{\mathbf{h}_v}$ *by its torsion submodule. Then*

$$\mathbb{T}_{\mathbf{h}_v}^\square \cong \mathrm{Ind}_K^{\mathbf{Q}} \Lambda_{\mathbf{Z}_p}(\Gamma^v)^\iota$$

as $\Lambda_{\mathbf{Z}_p}(\Gamma^v)[G_{\mathbf{Q}}]$*-modules.*

Proof. This is shown in Theorem 5.19 of [BSTW24]. □

3.2. **Zeta elements over** K**.** Let α and β be the roots of the Hecke polynomial $x^2 - a_p x + p^{k-1}$, and for $\xi \in \{\alpha, \beta\}$ let $f^\xi \in S_k(\Gamma_0(Np))$ be the p-stabilization of f with U_p-eigenvalue ξ. Upon enlarging $\mathscr{O}$ if necessary, we assume that $\alpha, \beta \in \mathscr{O}$. Recall from §2.1 the $\mathscr{O}$-lattice $T_f \subset V_f$. Working with level Np, we define $T_{f^\xi} \subset V_{f^\xi}$ similarly as before, and write $T_f^* = \mathrm{Hom}_{\mathscr{O}}(T_f, \mathscr{O})$ and likewise define $T_{f^\xi}^*$.

Replacing $\mathbf{Z}_p(1)$ by the p-adic étale sheaf $\mathrm{TSym}^{k-2}\mathscr{H}_r(1)$ on $X_1(Mp^r)$ and $Y_1(Mp^r)$ introduced in [KLZ17, §2.3], the inverse limits (3.1) define the $\mathfrak{h}_{Mp^\infty}^{\mathrm{ord}}[G_{\mathbf{Q}}]$- and $\mathfrak{H}_{Mp^\infty}^{\mathrm{ord}}[G_{\mathbf{Q}}]$-modules $H^1_{\mathrm{ord}}(Mp^\infty)^{[k-2]}$

and $\widetilde{H}^1_{\mathrm{ord}}(Mp^\infty)^{[k-2]}$. It follows from [KLZ17, Thm. 4.5.1, Rem. 4.5.3] that these are identified with the twists of $H^1_{\mathrm{ord}}(Mp^\infty)$ and $\widetilde{H}^1_{\mathrm{ord}}(Mp^\infty)$, respectively, by the $\mathbf{Z}_p$-linear twisting map

$$m_{k-2} : \Lambda_D \longrightarrow \Lambda_D \tag{3.3}$$

given by $z \mapsto z^{2-k}[z]$ for $z \in \mathbf{Z}_p^\times$. From these, we define $\mathbb{T}^{[k-2]}_{\mathbf{h}_v}$ and $\widetilde{\mathbb{T}}^{[k-2]}_{\mathbf{h}_v}$ by tensoring with φ_v as in (3.2), and let $\mathbb{T}^{\square,[k-2]}_{\mathbf{h}_v}$ and $\widetilde{\mathbb{T}}^{\square,[k-2]}_{\mathbf{h}_v}$ denote the quotient by the torsion submodules.

Put $M = ND_K$, fix an integer $c > 1$ with $(c, 6pM) = 1$, and let

$${}_c\mathcal{BF}^{\xi,\mathbf{h}_v} \in \mathrm{H}^1(\mathbf{Z}[1/p], T^*_{f^\xi} \hat{\otimes} (\widetilde{\mathbb{T}}^{\square,[k-2]}_{\mathbf{h}_v}) \hat{\otimes} \mathcal{H}_L(\widetilde{\Gamma})) \tag{3.4}$$

be the Beilinson–Flach element ${}_c\mathrm{BF}^{\xi,\mathbf{g}}_m$ of [BL21, Thm. 3.2] for $m = 1$ and $\mathbf{g} = \mathbf{h}_v$ the canonical CM family of §3.1; by construction, this interpolates for varying $n \geq 1$ and $0 \leq j \leq k-2$ the image of the modified Rankin–Eisenstein classes

$$\frac{(-1)^j\big((U_p')^{-n}, (U_p')^{-n}\big)}{j!\binom{k-2}{j}^2} {}_c\mathcal{RI}^{[j]}_{p^n, p^{n+1}M, 1}$$

of [KLZ17, Def. 5.1.5] (as well as [LZ16, Def. 3.2.1]) under the composite map

$$\begin{aligned} &\mathrm{H}^3_{\text{ét}}(Y(p^n, p^{n+1}M)^2, \Lambda(\mathscr{H}_{\mathbf{Z}_p}\langle t_{p^{n+1}M}\rangle)^{[j,j]}(2-j) \otimes L) \\ &\qquad\longrightarrow \mathrm{H}^1(\mathbf{Q}(\mu_{p^n}), T^*_{f^\xi} \hat{\otimes} \widetilde{H}^1_{\mathrm{ord}}(Mp^\infty)^{[k-2]}(-j) \otimes L) \\ &\qquad\longrightarrow \mathrm{H}^1(\mathbf{Q}(\mu_{p^n}), T^*_{f^\xi} \hat{\otimes} \widetilde{\mathbb{T}}^{\square,[k-2]}_{\mathbf{h}_v}(-j) \otimes L) \end{aligned} \tag{3.5}$$

of [BL21, p. 10685], where the second arrow in (3.5) is induced by the projection

$$\widetilde{H}^1_{\mathrm{ord}}(Mp^\infty)^{[k-2]} \longrightarrow \widetilde{\mathbb{T}}^{[k-2]}_{\mathbf{h}_v} \longrightarrow \widetilde{\mathbb{T}}^{\square,[k-2]}_{\mathbf{h}_v}.$$

By [KLZ17, Prop. 7.3.1] (see also [LLZ15, Prop. 4.3.6]), there is an isomorphism $(\mathrm{Pr}^\xi)_* : T^*_{f^\xi} \xrightarrow{\sim} T^*_f$ induced by a p-stabilisation map Pr^ξ; in the following we shall thus view the Beilinson–Flach elements (3.4) in $\mathrm{H}^1(\mathbf{Z}[1/p], T^*_f \hat{\otimes} (\widetilde{\mathbb{T}}^{\square,[k-2]}_{\mathbf{h}_v}) \hat{\otimes} \mathcal{H}_L(\widetilde{\Gamma})^\iota)$. Moreover, as observed in [BSTW24, §6.4], we may find c as above such that

$$\mathbf{c} := (c^2 - \epsilon_K(c) \otimes \langle c\rangle \gamma_v^{-r_c}) \otimes \gamma_{\mathrm{cyc}}^{2r_c}$$

is invertible, where $r_c = \log_p(c)$. Upon the choice of such c we then set

$$\mathcal{BF}^{\xi,\mathbf{h}_v} := \mathbf{c}^{-1} \cdot {}_c\mathcal{BF}^{\xi,\mathbf{h}_v} \in \mathrm{H}^1(\mathbf{Z}[1/p], T \hat{\otimes} (\widetilde{\mathbb{T}}^{\square,[k-2]}_{\mathbf{h}_v}) \hat{\otimes} \mathcal{H}_L(\widetilde{\Gamma})^\iota),$$

which is independent of c, and where we used the identification $T^*_f \cong T$ (since f has trivial nebentypus, and so the complex conjugate $\bar{f}$ is f itself).

Theorem 3.2. *There exist classes*

$$\mathcal{BF}^{1,\mathbf{h}_v},\ \mathcal{BF}^{2,\mathbf{h}_v} \in \mathrm{H}^1(\mathbf{Z}[1/p], T \hat{\otimes} (\widetilde{\mathbb{T}}^{\square,[k-2]}_{\mathbf{h}_v}) \hat{\otimes} \Lambda_{\mathbf{Z}_p}(\widetilde{\Gamma})^\iota) \otimes L$$

such that

$$\begin{pmatrix} \mathcal{BF}^{\alpha,\mathbf{h}_v} \\ \mathcal{BF}^{\beta,\mathbf{h}_v} \end{pmatrix} = Q_{\alpha,\beta}^{-1} M_{\log} \cdot \begin{pmatrix} \mathcal{BF}^{1,\mathbf{h}_v} \\ \mathcal{BF}^{2,\mathbf{h}_v} \end{pmatrix},$$

where $Q_{\alpha,\beta} = \frac{1}{\alpha-\beta}\left(\begin{smallmatrix} \alpha & -\beta \\ -p^{k-1} & p^{k-1} \end{smallmatrix}\right)$ *and* $M_{\log} \in M_{2\times 2}(\mathcal{H}_L(\Gamma))$ *is the logarithm matrix of Definition 2.3.*

Proof. This follows from [BL21, Thm. 3.7]. □

Fix some $s \geq 0$ such that for both $i \in \{1,2\}$ the classes

$$\mathcal{BF}^{i,\mathbf{h}_v}_{\mathrm{int}} := \varpi^s \cdot \mathcal{BF}^{i,\mathbf{h}_v} \tag{3.6}$$

land in $\mathrm{H}^1(\mathbf{Z}[1/p], T\hat{\otimes}(\widetilde{\mathbb{T}}^{\square,[k-2]}_{\mathbf{h}_v})\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\widetilde{\Gamma})^\iota)$ (the particular choice of s will be irrelevant for our arguments later). In the following, twisting by the inverse of the character m_{k-2} of (3.3), we shall view $\mathcal{BF}^{i,\mathbf{h}_v}_{\mathrm{int}}$ as living in $\mathrm{H}^1(\mathbf{Z}[1/p], T\hat{\otimes}(\widetilde{\mathbb{T}}^{\square}_{\mathbf{h}_v})\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\widetilde{\Gamma})^\iota)$.

3.3. **Two-variable Coleman maps.** Let F_∞ be the unramified $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$, and put

$$U := \mathrm{Gal}(F_\infty/\mathbf{Q}_p), \qquad G := \mathrm{Gal}(F_\infty(\mu_{p^\infty})/\mathbf{Q}_p) \cong \widetilde{\Gamma} \times U.$$

We begin by recalling the construction by Loeffler–Zerbes [LZ14] of a two-variable big logarithm map $\mathscr{L}_{V,F_\infty}$ interpolating Perrin-Riou's

$$\mathscr{L}_{V,F} : \mathrm{H}^1_{\mathrm{Iw}}(F(\mu_{p^\infty}), V) \longrightarrow \mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbf{Q}_p} \mathcal{H}_L(\widetilde{\Gamma})$$

(cf. (2.6)) over the finite extensions $F/\mathbf{Q}_p$ contained in F_∞.

For any finite unramified extension $F/\mathbf{Q}_p$, write $\mathbb{N}_F(T)$ for the Wach module attached to $T|_{G_F}$. By [LZ14, Prop. 4.5], the Fontaine–Berger isomorphism $\mathrm{H}^1_{\mathrm{Iw}}(F(\mu_{p^\infty}), T) \cong \mathbb{N}_F(T)^{\psi=1}$ extends to

$$\mathrm{H}^1_{\mathrm{Iw}}(F_\infty(\mu_{p^\infty}), T) \cong \mathbb{N}_{F_\infty}(T)^{\psi=1},$$

where $\mathbb{N}_{F_\infty}(T) = \varprojlim_F \mathbb{N}_F(T)$ with inverse limit relative to the trace maps, and with F running over the finite extensions of $\mathbf{Q}_p$ contained in F_∞; and by Proposition 3.12 in *op. cit.* we have

$$(\varphi^*\mathbb{N}_{F_\infty}(T))^{\psi=0} = (\varphi^*\mathbb{N}(T))^{\psi=0}\hat{\otimes}_{\mathbf{Z}_p} S_{F_\infty/\mathbf{Q}_p},$$

where $S_{F_\infty/\mathbf{Q}_p} \subset \Lambda_{\widehat{\mathscr{O}}_{F_\infty}}(U)$ is a free $\Lambda_{\mathbf{Z}_p}(U)$-module of rank one. Following [LZ14, Def. 4.6] (see also [BL21, §2.3]), letting $\Omega_{\mathbf{Q}_p}$ be a $\Lambda_{\mathbf{Z}_p}(U)$-basis of $S_{F_\infty/\mathbf{Q}_p}$, similarly as in (3.7) we define

$$\mathscr{L}_{V,F_\infty} : \mathrm{H}^1_{\mathrm{Iw}}(F_\infty(\mu_{p^\infty}), T) \longrightarrow \Omega_{\mathbf{Q}_p} \cdot \mathcal{H}_L(G) \otimes_{\mathbf{Q}_p} \mathbb{D}_{\mathrm{cris}}(V)$$

by the composition

$$\begin{aligned}\mathrm{H}^1_{\mathrm{Iw}}(F_\infty(\mu_{p^\infty}), T) \xrightarrow{\cong} \mathbb{N}_{F_\infty}(T)^{\psi=1} &\xrightarrow{1-\varphi} (\varphi^*\mathbb{N}(T))^{\psi=0}\hat{\otimes}_{\mathbf{Z}_p} S_{F_\infty/\mathbf{Q}_p}\\ &\hookrightarrow \mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbf{Q}_p} (\mathbb{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}\hat{\otimes}_{\mathbf{Z}_p} S_{F_\infty/\mathbf{Q}_p}\\ &\xrightarrow{\mathfrak{M}^{-1}} \mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbf{Q}_p} (\mathcal{H}_L(\widetilde{\Gamma})\hat{\otimes}_{\mathbf{Z}_p} S_{F_\infty/\mathbf{Q}_p})\\ &\hookrightarrow \Omega_{\mathbf{Q}_p} \cdot \mathcal{H}_L(G) \otimes_{\mathbf{Q}_p} \mathbb{D}_{\mathrm{cris}}(V),\end{aligned} \tag{3.7}$$

using the embedding of [LLZ11, Prop. 2.11] and the inclusion $\mathcal{H}_L(\widetilde{\Gamma})\hat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathbf{Z}_p}(U) \subset \mathcal{H}_L(\widetilde{\Gamma})\hat{\otimes}_{\mathbf{Z}_p}\mathcal{H}(U) = \mathcal{H}_L(G)$, respectively, for the unlabeled arrows.

Proposition 3.3. *Let (n_1, n_2) be a $\mathbb{A}^+_{\mathbf{Q}_p}$-basis of $\mathbb{N}(T)$ as in Lemma 2.2. Then we can write*

$$\mathscr{L}_{V,F_\infty} = (v_1\ ,\ v_2) \cdot M_{\log} \cdot \begin{pmatrix} \widetilde{\mathscr{C}}_{1,F_\infty} \\ \widetilde{\mathscr{C}}_{2,F_\infty} \end{pmatrix}, \tag{3.8}$$

where $\widetilde{\mathscr{C}}_{i,F_\infty}$ is defined as the composition

$$\begin{aligned}\widetilde{\mathscr{C}}_{i,F_\infty} : \mathrm{H}^1_{\mathrm{Iw}}(F_\infty(\mu_{p^\infty}), T) \xrightarrow{\cong} \mathbb{N}_{F_\infty}(T)^{\psi=1} &\xrightarrow{1-\varphi} (\varphi^*\mathbb{N}(T))^{\psi=0}\hat{\otimes}_{\mathbf{Z}_p} S_{F_\infty/\mathbf{Q}_p}\\ &\xrightarrow{\mathrm{pr}_i\circ\mathfrak{J}} \Lambda_{\mathscr{O}}(\widetilde{\Gamma})\hat{\otimes}_{\mathbf{Z}_p} S_{F_\infty/\mathbf{Q}_p}\\ &= \Lambda_{\mathscr{O}}(\widetilde{\Gamma})\hat{\otimes}_{\mathbf{Z}_p}(\Omega_{\mathbf{Q}_p} \cdot \Lambda_{\mathbf{Z}_p}(U))\\ &= \Omega_{\mathbf{Q}_p} \cdot \Lambda_{\mathscr{O}}(G),\end{aligned}$$

where the second arrow is given by taking the i-th coordinate with respect to the basis (n_1, n_2).

Proof. In the same way as in Proposition 2.6, this follows from the description of $\mathscr{L}_{V,F_\infty}$ in (3.7) and the definition of $M_{\log}$. ☐

Proposition 3.4. *Let $i \in \{1,2\}$. For every character $\eta : \Delta \to \mathbf{Z}_p^\times$, we have an inclusion*

$$\mathrm{Image}(e_\eta \widetilde{\mathscr{C}}_{i,F_\infty}) \subset \Omega_{\mathbf{Q}_p} \cdot \xi_{\eta,i} \Lambda_{\mathscr{O}}(G)$$

with finite index, where $\xi_{\eta,i}$ is the same factor of δ_{k-1} as in Proposition 2.8.

Proof. This follows readily from the construction of $\widetilde{\mathscr{C}}_{i,F_\infty}$ and Proposition 2.8 (see [BL21, Lem. 2.16] for the details). ☐

Similarly as in §2.3.2, letting

$$\mathscr{L}_{V,\xi,F_\infty} = \mathscr{L}_{V,(1+\pi)v_\xi^*,F_\infty} : \mathrm{H}^1_{\mathrm{Iw}}(F_\infty(\mu_{p^\infty}),T) \longrightarrow \Omega_{\mathbf{Q}_p} \cdot \mathcal{H}_L(G)$$

be the coordinate of $\mathscr{L}_{V,F_\infty}$ with respect to v_ξ, the decomposition (3.8) can be rewritten as

$$\begin{pmatrix} \mathscr{L}_{V,\alpha,F_\infty} \\ \mathscr{L}_{V,\beta,F_\infty} \end{pmatrix} = Q_{\alpha,\beta}^{-1} M_{\log} \cdot \begin{pmatrix} \widetilde{\mathscr{C}}_{1,F_\infty} \\ \widetilde{\mathscr{C}}_{2,F_\infty} \end{pmatrix}, \tag{3.9}$$

where $Q_{\alpha,\beta} = \frac{1}{\alpha-\beta}\left(\begin{smallmatrix} \alpha & -\beta \\ -p^{k-1} & p^{k-1} \end{smallmatrix}\right)$.

3.3.1. *Semi-local signed Coleman maps.* Let

$$\Gamma_K = \mathrm{Gal}(K_\infty/K) \cong \mathbf{Z}_p^2$$

be the Galois group of the $\mathbf{Z}_p^2$-extension K_∞/K. For every $w \in \{v, \overline{v}\}$, fix a prime $\widetilde{w}_0$ of K_∞ above w, and let $\Gamma_{\widetilde{w}_0} \subset \Gamma_K$ denote the associated decomposition group at w_0. For every character $\eta : \Delta \to \mathbf{Z}_p^\times$, the constructions of the preceding paragraphs give maps

$$\begin{aligned} e_\eta \mathscr{L}_{V,\xi,K_{\infty,\widetilde{w}_0}} &: \mathrm{H}^1_{\mathrm{Iw}}(K_{\infty,\widetilde{w}_0}, V(\eta)) \longrightarrow \Omega_{\mathbf{Q}_p} \cdot \mathcal{H}_L(\Gamma_{\widetilde{w}_0}), \\ e_\eta \widetilde{\mathscr{C}}_{i,K_{\infty,\widetilde{w}_0}} &: \mathrm{H}^1_{\mathrm{Iw}}(K_{\infty,\widetilde{w}_0}, T(\eta)) \longrightarrow \Omega_{\mathbf{Q}_p} \cdot \Lambda_{\mathscr{O}}(\Gamma_{\widetilde{w}_0}), \end{aligned} \tag{3.10}$$

where $V(\eta)$ denotes the twist $V \otimes \eta$, and likewise for $T(\eta)$. Using the Shapiro's lemma isomorphism

$$\begin{aligned} \mathrm{H}^1(K_w, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma_K)^\iota) &\cong \bigoplus_{\widetilde{w}|w} \mathrm{H}^1_{\mathrm{Iw}}(K_{\infty,\widetilde{w}}, T(\eta)) \\ &\cong \Lambda_{\mathscr{O}}(\Gamma_K) \otimes_{\Lambda_{\mathscr{O}}(\Gamma_{\widetilde{w}_0})} \mathrm{H}^1_{\mathrm{Iw}}(K_{\infty,\widetilde{w}_0}, T(\eta)), \end{aligned}$$

where $\widetilde{w}$ runs over the primes of K_∞ above w, we define

$$\begin{aligned} e_\eta \mathscr{L}_{V,\xi,w} &: \mathrm{H}^1(K_w, V(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma_K)^\iota) \longrightarrow \Omega_{\mathbf{Q}_p} \cdot \mathcal{H}_L(\Gamma_K), \\ e_\eta \widetilde{\mathscr{C}}_{i,w} &: \mathrm{H}^1(K_w, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma_K)^\iota) \longrightarrow \Omega_{\mathbf{Q}_p} \cdot \Lambda_{\mathscr{O}}(\Gamma_K) \end{aligned} \tag{3.11}$$

by tensoring (3.10) with $\Lambda_{\mathscr{O}}(\Gamma_K)$ over $\Lambda_{\mathscr{O}}(\Gamma_{\widetilde{w}_0})$. Then from (3.9) we obtain the relations

$$\begin{pmatrix} e_\eta \mathscr{L}_{V,\alpha,w} \\ e_\eta \mathscr{L}_{V,\beta,w} \end{pmatrix} = Q_{\alpha,\beta}^{-1} M_{\log,w} \cdot \begin{pmatrix} e_\eta \widetilde{\mathscr{C}}_{1,w} \\ e_\eta \widetilde{\mathscr{C}}_{2,w} \end{pmatrix}, \tag{3.12}$$

where $M_{\log,w} \in M_{2\times 2}(\mathcal{H}_L(\Gamma^w))$ is the image of the logarithm matrix $M_{\log}$ of Definition 2.3 under the inverse of the isomorphism given by the composition $\Gamma^w \hookrightarrow \Gamma_K \twoheadrightarrow \Gamma$.

3.4. **Two-variable Selmer groups.** For every character $\eta : \Delta \to \mathbf{Z}_p^\times$, consider the G_K-module

$$\mathbf{T}_K^\eta := T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma_K)^\iota, \qquad \mathbf{A}_K^\eta := T(\eta)^\vee(1) \otimes \Lambda_{\mathscr{O}}(\Gamma_K),$$

where we recall that $T = T_f(k-1)$. For $\bullet \in \{1, 2, \mathrm{rel}, \mathrm{str}\}$ and $w \in \{v, \overline{v}\}$ a prime of K above p, put

$$\mathrm{H}^1_\bullet(K_w, \mathbf{T}_K^\eta) := \begin{cases} \ker(e_\eta \widetilde{\mathscr{C}}_{\bullet,w}) & \text{if } \bullet \in \{1,2\}, \\ \mathrm{H}^1(K_w, \mathbf{T}_K^\eta) & \text{if } \bullet = \mathrm{rel}, \\ 0 & \text{if } \bullet = \mathrm{str}, \end{cases}$$

and let $\mathrm{H}^1_\bullet(K_w, \mathbf{A}_K^\eta)$ be the orthogonal complement of $\mathrm{H}^1_\bullet(K_w, \mathbf{A}_K^\eta)$ under the local Tate duality

$$\mathrm{H}^1(K_w, \mathbf{T}_K^\eta) \times \mathrm{H}^1(K_w, \mathbf{A}_K^\eta) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p.$$

For $(\bullet, \circ) \in \{1, 2, \mathrm{rel}, \mathrm{str}\}^{\oplus 2}$, define the (compact) Selmer group $\mathrm{H}^1_{\bullet,\circ}(K, \mathbf{T}_K^\eta)$ by setting

$$\mathrm{H}^1_{\bullet,\circ}(K, \mathbf{T}_K^\eta) := \ker\left\{ \mathrm{H}^1(K, \mathbf{T}_K^\eta) \longrightarrow \frac{\mathrm{H}^1(K_v, \mathbf{T}_K^\eta)}{\mathrm{H}^1_\bullet(K_v, \mathbf{T}_K^\eta)} \times \frac{\mathrm{H}^1(K_{\overline{v}}, \mathbf{T}_K^\eta)}{\mathrm{H}^1_\circ(K_{\overline{v}}, \mathbf{T}_K^\eta)} \times \prod_{w \nmid p} \mathrm{H}^1(K_w^{\mathrm{unr}}, \mathbf{T}_K^\eta) \right\};$$

and the *dual (discrete) Selmer group* $\mathrm{Sel}_{\bullet,\circ}(f_\eta/K_\infty) \subset \mathrm{H}^1(K, \mathbf{A}_K^\eta)$ by replacing $\mathbf{T}_K^\eta$ with $\mathbf{A}_K^\eta$ in the above definition. Finally, write $X_{\bullet,\circ}(f_\eta/K_\infty)$ for the Pontryagin dual of $\mathrm{Sel}_{\bullet,\circ}(f_\eta/K_\infty)$.

3.5. **Explicit reciprocity laws.** In this section we explain the relation between the classes $\mathcal{BF}^{i,\mathbf{h}_v}$ of Theorem 3.2 and two different types of Rankin–Selberg p-adic L-functions arising from the explicit reciprocity laws established in [LZ16].

3.5.1. *Filtrations.* By Ohta's work [Oht00], the modules $H^1_{\mathrm{ord}}(Mp^\infty)$ and $\widetilde{H}^1_{\mathrm{ord}}(Mp^\infty)$ introduced in §3.1 are equipped with $\mathfrak{H}^{\mathrm{ord}}_{Mp^\infty}[G_{\mathbf{Q}_p}]$-stable filtrations

$$\mathscr{F}^+ H^1_{\mathrm{ord}}(Mp^\infty) \subset \mathscr{F}^+ H^1_{\mathrm{ord}}(Mp^\infty),$$
$$\mathscr{F}^+ \widetilde{H}^1_{\mathrm{ord}}(Mp^\infty) \subset \mathscr{F}^+ \widetilde{H}^1_{\mathrm{ord}}(Mp^\infty)$$

with $\mathscr{F}^+ H^1_{\mathrm{ord}}(Mp^\infty) = \mathscr{F}^+ \widetilde{H}^1_{\mathrm{ord}}(Mp^\infty)$ isomorphic to $\mathfrak{h}^{\mathrm{ord}}_{Mp^\infty}$ as $\mathfrak{H}^{\mathrm{ord}}_{Mp^\infty}$-modules. Put

$$\mathbb{T}^+_{\mathbf{h}_v} := \mathscr{F}^+ H^1_{\mathrm{ord}}(D_K p^\infty) \otimes_{\mathfrak{h}^{\mathrm{ord}}_{D_K p^\infty}} \Lambda_{\mathbf{Z}_p}(\Gamma^v),$$
$$\widetilde{\mathbb{T}}^+_{\mathbf{h}_v} := \mathscr{F}^+ \widetilde{H}^1_{\mathrm{ord}}(D_K p^\infty) \otimes_{\mathfrak{h}^{\mathrm{ord}}_{D_K p^\infty}} \Lambda_{\mathbf{Z}_p}(\Gamma^v),$$

so we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{T}^+_{\mathbf{h}_v} \arrow[r] \arrow[d, equal] & \mathbb{T}^\square_{\mathbf{h}_v} \arrow[r] \arrow[d, hook] & \mathbb{T}^-_{\mathbf{h}_v} := \mathbb{T}^\square_{\mathbf{h}_v}/\mathbb{T}^+_{\mathbf{h}_v} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \widetilde{\mathbb{T}}^+_{\mathbf{h}_v} \arrow[r] & \widetilde{\mathbb{T}}^\square_{\mathbf{h}_v} \arrow[r] & \widetilde{\mathbb{T}}^-_{\mathbf{h}_v} := \widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}/\widetilde{\mathbb{T}}^+_{\mathbf{h}_v} \arrow[r] & 0. \end{tikzcd} \tag{3.13}$$

3.5.2. *Triangulations.* Let $\mathscr{R}$ denote the Robba ring, consisting of Laurent power series in $L(\!(\pi)\!)$ convergent in some annulus inside $\{|\pi| < 1\}$, and let $\mathbb{D}^\dagger_{\mathrm{rig}}$ be Fontaine's functor from p-adic $G_{\mathbf{Q}_p}$-representations to $(\varphi, \widetilde{\Gamma})$-modules over $\mathscr{R}$, as extended by Berger–Colmez [BC08] to p-adic families.

Put $\mathscr{F}^\pm \mathbf{D}^\dagger_{\mathrm{rig}}(\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}) = \mathbf{D}^\dagger_{\mathrm{rig}}(\mathscr{F}^\pm \widetilde{\mathbb{T}}^\square_{\mathbf{h}_v})$, and similarly with $\mathbb{T}^\square_{\mathbf{h}_v}$; and for $\xi \in \{\alpha, \beta\}$ and $\circ, \bullet \in \{+, -, \emptyset\}$ put

$$\mathscr{F}^{\circ,\bullet}_{\xi,\mathrm{ord}} \mathbb{D}^\dagger_{\mathrm{rig}}(V \otimes \widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}) := \mathscr{F}^\circ_\xi \mathbb{D}^\dagger_{\mathrm{rig}}(V) \otimes \mathscr{F}^\bullet \mathbb{D}^\dagger_{\mathrm{rig}}(\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}),$$

and similarly with $\mathbb{T}^\square_{\mathbf{h}_v}$ in place of $\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}$, where $\mathscr{F}^\pm_\xi$ denotes the filtered pieces in the triangulation

$$0 \longrightarrow \mathscr{F}^+_\xi \mathbb{D}^\dagger_{\mathrm{rig}}(V) \longrightarrow \mathbb{D}^\dagger_{\mathrm{rig}}(V) \longrightarrow \mathscr{F}^-_\xi \mathbb{D}^\dagger_{\mathrm{rig}}(V) \longrightarrow 0$$

arising from [Kis03] and its extension to Coleman families in [Liu15].

The regulator maps $e_\eta\mathscr{L}_{V,\xi,w}$ of (3.11) extend by linearity to

$$e_\eta\mathscr{L}_{V,\xi,w} : \mathrm{H}^1(K_w, T(\eta)\otimes\Lambda_{\mathscr{O}}(\Gamma_K)^\iota\otimes_{\Lambda_{\mathscr{O}}(\Gamma)}\mathcal{H}_L(\Gamma)) \longrightarrow \Omega_{\mathbf{Q}_p}\cdot\mathcal{H}_L(\Gamma_K) \tag{3.14}$$

using the isomorphism (see [LZ16, Prop. 2.4.2])

$$\mathrm{H}^1\big(K_w, V(\eta)\otimes\Lambda_{\mathscr{O}}(\Gamma_K)^\iota\big)\otimes_{\Lambda_{\mathscr{O}}(\Gamma)}\mathcal{H}_L(\Gamma)\cong\mathrm{H}^1\big(K_w, V(\eta)\otimes\Lambda_{\mathscr{O}}(\Gamma_K)^\iota\otimes_{\Lambda_{\mathscr{O}}(\Gamma)}\mathcal{H}_L(\Gamma)\big).$$

3.5.3. *Explicit reciprocity law for $w = v$.* As shown in [LZ16, Thm. 7.1.2], the image of $\mathrm{res}_p(\mathcal{BF}^{\xi,\mathbf{h}_v})$ under the composite map

$$\begin{aligned}\mathrm{H}^1(\mathbf{Q}_p, V\otimes(\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v})\otimes\Lambda_{\mathscr{O}}(\widetilde{\Gamma})^\iota)\otimes_{\Lambda_{\mathscr{O}}(\widetilde{\Gamma})}\mathcal{H}_L(\widetilde{\Gamma}) &\cong \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}))\\ &\longrightarrow \mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{-,\emptyset}_{\xi,\mathrm{ord}}\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}))\end{aligned}$$

is contained in the image of the injection

$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{-,+}_{\xi,\mathrm{ord}}\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}))\longrightarrow\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{-,\emptyset}_{\xi,\mathrm{ord}}\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v})).$$

On the other hand, under the isomorphism

$$\mathrm{H}^1(K_v, V(\eta)\otimes\Lambda_{\mathbf{Z}_p}(\Gamma^v)\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\Gamma)^\iota)\otimes_{\Lambda_{\mathscr{O}}(\Gamma)}\mathcal{H}_L(\Gamma)\cong\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{\emptyset,+}_{\xi,\mathrm{ord}}\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}))^\eta$$

induced by the identification $K_v = \mathbf{Q}_p$ and the isomorphism

$$\widetilde{\mathbb{T}}^+_{\mathbf{h}_v} = \mathbb{T}^+_{\mathbf{h}_v}\cong\Lambda_{\mathbf{Z}_p}(\Gamma^v)^\iota$$

as $G_{\mathbf{Q}_p}$-modules (cf. (3.13) and Theorem 3.10) defined by the Λ-adic Ohta's differential $\omega_{\mathbf{h}_v}$ attached to $\mathbf{h}_v$ as in [KLZ17, Prop. 10.1.1], the extended regulator map $e_\eta\mathscr{L}_{V,\xi,v}$ of (3.14) factors through the natural map

$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{\emptyset,+}_{\xi,\mathrm{ord}}\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}))^\eta\longrightarrow\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{-,+}_{\xi,\mathrm{ord}}\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}))^\eta,$$

and so $e_\eta\mathscr{L}_{V,\xi,v}$ can naturally be evaluated at $\mathrm{res}_p(\mathcal{BF}^{\xi,\mathbf{h}_v,\eta})$ for the η-isotypic component $\mathcal{BF}^{\xi,\mathbf{h}_v,\eta}$ of $\mathcal{BF}^{\xi,\mathbf{h}_v}$.

Definition 3.5. For $\xi\in\{\alpha,\beta\}$ and a character $\eta:\Delta\to\mathbf{Z}_p^\times$, put

$$L^{(\mathrm{I})}_{\xi,\xi}(f,\mathbf{h}_v)^\eta := e_\eta\mathscr{L}_{V,\xi,v}(\mathrm{res}_p(\mathcal{BF}^{\xi,\mathbf{h}_v,\eta})).$$

Put

$$\Xi^{(\mathrm{I})} := \big\{(1,\chi_2):\Gamma^v\hat{\otimes}\Gamma\to\overline{\mathbf{Q}}_p^\times\mid\chi_2=\langle\varepsilon\rangle^j\psi_\zeta,\ 0\le j\le k-2,\ \zeta\in\mu_{p^\infty}\big\},$$

where ψ_ζ is the character associated to ζ as in §2.4.3.

Theorem 3.6 (Reciprocity Law I)**.** *Let $\xi\in\{\alpha,\beta\}$ and $\eta:\Delta\to\mathbf{Z}_p^\times$ be a character. Then $L^{(\mathrm{I})}_{\xi,\xi}(f,\mathbf{h}_v)^\eta$ satisfies the following interpolation property: For all $\chi=(1,\chi_2)\in\Xi^{(\mathrm{I})}$ with $\chi_2=\langle\varepsilon\rangle^j\psi_\zeta$ for $0\le j\le k-2$ and ζ a primitive p^t-th root of unity for some $t\ge 0$, we have*

$$\phi_\chi\big(L^{(\mathrm{I})}_{\xi,\xi}(f,\mathbf{h}_v)^\eta\big)=e_{p,\xi,\eta}(f,\chi)^2\cdot\frac{(j!)^2}{2^{2j+k+1}\cdot(-i)^{k-1}}\cdot\frac{L(f\otimes\omega^j\eta\psi_\zeta^{-1},\mathbf{h}^\circ_{v,1},j+1)}{\mathfrak{g}(\omega^j\eta\psi_\zeta^{-1})^2\cdot\pi^{2j+2}\cdot\langle f,f\rangle_N},$$

where $e_{p,\xi,\eta}(f,\chi)$, $\mathfrak{g}(\omega^j\eta\psi_\zeta^{-1})$ are as in Theorem 2.15, and $\langle f,f\rangle_N$ is the Petersson norm of f with

$$\langle f_1,f_2\rangle_N := \int_{\Gamma_1(N)\backslash\mathfrak{H}} f_1(\tau)\overline{f_2(\tau)}y^{k-2}dxdy$$

for $\tau=x+iy\in\mathfrak{H}$.

Proof. This is essentially contained [LZ16, Thm. 7.1.5]; since the statement in *loc. cit.* only includes the case $\zeta = 1$, while for our purposes we will need the result for arbitrary p-power roots of unity ζ, we provide some details.

Let f^ξ denote the p-stabilisation of f with U_p-eigenvalue ξ, and let $\mathbf{f}^\xi$ denote the Coleman family passing through f^ξ. Let $L_p^{\rm Urb}(\mathbf{f}^\xi, \mathbf{h}_v, 1+\mathbf{j})$ be the 3-variable p-adic Rankin L-function constructed by Urban [Urb14] (see also [AI21, App. B]). On the other hand, let

$$L_p^{\rm geo}(\mathbf{f}^\xi, \mathbf{h}_v, 1+\mathbf{j}) := (-1)^{1+\mathbf{j}}\lambda_N(\mathbf{f}^\xi)\cdot\left\langle \mathcal{L}(\mathcal{BF}_{1,1}^{[\mathbf{f}^\xi,\mathbf{h}_v]}), \eta_{\mathbf{f}^\xi}\otimes\omega_{\mathbf{h}_v}\right\rangle$$

be the 3-variable *geometric* p-adic L-function introduced in [LZ16, Def. 9.1.1], so by Theorem 7.1.5 in [LZ16] one has the equality

$$L_p^{\rm geo}(\mathbf{f}^\xi, \mathbf{h}_v, 1+\mathbf{j}) = L_p^{\rm Urb}(\mathbf{f}^\xi, \mathbf{h}_v, 1+\mathbf{j})$$

(indeed, this follows from the Zariski-density of the crystalline points at which their corresponding specialisations agree by virtue of [KLZ20, Thm. 6.5.9]).

By the interpolation property satisfied by $L_p^{\rm Urb}(\mathbf{f}^\xi, \mathbf{h}_v, 1+\mathbf{j})$, it follows that if we let $L_p^{\rm geo}(f^\xi, \mathbf{h}_v, 1+\mathbf{j})$ denote the image of $L_p^{\rm geo}(\mathbf{f}^\xi, \mathbf{h}_v, 1+\mathbf{j})$ under the specialisation map at f^ξ, for all $\eta : \Delta \to \mathbf{Z}_p^\times$ and $\chi \in \Xi^{(\rm I)}$ as in the statement we have

$$\phi_{\eta\chi}\big(L_p^{\rm geo}(f^\xi, \mathbf{h}_v, 1+\mathbf{j})\big) = \frac{e_{p,\xi,\eta}(f,\chi)^2}{\mathcal{E}_{p,\xi}(f)\cdot\mathcal{E}^*_{p,\xi}(f)}\cdot\frac{(j!)^2}{2^{2j+k+1}\cdot(-i)^{k-1}}\cdot\frac{L(f\otimes\omega^j\eta\psi_\zeta^{-1}, \mathbf{h}^\circ_{v,1}, j+1)}{\mathfrak{g}(\omega^j\eta\psi_\zeta^{-1})^2\cdot\pi^{2j+2}\cdot\langle f,f\rangle_N},$$

where $\mathcal{E}_{p,\xi}(f) = \left(1-\frac{p^{k-2}}{\xi^2}\right)$, $\mathcal{E}^*_{p,\xi}(f) = \left(1-\frac{p^{k-1}}{\xi^2}\right)$.

Comparing the interpolation properties satisfied by $\mathscr{L}_V$ and $\mathcal{L}$ (see [LZ14, Thm. 4.15] and [LZ16, Thm. 7.1.4]), the result thus follows from the relation (2.12) and the fact that by [LZ16, Cor. 6.4.3], $\eta_{\mathbf{f}^\gamma}$ specialises at f^ξ to $\lambda_N(f)^{-1}\mathcal{E}_{p,\xi}(f)^{-1}\mathcal{E}^*_{p,\xi}(f)^{-1}\cdot\eta_\xi$. $\square$

3.5.4. *Cuspidality of Beilinson–Flach classes.* To motivate the discussion in the next two subsections, let us note that our next goal is the proof of a second explicit reciprocity law for the natural image of ${\rm res}_p(\mathcal{BF}^{\xi,\mathbf{h}_v,\eta})$ under the p-adic regulator map $e_\eta\mathscr{L}_{V,\xi,\overline{v}}$, yielding an analogue of Theorem 3.6 with $\overline{v}$ in place of v. This will be the essentially content of Theorem 3.17 below, but a difficulty that arises is that the above evaluation only makes sense after multiplying the classes $\mathcal{BF}^{\xi,\mathbf{h}_v,\eta}$ by T_v. Thus we shall first consider this modification, and building on the first explicit reciprocity law (as in the proof of Theorem 3.10) and a key computation by Büyükboduk–Lei (as in Corollary 3.13) arrive at the desired Theorem 3.17.

We begin by noting that as in the proof of [BSTW24, Thm. 6.22], multiplication by T_v annihilates the quotient $\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v}/\mathbb{T}^\square_{\mathbf{h}_v}$, and so the classes

$$\mathcal{BF}^{\xi,\mathbf{h}_v}_{\rm cusp} := T_v\cdot\mathcal{BF}^{\xi,\mathbf{h}_v} \tag{3.15}$$

land in the image of the natural map

$$\mathrm{H}^1(\mathbf{Q}, V\otimes(\mathbb{T}^\square_{\mathbf{h}_v})\otimes\Lambda_{\mathscr{O}}(\widetilde{\Gamma})^\iota)\otimes\mathcal{H}_L(\widetilde{\Gamma}) \longrightarrow \mathrm{H}^1(\mathbf{Q}, V\otimes(\widetilde{\mathbb{T}}^\square_{\mathbf{h}_v})\otimes\Lambda_{\mathscr{O}}(\widetilde{\Gamma})^\iota)\otimes\mathcal{H}_L(\widetilde{\Gamma}),$$

which is an injection. Viewing $\mathcal{BF}^{\xi,\mathbf{h}_v}_{\rm cusp}$ in the former space, and taking η-isotypic components $\mathcal{BF}^{\xi,\mathbf{h}_v,\eta}_{\rm cusp}$, from Theorem 3.2 (and its proof) we have classes

$$\mathcal{BF}^{i,\mathbf{h}_v,\eta}_{\rm cusp}\in\mathrm{H}^1(\mathbf{Z}[1/p], T(\eta)\hat{\otimes}(\mathbb{T}^\square_{\mathbf{h}_v})\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\Gamma)^\iota)\otimes L$$

such that

$$\begin{pmatrix}\mathcal{BF}^{\alpha,\mathbf{h}_v,\eta}_{\rm cusp}\\ \mathcal{BF}^{\beta,\mathbf{h}_v,\eta}_{\rm cusp}\end{pmatrix} = Q^{-1}_{\alpha,\beta}M_{\log}\cdot\begin{pmatrix}\mathcal{BF}^{1,\mathbf{h}_v,\eta}_{\rm cusp}\\ \mathcal{BF}^{2,\mathbf{h}_v,\eta}_{\rm cusp}\end{pmatrix}, \tag{3.16}$$

and similarly as in (3.6) we put

$$\mathcal{BF}^{i,\mathbf{h}_v,\eta}_{\mathrm{cusp,int}} := \varpi^s \cdot \mathcal{BF}^{i,\mathbf{h}_v,\eta}_{\mathrm{cusp}} \in \mathrm{H}^1(\mathbf{Z}[1/p], T(\eta)\hat{\otimes}(\mathbb{T}^{\square}_{\mathbf{h}_v})\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\Gamma)^{\iota}). \tag{3.17}$$

Since Theorem 3.10 together with Shapiro's lemma give isomorphisms

$$\begin{aligned}\mathrm{H}^1(\mathbf{Q}, T(\eta)\otimes(\mathbb{T}^{\square}_{\mathbf{h}_v})\otimes\Lambda_{\mathscr{O}}(\Gamma)^{\iota}) &\cong \mathrm{H}^1(\mathbf{Q}, T(\eta)\otimes \mathrm{Ind}_K^{\mathbf{Q}}(\Lambda_{\mathbf{Z}_p}(\Gamma^v)^{\iota})\otimes\Lambda_{\mathscr{O}}(\Gamma)^{\iota})\\ &\cong \mathrm{H}^1(K, T(\eta)\otimes\Lambda_{\mathscr{O}}(\Gamma_K)^{\iota}),\end{aligned}$$

for every prime $w\in\{v,\overline{v}\}$ of K above p we may consider the image of $\mathrm{res}_w(\mathcal{BF}^{\xi,\mathbf{h}_v,\eta}_{\mathrm{cusp}})$ and $\mathrm{res}_w(\mathcal{BF}^{i,\mathbf{h}_v,\eta}_{\mathrm{cusp}})$ under the maps (3.11) and (3.14).

Definition 3.7. For $\xi,\xi'\in\{\alpha,\beta\}$ and $i,j\in\{1,2\}$ put

$$\begin{aligned}\widetilde{L}^{(\mathrm{I})}_{\xi,\xi'}(f,\mathbf{h}_v)^{\eta} &:= e_\eta\mathscr{L}_{V,\xi,v}(\mathrm{res}_v(\mathcal{BF}^{\xi',\mathbf{h}_v,\eta}_{\mathrm{cusp}})), & \widetilde{L}^{(\mathrm{I})}_{i,j}(f,\mathbf{h}_v)^{\eta} &:= e_\eta\widetilde{\mathscr{C}}_{i,v}(\mathrm{res}_v(\mathcal{BF}^{j,\mathbf{h}_v,\eta}_{\mathrm{cusp,int}})),\\ \widetilde{L}^{(\mathrm{II})}_{\xi,\xi'}(\mathbf{h}_v,f)^{\eta} &:= e_\eta\mathscr{L}_{V,\xi,\overline{v}}(\mathrm{res}_{\overline{v}}(\mathcal{BF}^{\xi',\mathbf{h}_v,\eta}_{\mathrm{cusp}})), & \widetilde{L}^{(\mathrm{II})}_{i,j}(\mathbf{h}_v,f)^{\eta} &:= e_\eta\widetilde{\mathscr{C}}_{i,\overline{v}}(\mathrm{res}_{\overline{v}}(\mathcal{BF}^{j,\mathbf{h}_v,\eta}_{\mathrm{cusp,int}})),\end{aligned}$$

viewed as elements in $\mathcal{H}_L(\Gamma_K)$ and $\Lambda_{\mathscr{O}}(\Gamma_K)$, respectively.

Remark 3.8. For $\xi=\xi'$, we have

$$\widetilde{L}^{(\mathrm{I})}_{\xi,\xi}(f,\mathbf{h}_v)^{\eta} = T_v\cdot L^{(\mathrm{I})}_{\xi,\xi}(f,\mathbf{h}_v)^{\eta},$$

where $L^{(\mathrm{I})}_{\xi,\xi}(f,\mathbf{h}_v)^{\eta}$ is as in Definition 3.5.

In analogy with [Pol03], we have the following decomposition.

Corollary 3.9. *We have*

$$\varpi^s\cdot\begin{pmatrix}\widetilde{L}^{(\mathrm{I})}_{\alpha,\alpha}(f,\mathbf{h}_v)^{\eta} & \widetilde{L}^{(\mathrm{I})}_{\alpha,\beta}(f,\mathbf{h}_v)^{\eta}\\ \widetilde{L}^{(\mathrm{I})}_{\beta,\alpha}(f,\mathbf{h}_v)^{\eta} & \widetilde{L}^{(\mathrm{I})}_{\beta,\beta}(f,\mathbf{h}_v)^{\eta}\end{pmatrix} = Q_{\alpha,\beta}^{-1}M_{\log,v}\cdot\begin{pmatrix}\widetilde{L}^{(\mathrm{I})}_{1,1}(f,\mathbf{h}_v)^{\eta} & \widetilde{L}^{(\mathrm{I})}_{1,2}(f,\mathbf{h}_v)^{\eta}\\ \widetilde{L}^{(\mathrm{I})}_{2,1}(f,\mathbf{h}_v)^{\eta} & \widetilde{L}^{(\mathrm{I})}_{2,2}(f,\mathbf{h}_v)^{\eta}\end{pmatrix}\cdot(Q_{\alpha,\beta}^{-1}M_{\log,v})^{\mathrm{tr}},$$

and similarly

$$\varpi^s\cdot\begin{pmatrix}\widetilde{L}^{(\mathrm{II})}_{\alpha,\alpha}(\mathbf{h}_v,f)^{\eta} & \widetilde{L}^{(\mathrm{II})}_{\alpha,\beta}(\mathbf{h}_v,f)^{\eta}\\ \widetilde{L}^{(\mathrm{II})}_{\beta,\alpha}(\mathbf{h}_v,f)^{\eta} & \widetilde{L}^{(\mathrm{II})}_{\beta,\beta}(\mathbf{h}_v,f)^{\eta}\end{pmatrix} = Q_{\alpha,\beta}^{-1}M_{\log,\overline{v}}\cdot\begin{pmatrix}\widetilde{L}^{(\mathrm{II})}_{1,1}(\mathbf{h}_v,f)^{\eta} & \widetilde{L}^{(\mathrm{II})}_{1,2}(\mathbf{h}_v,f)^{\eta}\\ \widetilde{L}^{(\mathrm{II})}_{2,1}(\mathbf{h}_v,f)^{\eta} & \widetilde{L}^{(\mathrm{II})}_{2,2}(\mathbf{h}_v,f)^{\eta}\end{pmatrix}\cdot(Q_{\alpha,\beta}^{-1}M_{\log,v})^{\mathrm{tr}},$$

where $M_{\log,w}\in M_{2\times 2}(\mathcal{H}_L(\Gamma^w))$ is as in (3.12), and A^{tr} denotes the transpose of a matrix A.

Proof. Immediate from (3.12), (3.16), and (3.17). ☐

3.5.5. *Cuspidality of Beilinson–Flach classes, bis.*

Theorem 3.10. *Suppose $l\in\{1,2\}$ and $\eta:\Delta\to\mathbf{Z}_p^{\times}$ are such that*

$$(L^{(\mathrm{I})}_{l,l}(f,\mathbf{h}_v)^{\eta} \bmod I^-)\neq 0, \tag{3.18}$$

where $I^-=\ker(\Lambda_{\mathscr{O}}(\Gamma_K)\twoheadrightarrow\Lambda_{\mathscr{O}}(\Gamma^+))$. Then the η-component $\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{int}}$ of (3.6) is in the image of the natural injection

$$\mathrm{H}^1(\mathbf{Z}[1/p], T(\eta)\hat{\otimes}(\mathbb{T}^{\square}_{\mathbf{h}_v})\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\Gamma)^{\iota}) \longrightarrow \mathrm{H}^1(\mathbf{Z}[1/p], T(\eta)\hat{\otimes}(\widetilde{\mathbb{T}}^{\square}_{\mathbf{h}_v})\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\Gamma)^{\iota}).$$

Proof. Following the argument in the proof of [BSTW24, Cor. 6.22], the desired result follows from the nonvanishing of the composite map

$$e_\eta\widetilde{\mathscr{C}}_{l,v}\circ\mathrm{res}_v:\mathrm{H}^1_{\mathrm{rel},l}(K,\mathbb{T}_K^{\eta})\longrightarrow\Lambda_{\mathscr{O}}(\Gamma_K),$$

and this follows from Theorem 3.6 and the nonvanishing hypothesis. ☐

In view of Theorem 3.1 and Shapiro's lemma, we consider the composition

$$\begin{aligned}\mathrm{H}^1(\mathbf{Z}[1/p],T(\eta)\hat{\otimes}(\mathbb{T}^{\square}_{\mathbf{h}_v})\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\Gamma)^{\iota}) &\xrightarrow{\simeq} \mathrm{H}^1(\mathcal{O}_K[1/p],T(\eta)\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\Gamma^v)^{\iota}\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\Gamma)^{\iota}) \\ &\xrightarrow{\simeq} \mathrm{H}^1(\mathcal{O}_K[1/p],T(\eta)\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\Gamma_K)^{\iota}).\end{aligned} \tag{3.19}$$

Abusing notation, for $l\in\{1,2\}$ and $\eta:\Delta\to\mathbf{Z}_p^\times$ such that (3.18) holds, we shall still denote by

$$\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{int}}\in\mathrm{H}^1(\mathcal{O}_K[1/p],T(\eta)\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\Gamma_K)^{\iota})$$

the image of the η-component of (3.6) for $i=l$ under the composition (3.19).

Remark 3.11. In our application to Kato's main conjecture, K will be chosen so that

$$L(f\otimes\epsilon_K\omega^{k/2-1},1)\neq 0.$$

For such K, by virtue of [Shi76, Prop. 2] and [LLZ14, Prop. 3.28] the nonvanishing condition (3.18) for $\eta=\omega^{1-k/2}$ is satisfied for all $l\in\{1,2\}$ unless $k=2$ and $a_p\neq 0$, in which case Proposition 2.16 shows that the condition is satisfied for at least some $l\in\{1,2\}$, and this will suffice for our purposes.

3.5.6. *Explicit reciprocity law for $w=\overline{v}$.*

Proposition 3.12. *For every $\xi\in\{\alpha,\beta\}$ and $\eta:\Delta\to\mathbf{Z}_p^\times$, we have*

$$\widetilde{L}^{(\mathrm{II})}_{\xi,\xi}(\mathbf{h}_v,f)^\eta=0.$$

Moreover, if $\xi'\in\{\alpha,\beta\}$ is different from ξ, then

$$\widetilde{L}^{(\mathrm{II})}_{\xi,\xi'}(\mathbf{h}_v,f)^\eta=-\widetilde{L}^{(\mathrm{II})}_{\xi',\xi}(\mathbf{h}_v,f)^\eta.$$

Proof. By Theorem 3.10 we have G_K-module isomorphisms $\mathbb{T}^+_{\mathbf{h}_v}\cong\Lambda_{\mathbf{Z}_p}(\Gamma^v)^\iota$ and $\mathbb{T}^-_{\mathbf{h}_v}\cong c\cdot\mathbb{T}^+_{\mathbf{h}_v}$. Thus by Shapiro's lemma, $\mathrm{res}_{\overline{v}}(\mathcal{BF}^{\xi,\mathbf{h}_v,\eta}_{\mathrm{cusp}})$ is identified with the image of $\mathrm{res}_p(\mathcal{BF}^{\xi,\mathbf{h}_v,\eta}_{\mathrm{cusp}})$ under the natural map

$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\mathbb{T}^\square_{\mathbf{h}_v}))^\eta\longrightarrow\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{\emptyset,-}_{\xi,\mathrm{ord}}\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\mathbb{T}^\square_{\mathbf{h}_v}))^\eta. \tag{3.20}$$

On the other hand, by [LZ16, Thm. 7.1.2] the image of $\mathrm{res}_p(\mathcal{BF}^{\xi,\mathbf{h}_v,\eta}_{\mathrm{cusp}})$ under (3.20) is contained in the image of

$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{+,-}_{\xi,\mathrm{ord}}\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\mathbb{T}^\square_{\mathbf{h}_v}))^\eta\longrightarrow\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{\emptyset,-}_{\xi,\mathrm{ord}}\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\mathbb{T}^\square_{\mathbf{h}_v}))^\eta \tag{3.21}$$

Since under the above identifications the regulator map $e_\eta\mathscr{L}_{V,\xi,\overline{v}}$ factors through

$$\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{\emptyset,-}_{\xi,\mathrm{ord}}\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\mathbb{T}^\square_{\mathbf{h}_v}))^\eta\longrightarrow\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{-,-}_{\xi,\mathrm{ord}}\mathbb{D}^\dagger_{\mathrm{rig}}(V\otimes\mathbb{T}^\square_{\mathbf{h}_v}))^\eta,$$

whose composition with (3.21) is zero, the proof of the first claim follows. The second claim is shown in [BL21, Prop. 3.14]. ∎

Corollary 3.13. *For every $i\in\{1,2\}$ and $\eta:\Delta\to\mathbf{Z}_p^\times$, we have*

$$\widetilde{L}^{(\mathrm{II})}_{i,i}(\mathbf{h}_v,f)^\eta=0.$$

Moreover, writing $Q^{-1}_{\alpha,\beta}M_{\log}=\left(\begin{smallmatrix}\mathfrak{A}&\mathfrak{B}\\\mathfrak{C}&\mathfrak{D}\end{smallmatrix}\right)$ we have

$$\begin{aligned}\widetilde{L}^{(\mathrm{II})}_{1,2}(\mathbf{h}_v,f)^\eta\cdot(\mathfrak{A}\mathfrak{D}-\mathfrak{B}\mathfrak{C})&=-\widetilde{L}^{(\mathrm{II})}_{2,1}(\mathbf{h}_v,f)^\eta\cdot(\mathfrak{A}\mathfrak{D}-\mathfrak{B}\mathfrak{C})\\&=-\widetilde{L}^{(\mathrm{II})}_{\beta,\alpha}(\mathbf{h}_v,f)^\eta\cdot\varpi^s.\end{aligned}$$

Proof. Multiplying the second equality of Corollary 3.9 by $\left(\begin{smallmatrix}\mathfrak{D} & -\mathfrak{B}\\ -\mathfrak{C} & \mathfrak{A}\end{smallmatrix}\right)$ on the left and $\left(\begin{smallmatrix}\mathfrak{D} & -\mathfrak{C}\\ -\mathfrak{B} & \mathfrak{A}\end{smallmatrix}\right)$ on the right, and using Proposition 3.12 yields

$$\begin{aligned}(\mathfrak{A}\mathfrak{D}-\mathfrak{B}\mathfrak{C})^2&\begin{pmatrix}\widetilde{L}^{(\mathrm{II})}_{1,1}(\mathbf{h}_v,f)^\eta & \widetilde{L}^{(\mathrm{II})}_{1,2}(\mathbf{h}_v,f)^\eta\\ \widetilde{L}^{(\mathrm{II})}_{2,1}(\mathbf{h}_v,f)^\eta & \widetilde{L}^{(\mathrm{II})}_{2,2}(\mathbf{h}_v,f)^\eta\end{pmatrix}\\ &=\begin{pmatrix}0 & -(\mathfrak{A}\mathfrak{D}-\mathfrak{B}\mathfrak{C})\cdot\widetilde{L}^{(\mathrm{II})}_{\beta,\alpha}(\mathbf{h}_v,f)^\eta\cdot\varpi^s\\ (\mathfrak{A}\mathfrak{D}-\mathfrak{B}\mathfrak{C})\cdot\widetilde{L}^{(\mathrm{II})}_{\beta,\alpha}(\mathbf{h}_v,f)^\eta\cdot\varpi^s & 0\end{pmatrix},\end{aligned}$$

whence the result (cf. [BL21, Cor. 3.15]). □

Corollary 3.14. *Suppose $l\in\{1,2\}$ and $\eta:\Delta\to\mathbf{Z}_p^\times$ are such that (3.18) holds. Then $\widetilde{L}^{(\mathrm{II})}_{1,2}(\mathbf{h}_v,f)^\eta=-\widetilde{L}^{(\mathrm{II})}_{2,1}(\mathbf{h}_v,f)^\eta$ are divisible by T_v.*

Proof. Note that the stated equality follows from Corollary 3.13. By Theorem 3.10, under the stated assumption the class $\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{int}}$ is naturally contained in $\mathrm{H}^1(\mathcal{O}_K[1/p],T(\eta)\hat{\otimes}\Lambda_{\mathbf{Z}_p}(\Gamma_K)^\iota)$, and so $e_\eta\widetilde{\mathscr{C}}_{l',\overline{v}}$, for any $l'\in\{1,2\}$, may be evaluated at $\mathrm{res}_{\overline{v}}(\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{int}})$. Since by definition (see (3.15) and (3.16)) we have

$$T_v\cdot e_\eta\widetilde{\mathscr{C}}_{l',\overline{v}}(\mathrm{res}_{\overline{v}}(\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{int}}))=e_\eta\widetilde{\mathscr{C}}_{l',\overline{v}}(\mathrm{res}_{\overline{v}}(\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{cusp,int}}))=\widetilde{L}^{(\mathrm{II})}_{l',l}(\mathbf{h}_v,f)^\eta,$$

together with Corollary 3.13 this yields the result. □

Definition 3.15. In light of Corollary 3.14, for every character $\eta:\Delta\to\mathbf{Z}_p^\times$ we put

$$L^{(\mathrm{II})}_{1,2}(\mathbf{h}_v,f)^\eta:=\frac{\widetilde{L}^{(\mathrm{II})}_{1,2}(\mathbf{h}_v,f)^\eta}{T_v}\in\Lambda_{\mathscr{O}}(\Gamma_K).$$

Remark 3.16. As shown in [LLZ11, Cor. 3.2], the determinant of $M_{\log}$ is given up to a p-adic unit by

$$\det(M_{\log})\sim_p\frac{\log_{p,k-1}}{\delta_{k-1}}\sim_p\frac{p^{k-1}\prod_{j=1}^{k-1}\ell_{-j}}{\delta_{k-1}},$$

where $\ell_{-j}=\frac{\log_p(\gamma)}{\log_p\varepsilon(\gamma)}+j$ for $\gamma\in\Gamma$ any topological generator. Hence Corollary 3.13 shows that

$$\widetilde{L}^{(\mathrm{II})}_{i,j}(\mathbf{h}_v,f)^\eta\sim_p\frac{e_\eta\mathscr{L}_{V,\beta,\overline{v}}(\mathrm{res}_{\overline{v}}(\mathcal{BF}^{\alpha,\mathbf{h}_v,\eta}_{\mathrm{cusp}}))}{(\alpha-\beta)\cdot\prod_{j=1}^{k-1}\ell_{-j}}\cdot\delta_{k-1}\cdot\varpi^s.$$

To describe the interpolation property of $L_{1,2}(\mathbf{h}_v,f)^\eta$, put

$$\Xi^{(\mathrm{II})}:=\left\{(\chi_1,\chi_2):\Gamma^v\hat{\otimes}\Gamma\to\overline{\mathbf{Q}}_p^\times\;\middle|\;\begin{array}{c}\chi_1(\gamma_v^{h_p})=\langle\varepsilon(\gamma)\rangle^m,\;\;m\geq 1,\;\;m\equiv 0\ (\mathrm{mod}\ p-1)\\ \chi_2=\psi_\zeta\langle\varepsilon\rangle^n,\;\;0\leq n<m,\;\;\zeta\in\mu_{p^\infty}\end{array}\right\}.$$

Theorem 3.17 (Explicit Reciprocity Law II)**.** *For all $\chi=(\chi_1,\chi_2)\in\Xi^{(\mathrm{II})}$ with $\chi_2=\psi_\zeta\langle\varepsilon\rangle^n$ for a primitive p^t-th root of unity ζ for some $t\geq 0$, we have*

$$\begin{aligned}\phi_\chi\big(L^{(\mathrm{II})}_{1,2}(\mathbf{h}_v,f)^\eta\big)=\frac{\mathcal{E}_{p,\eta}(\mathbf{h}_v,f,\chi)}{\mathcal{E}_p(\mathbf{h}_v)\cdot\mathcal{E}_p^*(\mathbf{h}_v)}&\cdot\frac{n!(n+1-k)!}{\pi^{2n+3-k}\cdot(-i)^{m+1-k}\cdot 2^{2n+m+3-k}}\cdot\phi_\chi\left(\frac{c\cdot\omega_{\mathbf{h}_v}}{\eta_{\mathbf{h}_v}}\right)\cdot\phi_\chi(\delta_{k-1})\\ \times&\frac{L(\mathbf{h}^\circ_{v,\chi_1},f\otimes\eta\omega^n\psi_\zeta^{-1},n+1)}{\langle\mathbf{h}^\circ_{v,\chi_1},\mathbf{h}^\circ_{v,\chi_1}\rangle_{D_K}}\cdot\frac{\varpi^s}{[\varphi(\omega_f),\omega_f]_{\mathrm{dR}}},\end{aligned}$$

where

- $$\mathcal{E}_{p,\eta}(\mathbf{h}_v,f,\chi)=\begin{cases}\Big(1-\frac{p^n}{\chi_1(\overline{v})\alpha}\Big)\Big(1-\frac{p^n}{\chi_1(\overline{v})\beta}\Big)\Big(1-\frac{p^m\alpha}{p^{n+1}\chi_1(\overline{v})}\Big)\Big(1-\frac{p^m\beta}{p^{n+1}\chi_1(\overline{v})}\Big),\\ \Big(\frac{p^{t+1}}{\mathfrak{g}(\eta\omega^n\psi_\zeta^{-1})}\Big)^2\Big(\frac{p^{2n+1-k}}{\chi_1(\overline{v})^2}\Big)^{t+1},\end{cases}$$

according to whether or not $\eta = \omega^{-n}$ *and* $t = 0$*,*

- $\mathcal{E}_p(\mathbf{h}_v) = \left(1 - \frac{p^{m-1}}{\chi_1(\overline{v})^2}\right)$,
- $\mathcal{E}_p^*(\mathbf{h}_v) = \left(1 - \frac{p^{m}}{\chi_1(\overline{v})^2}\right)$,

and $\mathbf{h}^\circ_{v,\chi_1}$ *is the newform of level* D_K *and weight* $m+1$ *with ordinary* p*-stabilization* $\chi_1(\mathbf{h}_v)$*.*

Proof. With the same notations as in the proof of Theorem 3.6, but reversing the roles of $\mathbf{f}^\beta$ and $\mathbf{h}_v$, let $L_p^{\mathrm{Urb}}(\mathbf{h}_v, \mathbf{f}^\beta, 1+\mathbf{j})$ be Urban's 3-variable p-adic Rankin L-function, and put

$$L_p^{\mathrm{geo}}(\mathbf{h}_v, \mathbf{f}^\beta, 1+\mathbf{j}) := (-1)^{1+\mathbf{j}}\lambda_N(\mathbf{h}_v)\cdot\left\langle \mathcal{L}(\mathcal{BF}^{[\mathbf{f}^\beta,\mathbf{h}_v]}_{1,1}), \omega_{\mathbf{f}^\xi}\otimes\eta_{\mathbf{h}_v}\right\rangle$$

for the geometric p-adic L-function of Loeffler–Zerbes [LZ16]; so Theorem 7.1.5 in *op. cit.* gives

$$L_p^{\mathrm{geo}}(\mathbf{h}_v, \mathbf{f}^\beta, 1+\mathbf{j}) = L_p^{\mathrm{Urb}}(\mathbf{h}_v, \mathbf{f}^\beta, \mathbf{h}_v, 1+\mathbf{j}).$$

Letting $L_p^{\mathrm{geo}}(\mathbf{h}_v, f^\beta, 1+\mathbf{j})$ be the image of $L_p^{\mathrm{geo}}(\mathbf{h}_v, \mathbf{f}^\beta, 1+\mathbf{j})$ under the specialisation map at f^β, by the interpolation property of $L_p^{\mathrm{Urb}}(\mathbf{h}_v, \mathbf{f}^\beta, \mathbf{h}_v, 1+\mathbf{j})$, it follows that for all characters $\eta : \Delta \to \mathbf{Z}_p^\times$ and $\chi = (\chi_1, \chi_2) \in \Xi^{(\mathrm{II})}$ as in the statement, we have

$$\begin{aligned}\phi_{\eta\chi}\big(L_p^{\mathrm{geo}}(\mathbf{h}_v, f^\beta, 1+\mathbf{j})\big) &= \frac{\mathcal{E}_{p,\eta}(\mathbf{h}_v, f, \chi)}{\mathcal{E}_p(\mathbf{h}_v)\cdot\mathcal{E}_p^*(\mathbf{h}_v)}\cdot\frac{n!(n+1-k)!}{\pi^{2n+3-k}\cdot(-i)^{m+1-k}\cdot 2^{2n+m+3-k}}\\ &\quad\times\frac{L(\mathbf{h}^\circ_{v,\chi_1}, f\otimes\eta\omega^n\psi_\zeta^{-1}, n+1)}{\langle\mathbf{h}^\circ_{v,\chi_1}, \mathbf{h}^\circ_{v,\chi_1}\rangle_{D_K}}.\end{aligned} \tag{3.22}$$

Let ω_{f^β} denote the image of $\omega_{\mathbf{f}^\beta}$ under the specialisation map at f^β. Comparing the interpolation formulas of the maps $\mathscr{L}_V$ and $\mathcal{L}$ (as given in [LZ14, Thm. 4.15] and [LZ16, Thm. 7.1.4], respectively), and noting the relation up to a p-adic unit

$$\omega_{f^\beta} = \frac{[\varphi(\omega_f), \omega_f]_{\mathrm{dR}}}{\alpha-\beta}\cdot v_\beta^*,$$

as follows from a straightforward computation (and also noted in the proof of [BLLV19, Thm. 3.6.5]), from Remark 3.16 we thus find that

$$\begin{aligned}\phi_\chi(L_{1,2}^{(\mathrm{II})}(\mathbf{h}_v, f)^\eta) &= \phi_\chi\left(\frac{e_\eta\mathscr{L}_{V,\beta,\overline{v}}(\mathcal{BF}^{\alpha,\mathbf{h}_v,\eta}_{\mathrm{cusp}})}{T_v\cdot\prod_{j=1}^{k-1}\ell_{-j}}\right)\cdot\phi_\chi(\delta_{k-1})\cdot\frac{\varpi^s}{(\alpha-\beta)}\\ &= \phi_{\eta\chi}\left(\frac{(-1)^{1+\mathbf{j}}}{\lambda_N(\mathbf{h}_v)}\cdot L_p^{\mathrm{geo}}(\mathbf{h}_v, f^\beta, 1+\mathbf{j})\cdot\frac{c\cdot\omega_{\mathbf{h}_v}}{\eta_{\mathbf{h}_v}}\right)\cdot\phi_\chi(\delta_{k-1})\cdot\frac{\varpi^s}{[\varphi(\omega_f), \omega_f]_{\mathrm{dR}}}\end{aligned}$$

using that $\phi_\chi(\mathcal{BF}^{\alpha,\mathbf{h}_v,\eta}_{\mathrm{cusp}}) = \phi_\chi(T_v)\cdot\phi_\chi(\mathcal{BF}^{\alpha,\mathbf{h}_v,\eta})$ by definition, with $\phi_\chi(T_v)\neq 0$, for the last equality. Together with (3.22), this gives the result. □

We conclude this section with the following observation on the nonvanishing on $L_{i,j}^{(\mathrm{II})}(\mathbf{h}_v, f)^\eta$, which lies much less deep than that of $L_{i,i}^{(\mathrm{I})}(f, \mathbf{h}_v)^\eta$.

Corollary 3.18. *For* $i \neq j \in \{1,2\}$ *and* $\eta : \Delta \to \mathbf{Z}_p^\times$ *a character,* $L_{i,j}^{(\mathrm{II})}(\mathbf{h}_v, f)^\eta$ *is nonzero.*

Proof. One just needs to note that the range of p-adic interpolation in Theorem 3.17 contains values $L(\mathbf{h}^\circ_{v,\chi_1}, f\otimes\eta\omega^n\psi_\zeta^{-1}, n+1)$ for which the defining Euler product is absolutely convergent (cf. [BL21, Prop. 4.7]). □

4. Descent

In this section we relate the two-variable p-adic L-functions $L_{i,j}^{(\mathrm{II})}(\mathbf{h}_v, f)^\eta$ for $i \neq j$ (resp. $L_{i,i}^{(\mathrm{I})}(f, \mathbf{h}_v)^\eta$) to certain anticyclotomic (resp. cyclotomic and anticyclotomic) p-adic L-functions constructed independently.

4.1. **Anticyclotomic descent: Indefinite.** Let $\pi_f = \otimes_{\ell \leq \infty} \pi_{f,\ell}$ be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by f, and let $BC_K(\pi_f)$ denote its base change to $\mathrm{GL}_2(\mathbb{A}_K)$. Write

$$N = N^+ N^-$$

with N^+ (resp. N^-) divisible only by primes that are split or ramified (resp. inert) in K. Throughout this section, the we assume the following *generalised Heegner hypotheses*:

$$\text{(gen-H)} \qquad N^- = 1 \quad \text{and} \quad \varepsilon_v(BC_K(\pi_f)) = +1 \text{ for all } v \mid N^+ \text{ nonsplit in } K,$$

where $\varepsilon_v(BC_K(\pi_f))$ is the local root number at v (normalised as in [Hsi14, p. 710]).

The action of complex conjugation yields an eigenspace decomposition

$$\Gamma_K \cong \Gamma^+ \times \Gamma^-,$$

where Γ^+ (resp. Γ^-) is identified with the Galois group of the cyclotomic $\mathbf{Z}_p$-extension K_∞^+/K (resp. the anticyclotomic $\mathbf{Z}_p$-extension K_∞^-/K).

The following is a refinement and generalisation of the anticyclotomic p-adic L-function introduced by Bertolini–Darmon–Prasanna [BDP13].

Theorem 4.1. *There exists a unique "square-root" p-adic L-function*

$$\mathscr{L}_{\overline{v}}^{\mathrm{BDP}}(f/K) \in \Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma^-)$$

such that for every character ξ of Γ^- crystalline at both v and $\overline{v}$ corresponding to a Hecke character of K of infinity type $(-j, j)$ for some $j \geq k/2$ with $j \equiv 0 \pmod{p-1}$ we have

$$\mathscr{L}_{\overline{v}}^{\mathrm{BDP}}(f/K)^2(\xi) = \left(\frac{\Omega_p}{\Omega_\infty}\right)^{4j} \cdot (k/2 + j - 1)!(j - k/2)! \cdot \left(\frac{2\pi}{\sqrt{D_K}}\right)^{2j-1}$$
$$\times (1 - a_p \xi(v) p^{-k/2} + \xi(v)^2 p^{-1})^2 \cdot L(f/K, \xi, k/2),$$

where $(\Omega_\infty, \Omega_p) = (2\pi i \cdot \Omega_K, \Omega_p) \in \mathbf{C}^\times \times \mathcal{O}_{\mathbf{C}_p}^\times$ are CM periods as in [CH18a, §2.5].

Proof. This follows from the results of [CH18a, §3] as in Theorem 2.1.3 and Remark 2.1.4 of [Cas25] (whose notations we have largely adopted). ☐

We also need to recall the Katz p-adic L-function from [Kat78, dS87]. With notations as in [Cas25, Thm. 2.2.1], this is an element $\mathscr{L}_{\overline{v}}^{\mathrm{Katz}} \in \Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma_K)$ such that for every character ξ of Γ_K crystalline at both v and $\overline{v}$ corresponding to a Hecke character of infinity type (b, a) with $a < 0$ and $b \geq 1$ satisfies

$$\text{(4.1)} \qquad \mathscr{L}_{\overline{v}}^{\mathrm{Katz}}(\xi) = \left(\frac{\Omega_p}{\Omega_\infty}\right)^{a-b} \cdot (a-1)! \cdot \left(\frac{\sqrt{D_K}}{2\pi}\right)^b \cdot (1 - \xi^{-1}(\overline{v}) p^{-1})(1 - \xi(v)) \cdot L(\xi, 0).$$

Moreover, as shown in [dS87, Thm. II.6.4] we have the functional equation

$$\text{(4.2)} \qquad \mathscr{L}_{\overline{v}}^{\mathrm{Katz}}(\xi) = \mathscr{L}_{\overline{v}}^{\mathrm{Katz}}(\xi^{-1} \mathbf{N}^{-1}),$$

where the equality is up to a p-adic unit.

Definition 4.2. Denote by $\mathscr{L}_{\overline{v}}^{\mathrm{Katz},-}$ the image of $\mathscr{L}_{\overline{v}}^{\mathrm{Katz}}$ under map $\Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma_K) \to \Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma^-)$ induced by sending $\gamma \in \Gamma_K$ to the image of γ^{1-c} in Γ^-, and for every character $\eta : \Delta \to \mathbf{Z}_p^\times$ put

$$\mathcal{L}_{1,2}^{(\mathrm{II})}(f/K)^\eta := \frac{h_K}{w_K} \cdot \mathscr{L}_{\overline{v}}^{\mathrm{Katz},-} \cdot \frac{\eta_{\mathbf{h}_v}}{c \cdot \omega_{\mathbf{h}_v}} \cdot \frac{[\varphi(\omega_f), \omega_f]_{\mathrm{dR}}}{\delta_{k-1}} \cdot L_{1,2}^{(\mathrm{II})}(\mathbf{h}_v, f)^\eta,$$

where h_K is the class number of K and $w_K = \#(\mathcal{O}_K^\times)/2$.

Proposition 4.3. *The p-adic L-function $\varpi^{-s} \cdot \mathcal{L}_{1,2}^{(\mathrm{II})}(f/K)^{\eta}$ is integral, i.e. lives in $\Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma_K)$.*

Proof. This follows from a direct comparison with the interpolation property of the p-adic L-function $\mathcal{L}_{\overline{v}}(f/K)^{\eta} \in \Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma_K)$ in [CLW22, Prop. 8.2.2], yielding the equality

$$\varpi^{-s} \cdot \mathcal{L}_{1,2}^{(\mathrm{II})}(f/K)^{\eta} = \mathcal{L}_{\overline{v}}(f/K)^{\eta}$$

up to a unit. The details are virtually the same as in Proposition 4.4 below. □

Proposition 4.4. *Let I^+ be the kernel of the natural projection $\Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma_K) \to \Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma^-)$. Then*

$$\left(\varpi^{-s} \cdot \mathrm{Tw}_{k/2-1}(\mathcal{L}_{1,2}^{(\mathrm{II})}(f/K)^{\omega^{1-k/2}}) \bmod I^+\right) = \left(\mathscr{L}_{\overline{v}}^{\mathrm{BDP}}(f/K)^2\right)$$

as ideals in $\Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma^-)$, where $\mathrm{Tw}_{k/2-1} : \Lambda_{\mathscr{O}}(\Gamma_K) \to \Lambda_{\mathscr{O}}(\Gamma_K)$ denotes the $\mathscr{O}$-linear isomorphism given by $\gamma \mapsto \langle \varepsilon(\gamma) \rangle^{k/2-1}\gamma$ for $\gamma \in \Gamma^+$ and the identity on Γ^-.

Proof. This follows similarly as in [CGS25, Prop. 2.4.5], but we provide the details for the convenience of the reader. For $\eta = \omega^{1-k/2}$, $\chi = (\chi_1, \chi_2) \in \Xi^{(\mathrm{II})}$ with $m = 2j$ for $j \geq k/2$ with $j \equiv 0 \pmod{p-1}$, $n = k/2 + j - 1$, and $\zeta = 1$, the interpolation in Theorem 3.17 becomes

$$\begin{aligned}\phi_\chi(\mathrm{Tw}_{k/2-1}(L_{1,2}^{(\mathrm{II})}(\mathbf{h}_v, f)^{\omega^{1-k/2}})) = \frac{1}{(\mathrm{denom})} \cdot \left(1 - \frac{p^{j-k/2}\alpha}{\chi_1(\overline{v})}\right)^2 \left(1 - \frac{p^{j-k/2}\beta}{\chi_1(\overline{v})}\right)^2 \\ \times (k/2+j-1)!(j-k/2)! \cdot L(\mathbf{h}_{v,\chi_1}^\circ, f, k/2+j),\end{aligned} \tag{4.3}$$

where

$$(\mathrm{denom}) = \pi^{2j+1} \cdot (-i)^{2j+1+k} \cdot 2^{4j+1} \cdot \left(1 - \frac{p^{2j-1}}{\chi_1(\overline{v})^2}\right)\left(1 - \frac{p^{2j}}{\chi_1(\overline{v})^2}\right) \cdot \langle \mathbf{h}_{v,\chi_1}^\circ, \mathbf{h}_{v,\chi_1}^\circ \rangle.$$

Composed with the projection $\Gamma_K \twoheadrightarrow \Gamma^v$, the character $\chi_1 \varepsilon^j$ corresponds to a Hecke character ξ of infinity type $(-j, j)$ with

$$\left(1 - \frac{p^{2j-1}}{\chi_1(\overline{v})^2}\right)\left(1 - \frac{p^{2j}}{\chi_1(\overline{v})^2}\right) = \left(1 - \xi(v)^2 p^{-1}\right)\left(1 - \xi(v)^2\right). \tag{4.4}$$

In terms of ξ, Hida's adjoint L-value formula (see [HT93, Thm. 7.1]) combined with Dirichlet's class number formula give the equality up to a p-adic unit

$$\langle \mathbf{h}_{v,\chi_1}^\circ, \mathbf{h}_{v,\chi_1}^\circ \rangle \sim_p (2j)! \cdot \frac{1}{2^{4j-1} \cdot \pi^{2j+1}} \cdot \frac{h_K}{w_K} \cdot L(\xi^2, 1),$$

where we used the fact that $\xi/\xi^\tau = \xi^2$. Thus together with (4.1) (expressing $L(\xi^2,1)$ in terms of the value of $\mathscr{L}_{\overline{v}}^{\mathrm{Katz}}$ at $\xi^2\mathbf{N}^{-1}$, of infinity type $(1-2j, 1+2j)$) and (4.4) we obtain

$$\begin{aligned}(\mathrm{denom}) &= \left(1 - \xi(v)^2 p^{-1}\right)\left(1 - \xi(v)^2\right) \cdot \langle \mathbf{h}_{v,\chi_1}^\circ, \mathbf{h}_{v,\chi_1}^\circ \rangle \cdot \pi^{2j+1} \cdot (-i)^{2j+1-k} \cdot 2^{4j+1} \\ &= \frac{\frac{h_K}{w_K} \cdot \mathscr{L}_{\overline{v}}^{\mathrm{Katz}}(\xi^2\mathbf{N}^{-1}) \cdot (-i)^{2j+1-k} \cdot 2^4}{\left(\frac{\Omega_p}{\Omega_\infty}\right)^{4j} \cdot \left(\frac{2\pi}{\sqrt{D_K}}\right)^{2j-1}}.\end{aligned} \tag{4.5}$$

Taking the product of (4.3) and (4.5), using the relation

$$\left(1 - \frac{p^{j-k/2}\alpha}{\chi_1(\overline{v})}\right)^2 \left(1 - \frac{p^{j-k/2}\beta}{\chi_1(\overline{v})}\right)^2 = \left(1 - a_p\xi(v)p^{-k/2} + \xi(v)^2 p^{-1}\right)^2$$

and the functional equation (4.2), the interpolation property in Theorem 4.1 yields the result. □

4.2. Cyclotomic descent.

Definition 4.5. For $\xi \in \{\alpha, \beta\}$, $i \in \{1,2\}$, and a character $\eta : \Delta \to \mathbf{Z}_p^\times$, put

$$\mathcal{L}^{(\mathrm{I})}_{\xi,\xi}(f/K)^\eta := c_f \cdot L^{(\mathrm{I})}_{\xi,\xi}(f, \mathbf{h}_v)^\eta, \qquad \mathcal{L}^{(\mathrm{I})}_{i,i}(f/K)^\eta := c_f \cdot L^{(\mathrm{I})}_{i,i}(f, \mathbf{h}_v)^\eta,$$

where c_f is the congruence number of f.

Remark 4.6. Note that the specialisation of $\mathbf{h}_v$ at the trivial character is the ordinary p-stabilisation of the weight one Eisenstein series

$$\mathbf{h}^\circ_{v,1} = E_1(1, \epsilon_K).$$

Thus from Theorem 3.6 we see that the L-values interpolated by $\phi_\chi(\mathcal{L}^{(\mathrm{I})}_{\xi,\xi}(f/K)^\eta)$ for $\chi = (1, \chi_2) \in \Xi^{(\mathrm{I})}$ with $\chi_2 = \psi_\zeta \langle \varepsilon \rangle^j$ (for $0 \le j \le k-2$ and $\zeta \in \mu_{p^\infty}$) are given by

$$\frac{L(f \otimes \omega^j \eta \psi_\zeta^{-1}, j+1) \cdot L(f \otimes \epsilon_K \omega^j \eta \psi_\zeta^{-1}, j+1)}{\pi^{2j-k+2} \cdot (-i)^{k-1} \cdot 2^{2j-k+1} \cdot \Omega_f^{\mathrm{can}}}, \tag{4.6}$$

where

$$\Omega_f^{\mathrm{can}} := \frac{(4\pi)^k \cdot \langle f, f \rangle_N}{c_f}$$

is Hida's canonical period (see e.g. [CH18b, §6]).

In the following, we shall often make the identification $\Gamma^+ \cong \Gamma$.

Proposition 4.7. *Let I^- be the kernel of the natural projection $\Lambda_L(\Gamma_K) \twoheadrightarrow \Lambda_L(\Gamma)$. Then for every $i \in \{1,2\}$ and $\eta : \Delta \to \mathbf{Z}_p^\times$ we have*

$$\left(\varpi^{-s} \cdot \mathcal{L}^{(\mathrm{I})}_{i,i}(f/K)^\eta \bmod I^-\right) = \left(L_i(f)^\eta \cdot L_i(f \otimes \epsilon_K)^\eta\right)$$

as ideals in $\Lambda_{\mathscr{O}}(\Gamma)$.

Proof. This can be shown in different ways (see e.g. [Lei25] for an approach in the weight 2 case with $v_p(a_p) > 1/p$). Here we adapt the argument given in [IS24, Lem. 4.23] for rational elliptic curves.

Write $Q^{-1}_{\alpha,\beta} M_{\log} = \left(\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right)$. The decomposition (2.13) and its analogue for $f^K := f \otimes \epsilon_K$ give

$$\begin{aligned} L_\alpha(f)^\eta \cdot L_\alpha(f^K)^\eta = \mathfrak{A}^2 \cdot L_1(f)^\eta \cdot L_1(f^K)^\eta + \mathfrak{A}\mathfrak{B} \cdot \left(L_1(f)^\eta \cdot L_2(f^K)^\eta + L_2(f)^\eta \cdot L_1(f^K)^\eta\right) \\ + \mathfrak{B}^2 \cdot L_2(f)^\eta \cdot L_2(f^K)^\eta. \end{aligned} \tag{4.7}$$

On the other hand, from Corollary 3.9 we have

$$\begin{aligned} \varpi^s \cdot \mathcal{L}^{(\mathrm{I})}_{\alpha,\alpha}(f/K)^\eta = \mathfrak{A}^2 \cdot \mathcal{L}^{(\mathrm{I})}_{1,1}(f/K)^\eta + \mathfrak{A}\mathfrak{B} \cdot \left(\mathcal{L}^{(\mathrm{I})}_{1,2}(f/K)^\eta + \mathcal{L}^{(\mathrm{I})}_{2,1}(f/K)^\eta\right) \\ + \mathfrak{B}^2 \cdot \mathcal{L}^{(\mathrm{I})}_{2,2}(f/K)^\eta. \end{aligned} \tag{4.8}$$

By [Bel12] and [LZ16], respectively, there exist p-adic L-functions interpolating $L_\alpha(f)^\eta$ and $L_\alpha(f^K)^\eta$ (resp. $\mathcal{L}^{(\mathrm{I})}_{\alpha,\alpha}(f/K)^\eta$) along the Coleman family passing through the p-stabilisation f^α of f with U_p-eigenvalue α, and a direct comparison of the interpolation formulas in Theorem 2.15 and Theorem 3.6, using the equalities up to a p-adic unit

$$\Omega_f^{\mathrm{can}} \sim_p \Omega^+_{\omega_f} \cdot \Omega^-_{\omega_f}, \qquad \Omega^\pm_{\omega_f} \sim_p \Omega^\mp_{\omega_{f^K}}$$

(see [DDT94, §4.4] and [SZ14, Lem. 9.6]) and Remark 4.6 shows that

$$\left(\mathcal{L}^{(\mathrm{I})}_{\xi,\xi}(f/K)^\eta \bmod I^-\right) = \left(L_\xi(f)^\eta \cdot L_\xi(f^K)^\eta\right). \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we thus see that it suffices to show that $\mathfrak{A}^2$, $\mathfrak{A}\mathfrak{B}$, and $\mathfrak{B}^2$ are linearly independent over $\Lambda_{\mathscr{O}}(\Gamma)$. Since $\mathfrak{A}$ and $\mathfrak{B}$ have only finitely many zeros in common (indeed, this follows from the fact that by [LLZ11, Cor. 3.2] such common zeroes are of the form $z = \langle \varepsilon(\gamma) \rangle^j \zeta$

for $0 \leq j \leq k-2$ and ζ a p-power root of unity, but the relation $L_\alpha(f)^\eta = \mathfrak{A} \cdot L_1(f)^\eta + \mathfrak{B} \cdot L_2(f)^\eta$ from (2.8) and the main result of [Roh88] imply that $\mathfrak{A}$ and $\mathfrak{B}$ can vanish simultaneously at only finitely many such z), the $\Lambda_{\mathscr{O}}(\Gamma)$-linear independence of $\mathfrak{A}^2$, $\mathfrak{A}\mathfrak{B}$, and $\mathfrak{B}^2$ is readily checked by a case by case analysis, whence the result. □

Remark 4.8. Note that *a priori* $\varpi^{-s} \cdot \mathcal{L}_{i,i}^{(\mathrm{I})}(f/K)$ has bounded denominators, i.e. it lands in $\Lambda_L(\Gamma_K)$ rather than $\Lambda_{\mathscr{O}}(\Gamma_K)$. Proposition 4.7 above shows that its projection to $\Lambda_L(\Gamma)$ is contained in $\Lambda_{\mathscr{O}}(\Gamma)$, and later we shall show that in fact $\varpi^{-s} \cdot \mathcal{L}_{i,i}^{(\mathrm{I})}(f/K) \in \Lambda_{\mathscr{O}}(\Gamma_K)$ (see the end of proof of Corollary 5.5).

Recall that by Proposition 2.16 for every $\eta : \Delta \to \mathbf{Z}_p^\times$ we have $L_i(f)^\eta \neq 0$ for some $i \in \{1, 2\}$.

Corollary 4.9. *Suppose $i \in \{1, 2\}$ and $\eta : \Delta \to \mathbf{Z}_p^\times$ are such that $L(f \otimes \epsilon_K \eta, 1) \neq 0$ and $L_i(f)^\eta \neq 0$. Then*

$$(\mathcal{L}_{i,i}^{(\mathrm{I})}(f/K)^\eta \bmod I^-) \neq 0,$$

and so condition (3.18) holds.

Proof. By [LLZ10, Prop. 3.28], the p-adic L-function $L_i(f \otimes \epsilon_K)^\eta$ interpolates a nonzero multiple of $L(f \otimes \epsilon_K \eta, 1)$ at the trivial character, so Proposition 4.7 yields the result. □

Remark 4.10. As is well-known (see e.g. [Shi76, Prop. 2]), the condition in Corollary 4.9 is automatic unless $k = 2$.

Replacing K_∞ by K_∞^+ and $e_\eta \widetilde{\mathscr{C}}_{i,w}$ by its reduction modulo I^-, the Selmer groups $X_{i,i}(f_\eta/K_\infty^+)$ are defined in the same manner as in §3.4. We conclude this section with a counterpart of Proposition 4.7 for algebraic p-adic L-functions (see Corollary 4.13 below).

Proposition 4.11. *Let $i \in \{1, 2\}$ and $\eta : \Delta \to \mathbf{Z}_p^\times$ be a character. Then*

$$\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X_{i,i}(f_\eta/K_\infty^+)\big) = \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\big) \cdot \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X_i(f \otimes \epsilon_K/\mathbf{Q}(\mu_{p^\infty}))^\eta\big).$$

Proof. Let Σ be the set of rational primes dividing N. We begin by showing that the restriction map from $G_{\mathbf{Q}}$ to G_K yields a $\Lambda_{\mathscr{O}}(\Gamma)$-module isomorphism

$$X_{i,i}^\Sigma(f_\eta/K_\infty^+) \cong X_i^\Sigma(f/\mathbf{Q}(\mu_{p^\infty}))^\eta \oplus X_i^\Sigma(f \otimes \epsilon_K/\mathbf{Q}(\mu_{p^\infty}))^\eta \tag{4.10}$$

for the Σ-imprimitive variants (obtained by removing the local conditions at the primes above Σ) of the Selmer groups in the statement.

The action of complex conjugation yields a decomposition

$$\mathrm{H}^1(K, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota) = \mathrm{H}^1(K, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota)^+ \oplus \mathrm{H}^1(K, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota)^-,$$

and from inflation-restriction we see that the restriction map from $G_{\mathbf{Q}}$ to G_K induces $\Lambda_{\mathscr{O}}(\Gamma)$-module isomorphisms

$$\mathrm{H}^1(\mathbf{Q}, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota) \cong \mathrm{H}^1(K, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota)^+,$$
$$\mathrm{H}^1(\mathbf{Q}, T(\eta) \otimes \epsilon_K \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota) \cong \mathrm{H}^1(K, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota)^-.$$

Thus, putting $K_p = K \otimes_{\mathbf{Q}} \mathbf{Q}_p = K_v \oplus K_{\overline{v}}$, on which $\mathrm{Gal}(K/\mathbf{Q})$ act by interchanging the factors, it suffices to show that the restriction map from $G_{\mathbf{Q}}$ to G_K induces $\Lambda_{\mathscr{O}}(\Gamma)$-module isomorphisms

$$\mathrm{H}^1_i(\mathbf{Q}_p, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota) \cong \mathrm{H}^1_i(K_p, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota)^+,$$
$$\mathrm{H}^1_i(\mathbf{Q}_p, T(\eta) \otimes \epsilon_K \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota) \cong \mathrm{H}^1_i(K_p, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota)^-;$$

but this is clear from the fact that

$$z_p = (z_v, z_{\overline{v}}) \in \mathrm{H}^1(K_p, T(\eta) \otimes \Lambda_{\mathscr{O}}(\Gamma)^\iota)^\pm \iff c \cdot z_v = \pm z_v,$$

and via the identification $K_v = \mathbf{Q}_p$ we have $\mathrm{H}^1(K_v, T(\eta)\otimes\Lambda_{\mathscr{O}}(\Gamma)^\iota) = \ker(e_\eta\widetilde{\mathscr{C}}_i)$, and $\mathrm{H}^1(K_{\overline{v}}, T(\eta)\otimes\Lambda_{\mathscr{O}}(\Gamma)^\iota) = \ker(e_\eta\widetilde{\mathscr{C}}_i\circ c)$. Thus we have (4.10), and so

$$\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X^{\Sigma}_{i,i}(f_\eta/K^+_\infty)\big) = \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X^{\Sigma}_{i}(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\big)\cdot \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X^{\Sigma}_{i}(f\otimes\epsilon_K/\mathbf{Q}(\mu_{p^\infty}))^\eta\big).$$

That this equality of characteristic ideals then also holds for the primitive Selmer groups (interpreted as the equality "0 = 0" when either of the Selmer groups involved is not $\Lambda_{\mathscr{O}}(\Gamma)$-torsion) follows by a standard application of [GV00, Prop. 2.4] as in e.g. the proof of [JSW17, Thm. 6.1.6]. □

Proposition 4.12. *Let $i\in\{1,2\}$ and $\eta:\Delta\to\mathbf{Z}_p^\times$ be a character. Then we have the divisibility*

$$(\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\big(X_{i,i}(f_\eta/K_\infty)\big) \bmod I^-) \supset \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X_{i,i}(f_\eta/K^+_\infty)\big),$$

where $I^- = \ker(\Lambda_{\mathscr{O}}(\Gamma_K)\to\Lambda_{\mathscr{O}}(\Gamma))$.

Proof. With notations as in the proof of Proposition 4.11, we begin by noting that the restriction map from $G_{K^+_\infty}$ to G_{K_∞} yields a $\Lambda_{\mathscr{O}}(\Gamma)$-module isomorphism

$$X^{\Sigma}_{i,i}(f_\eta/K_\infty)\otimes_{\Lambda_{\mathscr{O}}(\Gamma_K)}\Lambda_{\mathscr{O}}(\Gamma)\cong X^{\Sigma}_{i,i}(f_\eta/K^+_\infty),$$

(Indeed, this follows from a standard application of the Snake Lemma using [LZ14, Thm. 4.7].) By part (ii) of [SU14, Cor. 3.8] it follows that

$$(\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\big(X^{\Sigma}_{i,i}(f_\eta/K_\infty)\big) \bmod I^-) \supset \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X^{\Sigma}_{i,i}(f_\eta/K^+_\infty)\big),$$

from where the result follows similarly as in Proposition 4.11. □

Corollary 4.13. *For every $i\in\{1,2\}$ and every character $\eta:\Delta\to\mathbf{Z}_p^\times$ we have the divisibility*

$$\begin{aligned}&(\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\big(X_{i,i}(f_\eta/K_\infty)\big) \bmod I^-)\\ &\qquad\supset\ \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X_{i}(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\big)\cdot\mathrm{char}_{\Lambda}\big(X_{i}(f\otimes\epsilon_K/\mathbf{Q}(\mu_{p^\infty}))^\eta\big).\end{aligned}$$

Proof. This is the combination of Propositions 4.11 and 4.12. □

4.3. **Anticyclotomic descent: Definite.** In this section we restrict to the case[7] $k=2$, and assume $(D_K,N)=1$ and

(def) $\qquad N^-$ is the squarefree product of an *odd* number of primes.

Let $\gamma_{\mathrm{ac}}\in\Gamma^-$ be a topological generator, and for any $n\geq 1$ put $\Phi_n=\frac{\omega_n}{\omega_{n-1}}$, where $\omega_n=\gamma_{\mathrm{ac}}^{p^n}-1$, for the p^n-th cyclotomic polynomial in γ_{ac}. Following [BBL24, Def. 3.3] define $\mathscr{M}_{\log,\mathrm{ac}}\in M_{2\times 2}(\mathcal{H}_L(\Gamma^-))$ by

$$\mathscr{M}_{\log,\mathrm{ac}}:=\lim_{n\to\infty}\begin{pmatrix}a_p & 1\\ -p & 0\end{pmatrix}^{-(n+1)}\begin{pmatrix}a_p & 1\\ -\Phi_n & 0\end{pmatrix}\cdots\begin{pmatrix}a_p & 1\\ -\Phi_1 & 0\end{pmatrix}, \tag{4.11}$$

Noting the congruence $\Phi_{n+1}\equiv p \pmod{\omega_n}$, the convergence of the limit in (4.11) to an element in $M_{2\times 2}(\mathcal{H}_L(\Gamma^-))$ follows from [PR94, Lem. 1.2.1].

For every p-power root of unity ζ, let $\chi_\zeta:\Gamma^-\to\overline{\mathbf{Q}}_p^\times$ denote the finite order character determined by $\chi_\zeta(\gamma_{\mathrm{ac}})=\zeta$. The following is a refinement and generalisation of the anticyclotomic p-adic L-function introduced by Bertolini–Darmon [BD96].

Theorem 4.14. *For every $\xi\in\{\alpha,\beta\}$, there exists a "square-root" p-adic L-function*

$$\Theta^{\mathrm{BD}}_\xi(f/K)\in\mathcal{H}_L(\Gamma^-)$$

such that for every p^t-th root of unity ζ with $t>0$ we have

$$\Theta^{\mathrm{BD}}_\xi(f/K)^2(\chi_\zeta)=\frac{1}{\xi^{2t}}\cdot\frac{L(f/K,\chi_\zeta,1)}{\Omega_{f,N^-}}\cdot\sqrt{D_K}p^t\cdot w_K^2,$$

[7] Largely for simplicity; the general case $2\leq k\leq p-1$ should be similar.

where

$$\Omega_{f,N^-} = \frac{(4\pi)^2 \cdot \langle f, f\rangle_N}{\eta_{f,N^-}}$$

is Gross's period as in [CH18a, (4.3)]*. Moreover, for $i \in \{1,2\}$ there exist $\Theta_i^{\mathrm{BD}}(f/K) \in \Lambda_{\mathscr{O}}(\Gamma^-)$ such that*

$$\left(\begin{array}{c} \Theta_\alpha^{\mathrm{BD}}(f/K) \\ \Theta_\beta^{\mathrm{BD}}(f/K) \end{array} \right) = Q_{\alpha,\beta}^{-1} \mathscr{M}_{\mathrm{log,ac}} \cdot \left(\begin{array}{c} \Theta_1^{\mathrm{BD}}(f/K) \\ \Theta_2^{\mathrm{BD}}(f/K) \end{array} \right), \tag{4.12}$$

where $Q_{\alpha,\beta} = \frac{1}{\alpha-\beta} \left(\begin{smallmatrix} \alpha & -\beta \\ -p & p \end{smallmatrix} \right)$.

Proof. For each $n \geq 1$ let $\Gamma_n^- = \mathrm{Gal}(K_n^-/K)$ denote the Galois group of the unique subfield $K_n^- \subset K_\infty^-$ with $[K_n^- : K] = p^n$. From [CH18b] we have elements $\theta_n^{\mathrm{BD}}(f/K) \in \mathscr{O}[\Gamma_n^-]$ satisfying the three-term norm-relation

$$\pi_{n+1,n}(\theta_{n+1}^{\mathrm{BD}}(f/K)) = a_p \cdot \theta_n^{\mathrm{BD}}(f/K) - \nu_{n-1,n}(\theta_{n-1}^{\mathrm{BD}}(f/K)), \tag{4.13}$$

where $\pi_{n+1,n} : \mathscr{O}[\Gamma_{n+1}^-] \to \mathscr{O}[\Gamma_n^-]$ is the natural projection and $\nu_{n-1,n} : \mathscr{O}[\Gamma_{n-1}^-] \to \mathscr{O}[\Gamma_n^-]$ is given by $\sigma \mapsto \sum_\tau \tau$ with τ running over all the elements of Γ_n^- with $\tau|_{K_{n-1}^-} = \sigma$. Setting

$$\theta_{\xi,n}^{\mathrm{BD}}(f/K) := \theta_n^{\mathrm{BD}}(f/K) - \frac{1}{\xi} \cdot \nu_{n-1,n}(\theta_{n-1}^{\mathrm{BD}}(f/K)),$$

the compatibility $\pi_{n+1,n}(\theta_{n+1,\xi}^{\mathrm{BD}}(f/K)) = \xi \cdot \theta_{n,\xi}^{\mathrm{BD}}(f/K)$ follows readily from (4.13). Thus by [PR94, Lem. 1.2.1] there exist elements $\Theta_\xi^{\mathrm{BD}}(f/K) \in \mathcal{H}_L(\Gamma^-)$ projecting to $\xi^{-n} \cdot \theta_{\xi,n}^{\mathrm{BD}}(f/K)$ under $\mathcal{H}_L(\Gamma^-) \to L[\Gamma_n^-]$ for all $n \geq 1$ and interpolating the special values $L(f/K, \chi_\zeta, 1)$ as in the statement by virtue of [CH18b, Prop. 4.3].

On the other hand, as shown in [BBL24, Thm. 3.4], from (4.13) we obtain elements $\Theta_i^{\mathrm{BD}}(f/K) \in \Lambda_{\mathscr{O}}(\Gamma^-)$, $i \in \{1,2\}$, satisfying

$$\left(\begin{array}{cc} a_p & 1 \\ -\Phi_n & 0 \end{array} \right) \cdots \left(\begin{array}{cc} a_p & 1 \\ -\Phi_1 & 0 \end{array} \right) \left(\begin{array}{c} \Theta_1^{\mathrm{BD}}(f/K) \\ \Theta_2^{\mathrm{BD}}(f/K) \end{array} \right) \equiv \left(\begin{array}{c} \theta_n^{\mathrm{BD}}(f/K) \\ -\nu_{n-1,n}(\theta_{n-1}^{\mathrm{BD}}(f/K)) \end{array} \right) \pmod{\omega_n} \tag{4.14}$$

for all $n \geq 1$, and the relation (4.12) then follows as in the proof of Theorem 3.9 in *op. cit.* (where we note that there is a typo in the first displayed equation). $\square$

Proposition 4.15. *Let $i \in \{1,2\}$ and I^+ be the kernel of the natural projection $\Lambda_L(\Gamma_K) \to \Lambda_L(\Gamma^-)$. Then*

$$\left(\varpi^{-s} \cdot \mathcal{L}_{i,i}^{(\mathrm{I})}(f/K)^\eta \bmod I^+\right) = \left(\Theta_i^{\mathrm{BD}}(f/K)^2 \cdot \frac{c_f}{\eta_{f,N^-}}\right)$$

as ideals in $\Lambda_{\mathscr{O}}(\Gamma^-)$, where η is the trivial character of Δ.

Proof. This can be shown similarly as in the proof of Proposition 4.7. Writing $Q_{\alpha,\beta}^{-1} \mathscr{M}_{\mathrm{log,ac}} = \left(\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix} \right)$, from (4.12) we have

$$\Theta_\alpha^{\mathrm{BD}}(f/K)^2 = \mathfrak{A}^2 \cdot \Theta_1^{\mathrm{BD}}(f/K)^2 + 2 \cdot \mathfrak{A}\mathfrak{B} \cdot \Theta_1^{\mathrm{BD}}(f/K) \cdot \Theta_2^{\mathrm{BD}}(f/K) + \mathfrak{B}^2 \cdot \Theta_2^{\mathrm{BD}}(f/K)^2. \tag{4.15}$$

On the other hand, from Corollary 3.9 we have

$$\varpi^s \cdot \mathcal{L}_{\alpha,\alpha}^{(\mathrm{I})}(f/K)^\eta = \mathfrak{A}^2 \cdot \mathcal{L}_{1,1}^{(\mathrm{I})}(f/K)^\eta + \mathfrak{A}\mathfrak{B} \cdot \left(\mathcal{L}_{1,2}^{(\mathrm{I})}(f/K)^\eta + \mathcal{L}_{2,1}^{(\mathrm{I})}(f/K)^\eta\right) + \mathfrak{B}^2 \cdot \mathcal{L}_{2,2}^{(\mathrm{I})}(f/K)^\eta. \tag{4.16}$$

As noted in the proof of Proposition 4.7, by [LZ16] $\mathcal{L}_{\alpha,\alpha}^{(\mathrm{I})}(f/K)^\eta$ can be interpolated along the Coleman family passing through the p-stabilisation f^α of f with U_p-eigenvalue α, and similarly as in [CL16][8],

[8] Where the p-ordinary case is considered, but whose extension to Coleman families can be easily obtained using the techniques in e.g. [BDI10].

the same holds for $\Theta_\alpha^{\rm BD}(f/K)$. A direct comparison of the interpolation formulas of Theorem 3.6 (as extended in [Loe18]) and Theorem 4.14 then shows that

$$\big(\mathcal{L}^{(\rm I)}_{\xi,\xi}(f/K)^\eta \bmod I^+\big) = \Big(\Theta^{\rm BD}_\xi(f/K)^2\cdot\frac{c_f}{\eta_{f,N^-}}\Big). \tag{4.17}$$

Combining (4.15), (4.16), and (4.17), we are reduced to showing that $\mathfrak{A}^2$, $\mathfrak{A}\mathfrak{B}$, and $\mathfrak{B}^2$ are linearly independent over $\Lambda_{\mathscr{O}}(\Gamma^-)$; and this follows by the same argument as in the proof of Proposition 4.12, replacing the appeal to [Roh88] by an appeal to the nonvanishing result of [CH18b, Thm. D]. ☐

We conclude this section with the following result allowing us to deduce the vanishing of Iwasawa's μ-invariant for $\mathcal{L}^{(\rm I)}_{i,i}(f/K)^\eta$ under suitable ramification hypotheses on $\bar\rho_f$ (see Corollary 5.5).

Theorem 4.16. *For every $i\in\{1,2\}$ we have*

$$\mu(\Theta_i^{\rm BD}(f/K))=0.$$

Proof. Since $a_n\equiv 0 \pmod{\varpi}$, for each $n\geq 1$ we have the congruence

$$\begin{pmatrix} a_p & 1\\ -\Phi_n & 0\end{pmatrix}\cdots\begin{pmatrix} a_p & 1\\ -\Phi_1 & 0\end{pmatrix}\equiv\begin{cases}\begin{pmatrix} 0 & (-1)^{\frac{n-1}{2}}\Phi_n^+\\ (-1)^{\frac{n-1}{2}}\Phi_n^- & 0\end{pmatrix} \pmod{\varpi} & \text{if } n \text{ is odd,}\\ \begin{pmatrix} (-1)^{\frac{n}{2}}\Phi_n^- & 0\\ 0 & (-1)^{\frac{n}{2}}\Phi_n^+\end{pmatrix} \pmod{\varpi} & \text{if } n \text{ is even,}\end{cases} \tag{4.18}$$

where $\Phi_n^\pm$ are appropriate products of the Φ_i for $1\leq i\leq n$ of the corresponding parity (cf. [GLL26, Prop. 3.8]). Since the proof of [CH18b, Thm. 5.7][9] shows that $\mu(\theta_n^{\rm BD}(f/K))=0$ for $n\gg 0$, combining (4.14) and (4.18) the result follows. ☐

5. Proof of the main results

In this section we piece everything together to conclude the proof of our main result toward Kato's main Conjecture 1.1.

5.1. **Signed main conjectures of Lei–Loeffler–Zerbes.** By Propositions 2.16 and 2.22 (see also Remark 2.20), the proof of Theorem A is reduced to the following result.

Theorem 5.1. *Let $f\in S_k(\Gamma_0(N))$ be a newform and $p>2$ be a prime of good nonordinary reduction. Suppose*

- *$2\leq k<p$;*
- *N is squarefree;*
- *f is p-*regular*;*
- *either* $\begin{cases} k=2, \textit{or} \\ \textit{Condition (rtn) holds.}\end{cases}$

Let $\eta=\omega^{1-k/2}$ and $i\in\{1,2\}$ be such that $L_i(f)^\eta\neq 0$. Then $X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta$ is $\Lambda_{\mathscr{O}}(\Gamma)$-torsion, with

$$(\xi_{\eta,i})\cdot{\rm char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\big)=\big(L_i(f)^\eta\big)$$

as ideals in $\Lambda_{\mathscr{O}}(\Gamma)$.

The rest of this section is devoted to the proof of Theorem 5.1.

[9]Note that the absolute irreducibility hypothesis of *loc. cit.* holds in our case by [Edi92, Thm. 2.6] and our running assumption that p splits in K.

5.2. **Kato's divisibility.** As shown in [LLZ10, LLZ11], a divisibility towards Conjecture 2.21 follows from Kato's work.

Theorem 5.2. *Let $f \in S_k(\Gamma_0(N))$ be a newform and $p > 2$ be a prime of good nonordinary reduction for f. Let $\eta = \omega^{1-k/2}$ and $i \in \{1, 2\}$ be such that $L_i(f)^\eta \neq 0$. Then $X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta$ is $\Lambda_{\mathscr{O}}(\Gamma)$-torsion, and*

$$(\xi_{\eta,i}) \cdot \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\big) \supset \big(L_i(f)^\eta\big) \tag{5.1}$$

as ideals in $\Lambda_{\mathscr{O}}(\Gamma) \otimes \mathbf{Q}_p$. If in addition

(i) *$\bar{\rho}_f$ is irreducible; and*
(ii) *$\bar{\rho}_f$ is ramified at some prime $\ell \| N$,*

then the divisibility (5.1) holds in $\Lambda_{\mathscr{O}}(\Gamma)$.

Proof. By [Kat04, Thm. 12.5], $X_{\mathrm{str}}(f/\mathbf{Q}(\mu_{p^\infty}))^\eta$ is $\Lambda_{\mathscr{O}}(\Gamma)$-torsion, with

$$\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\big(X_{\mathrm{str}}(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\big) \supset \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma)}\left(\frac{\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), T)^\eta}{(\mathbf{z}^{\mathrm{int}}_{f,k-1})^\eta}\right) \tag{5.2}$$

in $\Lambda_{\mathscr{O}}(\Gamma) \otimes \mathbf{Q}_p$. That the divisibility (5.2) holds in $\Lambda_{\mathscr{O}}(\Gamma)$ under the additional hypotheses (i)-(ii) is explained in [Ski16, §2.5]. Together with Proposition 2.22, this yields the result. □

5.3. **Eisenstein congruence divisibility over K.** Ultimately, a converse to Kato's divisibility will be deduced from the following main result from our previous work [CLW22].

Recall the equality (up to a unit) of p-adic L-functions

$$\varpi^{-s} \cdot \mathcal{L}^{(\mathrm{II})}_{1,2}(f/K)^\eta = \mathcal{L}_{\overline{v}}(f/K)^\eta \tag{5.3}$$

from Proposition 4.3. Here and throughout the following, we put

$$\eta = \omega^{1-k/2}$$

for the ease of notation.

Theorem 5.3. *Let $f \in S_k(\Gamma_0(N))$ be a newform of squarefree level N and weight $k \geq 2$, and let p be an odd prime of good nonordinary reduction for f. Let K be an imaginary quadratic field satisfying:*

(i) *p splits in K;*
(ii) *if N is odd, then 2 splits in K;*
(iii) *some prime factor of N is nonsplit in K;*
(iv) *$\bar{\rho}_f|_{G_K}$ is irreducible.*

Then

$$\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\big(X_{\mathrm{str,rel}}(f_\eta/K_\infty)\big)\Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma_K) \subset \big(\mathcal{L}_{\overline{v}}(f/K)^\eta\big) \tag{5.4}$$

in $\Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma_K) \otimes_{\Lambda_{\mathscr{O}}(\Gamma_K)} \mathrm{Frac}(\Lambda_{\mathscr{O}}(\Gamma))$. If in addition

(v) *$\varepsilon_v(BC_K(\pi_f)) = +1$ for every prime v factor of N nonsplit in K,*

then the divisibility (5.4) holds in $\Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma_K)$.

Proof. The divisibility (5.4) is proved in [CLW22, Thm. 8.21] in weight 2; the required modifications of the argument to extend the result to higher weights are explained in [Wan16, §4] (cf. [FW21, §7]).

Under the additional hypothesis (v) we have $\mu(\mathscr{L}^{\mathrm{BDP}}_{\overline{v}}(f/K)) = 0$ by [Hsi14, Thm. B]. By Proposition 4.4 and (5.3), this implies that $\mathcal{L}_{\overline{v}}(f/K)^\eta$ is not divisible by any height one prime of $\Lambda_{\mathscr{O}}(\Gamma_K)$ from the inverse image of the projection $\Lambda_{\mathscr{O}}(\Gamma_K) \to \Lambda_{\mathscr{O}}(\Gamma)$, whence the result. □

5.4. **Transferring divisibilities.** We next deduce from the lower bound divisibility in Theorem 5.3 a proof of corresponding divisibilities in related Iwasawa main conjectures; the main result is Corollary 5.5. With the explicit reciprocity laws of the preceding sections at hand, the required arguments go along similar lines as those in e.g. [CGS25, Prop. 3.2.1]; however, since several additional complications arise in our setting, we provide the details.

Corollary 5.4. *In the setting of and under hypotheses* (i)–(iv) *of Theorem 5.3, if $l \in \{1,2\}$ is such that condition* (3.18) *holds, then*

$$(\varpi^s \cdot \xi_{\eta,l}) \cdot \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\big(X_{\mathrm{str},l}(f_\eta/K_\infty)\big) \subset \mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\left(\frac{\mathrm{H}^1_{\mathrm{rel},l}(K,\mathbf{T}_K^\eta)}{(\mathcal{BF}_{\mathrm{int}}^{l,\mathbf{h}_v,\eta})}\right) \tag{5.5}$$

as ideals in $\Lambda_{\mathscr{O}}(\Gamma_K)\otimes_{\Lambda_{\mathscr{O}}(\Gamma_K)}\mathrm{Frac}(\Lambda_{\mathscr{O}}(\Gamma))$. If in addition hypothesis (v) *in Theorem 5.3 holds, then the divisibility* (5.5) *holds in $\Lambda_{\mathscr{O}}(\Gamma_K)$.*

Proof. For $i,j\in\{1,2\}$ put $\mathrm{H}^1_i(K_{\overline{v}})^\eta = \mathrm{H}^1_i(K_{\overline{v}},\mathbf{T}_K^\eta)$ for the ease of notation, let $\mathrm{H}^1_{\mathrm{rel},i\cap j}(K,\mathbf{T}_K^\eta)$ denote the submodule of $\mathrm{H}^1_{\mathrm{rel},j}(K,\mathbf{T}_K^\eta)$ consisting of classes c satisfying $\mathrm{res}_{\overline{v}}(c)\in \mathrm{H}^1_i(K_{\overline{v}})^\eta\cap \mathrm{H}^1_j(K_{\overline{v}})^\eta$, and let $X_{\mathrm{str},i+j}(f_\eta/K_\infty)$ be the Pontryagin dual of the Selmer group dual to $\mathrm{H}^1_{\mathrm{rel},i\cap j}(K,\mathbf{T}_K^\eta)$ in the sense of §3.4. Then Poitou–Tate duality gives rise to the exact sequence

$$\begin{aligned}0\longrightarrow \mathrm{H}^1_{\mathrm{rel},i\cap j}(K,\mathbf{T}_K^\eta)\longrightarrow &\mathrm{H}^1_{\mathrm{rel},j}(K,\mathbf{T}_K^\eta)\xrightarrow{\mathrm{res}_{\overline{v}}}\frac{\mathrm{H}^1_j(K_{\overline{v}})^\eta}{\mathrm{H}^1_i(K_{\overline{v}})^\eta\cap \mathrm{H}^1_j(K_{\overline{v}})^\eta}\\ &\longrightarrow X_{\mathrm{str},i+j}(f_\eta/K_\infty)\longrightarrow X_{\mathrm{str},j}(f_\eta/K_\infty)\longrightarrow 0.\end{aligned}\tag{5.6}$$

For the proof of the stated divisibility, it suffices to consider the case where $X_{\mathrm{str},l}(f/K_\infty)^\eta$ is $\Lambda_{\mathscr{O}}(\Gamma_K)$-torsion (otherwise the result is obvious). Letting l' denote the element in $\{1,2\}$ different from l, and taking $(i,j)=(l',l)$ in (5.6), the nonvanishing of $L^{(\mathrm{II})}_{l',l}(\mathbf{h}_v,f)^\eta$ (see Corollary 3.18) and Theorem 3.17 implies that $\mathrm{res}_{\overline{v}}$ has $\Lambda_{\mathscr{O}}(\Gamma_K)$-torsion cokernel. Moreover, by [LZ14, Prop. 4.11] the map $e_\eta\mathscr{L}_{V,\overline{v}}$ is injective, and so by the decomposition (3.12) so is $e_\eta\widetilde{\mathscr{C}_1}\oplus e_\eta\widetilde{\mathscr{C}_2}$. Thus

$$\mathrm{H}^1_1(K_{\overline{v}})^\eta\cap \mathrm{H}^1_2(K_{\overline{v}})^\eta=0,$$

which implies $\mathrm{H}^1_{\mathrm{rel},l'\cap l}(K,\mathbf{T}_K^\eta)=\mathrm{H}^1_{\mathrm{rel},\mathrm{str}}(K,\mathbf{T}_K^\eta)=0$, and so (5.6) reduces to

$$0\longrightarrow \mathrm{H}^1_{\mathrm{rel},l}(K,\mathbf{T}_K^\eta)\xrightarrow{\mathrm{res}_{\overline{v}}}\mathrm{H}^1_l(K_{\overline{v}})^\eta\longrightarrow X_{\mathrm{str},\mathrm{rel}}(f_\eta/K_\infty)\longrightarrow X_{\mathrm{str},l}(f_\eta/K_\infty)\longrightarrow 0.\tag{5.7}$$

By the construction in [KLZ20, Prop. 10.1.1] the differential $\eta_{\mathbf{h}_v}$ has denominators controlled by the congruence power series $H_v\in\mathrm{Frac}(\Lambda_{\mathbf{Z}_p}(\Gamma^v))$ associated to $\mathbf{h}_v$ as in [BSTW24, §4.3]. By Theorem 4.20 in *loc. cit.*, we have

$$(H_v)=\Big(\frac{h_K}{w_K}\cdot\mathscr{L}_{\overline{v}}^{\mathrm{Katz},-}\Big)$$

as ideals in $\Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma^v)$. Thus together with the argument in [Wan14b, Lem. 4.5] showing the divisibility $(H_v)\subset(\frac{c\cdot\omega_{\mathbf{h}_v}}{\eta_{\mathbf{h}_v}})$, we deduce that Theorem 5.3 (see also Definition 4.2 and (5.3)) implies the divisibility

$$(\delta_{k-1})\cdot\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\big(X_{\mathrm{str},\mathrm{rel}}(f_\eta/K_\infty)\big)\subset\big(\varpi^{-s}\cdot L^{(\mathrm{II})}_{1,2}(\mathbf{h}_v,f)^\eta\big)\tag{5.8}$$

in $\Lambda_{\mathscr{O}}(\Gamma_K)\otimes_{\Lambda_{\mathscr{O}}(\Gamma_K)}\mathrm{Frac}(\Lambda_{\mathscr{O}}(\Gamma))$, and even in $\Lambda_{\mathscr{O}}(\Gamma_K)$ if hypothesis (v) of Theorem 5.3 holds. On the other hand, consider the tautological exact sequence

$$0\longrightarrow \mathrm{H}^1_l(K_{\overline{v}})^\eta\cong\frac{\mathrm{H}^1_l(K_{\overline{v}})^\eta+\mathrm{H}^1_{l'}(K_{\overline{v}})^\eta}{\mathrm{H}^1_{l'}(K_{\overline{v}})^\eta}\longrightarrow\frac{\mathrm{H}^1(K_{\overline{v}})^\eta}{\mathrm{H}^1_{l'}(K_{\overline{v}})^\eta}\longrightarrow\frac{\mathrm{H}^1(K_{\overline{v}})^\eta}{\mathrm{H}^1_l(K_{\overline{v}})^\eta+\mathrm{H}^1_{l'}(K_{\overline{v}})^\eta}\longrightarrow 0,\tag{5.9}$$

using that $\mathrm{H}^1_l(K_{\overline{v}})^\eta\cap\mathrm{H}^1_{l'}(K_{\overline{v}})^\eta=0$, as shown as above, for the left isomorphism. By [BL21, Prop. 2.17] and the discussion following *loc. cit.*, the right-most term of (5.9) is $\Lambda_{\mathscr{O}}(\Gamma_K)$-torsion, with characteristic ideal generated by $\varpi^n\cdot\delta_{k-1}/\xi_{\eta,2}$ for some integer n. Thus from Proposition 3.4 and Theorem 3.17

we find

$$(\varpi^n\cdot\delta_{k-1})\cdot\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\biggl(\frac{\mathrm{H}^1_l(K_{\overline{v}})^\eta}{(\mathrm{res}_{\overline{v}}(\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{int}}))}\biggr)=(L_{1,2}(\mathbf{h}_v,f)^\eta)\cdot\frac{\xi_{\eta,2}}{\xi_{\eta,l'}}\tag{5.10}$$

in $\Lambda_{\mathscr{O}}(\Gamma_K)$, and we note that since $\xi_{\eta,1}=1$ for all η (see Remark 2.9) we have the equality $\xi_{\eta,2}/\xi_{\eta,l'}=\xi_{\eta,l}$. Thus from (5.10) we see that (5.8) amounts to the divisibility

$$(\varpi^s\cdot\xi_{\eta,l})\cdot\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\bigl(X_{\mathrm{str,rel}}(f_\eta/K_\infty)\bigr)\subset(\varpi^n)\cdot\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\biggl(\frac{\mathrm{H}^1_l(K_{\overline{v}})^\eta}{(\mathrm{res}_{\overline{v}}(\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{int}}))}\biggr),$$

which together with (5.7) and multiplicativity of characteristic ideals yields the result. □

Corollary 5.5. *In the setting of and under hypotheses* (i)–(iv) *of Theorem 5.3, if $l\in\{1,2\}$ is such that condition* (3.18) *holds, then*

$$(\xi_{\eta,l})^2\cdot\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\bigl(X_{l,l}(f_\eta/K_\infty)\bigr)\subset\bigl(\varpi^{-s}\cdot\mathcal{L}^{(\mathrm{I})}_{l,l}(f/K)^\eta\bigr)$$

as ideals in $\Lambda_{\mathscr{O}}(\Gamma_K)\otimes_{\Lambda_{\mathscr{O}}(\Gamma_K)}\mathrm{Frac}(\Lambda_{\mathscr{O}}(\Gamma))$. Moreover, if either:

- (v) *hypothesis* (v) *in Theorem 5.3 holds, or*
- (v)' *$k=2$ and $\bar\rho_f$ is ramified at every prime $\ell|N^-$,*

then the divisibility (5.13) *holds in $\Lambda_{\mathscr{O}}(\Gamma_K)$.*

Proof. With similar abbreviated notations as in the proof of Corollary 5.4, Poitou–Tate duality gives rise to the exact sequence

$$\begin{aligned}0\to\mathrm{H}^1_{l,l}(K,\mathbf{T}^\eta_K)\longrightarrow&\mathrm{H}^1_{\mathrm{rel},l}(K,\mathbf{T}^\eta_K)\xrightarrow{\mathrm{res}_v}\frac{\mathrm{H}^1(K_v)^\eta}{\mathrm{H}^1_l(K_v)^\eta}\\&\longrightarrow X_{l,l}(f_\eta/K_\infty)\longrightarrow X_{\mathrm{str},l}(f_\eta/K_\infty)\longrightarrow 0.\end{aligned}\tag{5.11}$$

As before, it suffices to consider the case where $X_{l,l}(f_\eta/K_\infty)$ is $\Lambda_{\mathscr{O}}(\Gamma_K)$-torsion. Then $X_{\mathrm{str},l}(f_\eta/K_\infty)$ is also $\Lambda_{\mathscr{O}}(\Gamma_K)$-torsion, and from (5.6) it follows that $\mathrm{H}^1_{\mathrm{rel},l}(K,\mathbf{T}^\eta_K)$ has $\Lambda_{\mathscr{O}}(\Gamma_K)$-rank one. By Theorem 3.6 and the nonvanishing of $\mathcal{L}^{(\mathrm{I})}_{l,l}(f/K)^\eta$ (by our assumption on l), it follows that $\mathrm{H}^1_{l,l}(K,\mathbf{T}^\eta_K)=0$, and so after quotienting out by the image of $\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{int}}$ the exact sequence (5.11) reduces to

$$0\longrightarrow\frac{\mathrm{H}^1_{\mathrm{rel},l}(K,\mathbf{T}^\eta_K)}{(\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{int}})}\longrightarrow\frac{\mathrm{H}^1(K_v)^\eta/\mathrm{H}^1_l(K_v)^\eta}{(\mathrm{res}_v(\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{int}}))}\longrightarrow X_{l,l}(f_\eta/K_\infty)\longrightarrow X_{\mathrm{str},l}(f_\eta/K_\infty)\longrightarrow 0.\tag{5.12}$$

Since by Proposition 3.4 and Theorem 3.6 we have the equality

$$\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\biggl(\frac{\mathrm{H}^1(K_v)^\eta/\mathrm{H}^1_l(K_v)^\eta}{(\mathrm{res}_v(\mathcal{BF}^{l,\mathbf{h}_v,\eta}_{\mathrm{int}}))}\biggr)=\frac{(\mathcal{L}^{(\mathrm{I})}_{l,l}(f/K)^\eta)}{(\xi_{\eta,l})},$$

taking characteristic ideals in (5.12), the divisibility

$$(\xi_{\eta,l})^2\cdot\mathrm{char}_{\Lambda_{\mathscr{O}}(\Gamma_K)}\bigl(X_{l,l}(f_\eta/K_\infty)\bigr)\subset\bigl(\varpi^{-s}\cdot\mathcal{L}^{(\mathrm{I})}_{l,l}(f/K)^\eta\bigr)\tag{5.13}$$

as ideals in $\Lambda_{\mathscr{O}}(\Gamma_K)\otimes_{\Lambda_{\mathscr{O}}(\Gamma_K)}\mathrm{Frac}(\Lambda_{\mathscr{O}}(\Gamma))$, and even $\Lambda_{\mathscr{O}}(\Gamma_K)$ under the additional hypothesis (v) of Theorem 5.3, follows from Corollary 5.4.

Finally, it remains to show that that ambiguity by primes from $\Lambda_{\mathscr{O}}(\Gamma)$ is (5.13) can also be removed under the additional hypothesis (v)' in the statement. For this, we first argue the inclusion

$$\varpi^{-s}\cdot\mathcal{L}^{(\mathrm{I})}_{l,l}(f/K)^\eta\in\Lambda_{\mathscr{O}}(\Gamma_K).\tag{5.14}$$

We proceed by contradiction. Suppose we had $\varpi^{-s}\cdot\mathcal{L}^{(\mathrm{I})}_{l,l}(f/K)^\eta\in\varpi^{-t}\Lambda_{\mathscr{O}}(\Gamma_K)\smallsetminus\Lambda_{\mathscr{O}}(\Gamma_K)$ for some $t>0$. Then after multiplying $\varpi^{-s}\cdot\mathcal{L}^{(\mathrm{I})}_{l,l}(f/K)^\eta$ by an appropriate product of elements of $\Lambda_{\mathscr{O}}(\Gamma_K)$

supported at height one primes from $\Lambda_{\mathscr{O}}(\Gamma)$, using Proposition 4.12 and Corollary 4.13 we can deduce from the divisibility (5.13) a proof of the *strict* divisibility

$$(\xi_{\eta,l})^2 \cdot \mathrm{char}_\Lambda\big(X_l(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\big) \cdot \mathrm{char}_\Lambda\big(X_l(f\otimes\epsilon_K/\mathbf{Q}(\mu_{p^\infty}))^\eta\big) \subsetneq \big(L_l(f)^\eta \cdot L_i(f\otimes\epsilon_K)^\eta\big),$$

but this contradicts the divisibility obtained from Theorem 5.2 applied to f and $f\otimes\epsilon_K$, and so (5.14) holds. Now, by the formula for $\mathrm{ord}_\varpi(c_f/\eta_{f,N^-})$ in terms of Tamagawa numbers in [PW11, Thm. 6.8] (see also [KO23, Cor. 6.7]), from Proposition 4.15 and Theorem 4.16 we see that if $k=2$ (in which case η is the trivial character of Δ) and $\bar\rho_f$ is ramified at every prime $\ell|N^-$ then

$$\mu(\varpi^{-s}\cdot\mathcal{L}^{(\mathrm{I})}_{l,l}(f/K)^\eta \bmod I^+)=0,$$

and hence together with (5.14) we conclude that the divisibility (5.13) holds in $\Lambda_{\mathscr{O}}(\Gamma_K)$. □

5.5. **Proof of Theorem A.** We can now complete the proof building on Theorems 5.2 and 5.3.

By Ribet's level-lowering result [Rib90], there exists a prime $\ell\|N$ (recall that N is assumed to be squarefree) at which $\bar\rho_f$ is ramified. If $k=2$, we fix such a prime ℓ; while if $k>2$ we let $\ell\|N$ be a prime as in Condition (rtn). Choose an imaginary quadratic field K satisfying:

(a) ℓ is inert (resp. ramified) in K if $k=2$ (resp. $k>2$);
(b) every prime dividing N/ℓ splits in K;
(c) if N is odd, then 2 splits in K;
(d) p splits in K;
(e) $L(f\otimes\epsilon_K\eta^{-1},1)\neq 0$, where $\eta=\omega^{1-k/2}$.

The existence of infinitely many K satisfying these properties is ensured by [FH95, Thm. B] (in the case $k=2$ where the nonvanishing $L(f\otimes\epsilon_K\eta^{-1},1)$ is non-trivial; for $k>2$ condition (e) follows from [Shi76, Prop. 2]).

Put $\Lambda_K=\Lambda_{\mathscr{O}}(\Gamma_K)$, $\Lambda_K^{\mathrm{ur}}=\Lambda_{\mathscr{O}^{\mathrm{ur}}}(\Gamma_K)$, and $\Lambda=\Lambda_{\mathscr{O}}(\Gamma)$ for the ease of notation. By Corollary 4.9, the nonvanishing of $L_i(f)^\eta$ and condition (e) imply that $\mathcal{L}^{(\mathrm{I})}_{i,i}(f/K)^\eta\neq 0$, and hence by Corollary 5.5 we have the divisibility

$$\big(\varpi^{-s}\cdot\mathcal{L}^{(\mathrm{I})}_{i,i}(f/K)^\eta\big)\supset(\xi_{\eta,i})^2\cdot\mathrm{char}_{\Lambda_K}\big(X_{i,i}(f_\eta/K_\infty)\big)$$

in Λ_K. (Note that hypotheses (i)–(iii) from Theorem 5.3 are embedded in the choice of K, while (iv) follows from the irreducibility of $\bar\rho_f|_{G_{\mathbf{Q}_p}}$ and the fact that p splits in K; and the additional hypothesis (v) (resp. (v)') in Corollary 5.5 follows from the above choice of K (resp. Condition (rtn) and the discussion in [Ski20, p. 336]).) Together with Proposition 4.7 and Corollary 4.13, it follows that

$$\begin{aligned}\big(L_i(f)^\eta\cdot L_i(f\otimes\epsilon_K)^\eta\big)&=\big(\varpi^{-s}\cdot\mathcal{L}^{(\mathrm{I})}_{i,i}(f/K)^\eta \bmod I^-\big)\\&\supset(\xi_{\eta,i})^2\cdot\mathrm{char}_\Lambda\big(X_{i,i}(f_\eta/K_\infty)\bmod I^-\big)\\&\supset(\xi_{\eta,i})^2\cdot\mathrm{char}_\Lambda\big(X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\big)\cdot\mathrm{char}_\Lambda\big(X_i(f\otimes\epsilon_K/\mathbf{Q}(\mu_{p^\infty}))^\eta\big).\end{aligned}\tag{5.15}$$

Now we can conclude similarly as in [SU14, Thm. 3.29]: From Theorem 5.2 we have the divisibilities

$$(\xi_{\eta,i})\cdot\mathrm{char}_\Lambda\big(X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\big)\supset\big(L_i(f)^\eta\big),$$
$$(\xi_{\eta,i})\cdot\mathrm{char}_\Lambda\big(X_i(f\otimes\epsilon_K/\mathbf{Q}(\mu_{p^\infty}))^\eta\big)\supset\big(L_i(f\otimes\epsilon_K)^\eta\big)$$

in Λ. If the first of these inclusions was not an equality, they would give

$$(\xi_{\eta,i})^2\cdot\mathrm{char}_\Lambda\big(X_i(f/\mathbf{Q}(\mu_{p^\infty}))^\eta\big)\cdot\mathrm{char}_\Lambda\big(X_i(f\otimes\epsilon_K/\mathbf{Q}(\mu_{p^\infty}))^\eta\big)\supsetneq\big(L_i(f)^\eta\cdot L_i(f\otimes\epsilon_K)^\eta\big),$$

contradicting (5.15). Thus we deduce that equality holds, concluding the proof of Theorem 5.1, and hence of Theorem A in the Introduction.

5.6. **Proof of Theorem B and Corollary C.**

5.6.1. *p-part of the Tamagawa Number Conjecture.* For the proof of Theorem B, we essentially combine Theorem A with some of the calculations in [Kat04, §14], using a reformulation of Kato's main conjecture in terms of determinants of arithmetic complexes. Put

$$T=T_f(k/2),\quad A=T_f^\vee(1-k/2),\quad \Lambda=\Lambda_{\mathscr{O}}(\Gamma),\quad \mathbf{T}=T\otimes_{\mathscr{O}}\Lambda^\iota$$

for the ease of notation. Let S be the set of primes dividing Np, and let $G_{\mathbf{Q},S}=\mathrm{Gal}(\mathbf{Q}^S/\mathbf{Q})$ denote the Galois group of the maximal extension of $\mathbf{Q}$ unramified outside S.

Let $\mathfrak{Sel}(\mathbf{Q},T)$ be the subgroup of $\mathrm{H}^1(G_{\mathbf{Q},S},T)$ consisting of classes crystalline at p and unramified at the primes $\ell|N$, and let $\mathfrak{Sel}(\mathbf{Q},A)$ denote the Selmer group group dual to $\mathfrak{Sel}(\mathbf{Q},T)$. Poitou–Tate duality then gives rise to the exact sequence

(5.16)

$$\begin{aligned}0\to\mathfrak{Sel}(\mathbf{Q},T)\longrightarrow\mathrm{H}^1(G_{\mathbf{Q},S},T)\longrightarrow\frac{\mathrm{H}^1(\mathbf{Q}_p,T)}{\mathrm{H}^1_f(\mathbf{Q}_p,T)}\oplus\bigoplus_{\ell|N}\frac{\mathrm{H}^1(\mathbf{Q}_\ell,T)}{\mathrm{H}^1_{\mathrm{unr}}(\mathbf{Q}_\ell,T)}\longrightarrow\mathfrak{Sel}(\mathbf{Q},A)^\vee\\ \longrightarrow\mathrm{H}^2(G_{\mathbf{Q},S},T)\longrightarrow\bigoplus_{\ell|Np}\mathrm{H}^2(\mathbf{Q}_\ell,T)\longrightarrow 0.\end{aligned}$$

It is easy to see that $\mathrm{Sel}(\mathbf{Q},A)$ is contained in $\mathfrak{Sel}(\mathbf{Q},A)$ with finite index (see (5.19) below); since the former is finite by [Kat04, Thm. 14.2] and $\mathrm{H}^1(G_{\mathbf{Q},S},T)$ is torsion-free by the irreducibility of $\bar\rho_f$, it follows that $\mathfrak{Sel}(\mathbf{Q},T)=0$.

For a prime $\ell\nmid p$, let $P_\ell(A,\ell^{-s})$, where $P_\ell(A,X)=(1-\alpha_\ell X)(1-\beta_\ell X)$ (with one of both of α_ℓ,β_ℓ possibly zero), be the Euler factor of $L(f,s)$ at ℓ, and put

$$\mathbf{z}_f^{\{N\}}:=\mathbf{z}_{f,k/2}^{\mathrm{int}}\cdot\prod_{\ell|N}\mathcal{P}_\ell\in\mathrm{H}^1(G_{\mathbf{Q},S},\mathbf{T}),\tag{5.17}$$

where $\mathbf{z}_{k/2}^{\mathrm{int}}$ denotes the image of $\mathbf{z}_{f,k-1}^{\mathrm{int}}$ under the composite map

$$\begin{aligned}\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),T_f(k-1))\xrightarrow{\otimes(\zeta_{p^n})_{n\geq0}^{\otimes(1-k/2)}}&\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),T_f(k/2))\\ &\longrightarrow\mathrm{H}^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),T_f(k/2))^\Delta\cong\mathrm{H}^1(G_{\mathbf{Q},S},\mathbf{T}),\end{aligned}$$

and $\mathcal{P}_\ell:=(1-\alpha_\ell\ell^{-k/2}\gamma_\ell)(1-\beta_\ell\ell^{-k/2}\gamma_\ell)\in\Lambda$ with $\gamma_\ell\in\Gamma$ a Frobenius element at ℓ.

For the primes $\ell|N$ (hence coprime to p) we have

$$\mathrm{H}^1_f(\mathbf{Q}_\ell,T)=\mathrm{H}^1(\mathbf{Q}_\ell,T)_{\mathrm{tors}}=\mathrm{H}^1(\mathbf{Q}_\ell,T),$$

and therefore quotienting out in (5.16) by the image of $\mathbf{z}_{f,0}^{\{N\}}$ we obtain

$$\#\mathfrak{Sel}(\mathbf{Q},A)=\frac{\#\mathrm{H}^2(G_{\mathbf{Q},S},T)}{\left[\mathrm{H}^1(G_{\mathbf{Q},S},T):\mathbf{z}_{f,0}^{\{N\}}\right]}\cdot\frac{\left[\mathrm{H}^1_{/f}(\mathbf{Q}_p,T):\mathbf{z}_{f,0}^{\{N\}}\right]}{\#\mathrm{H}^2(\mathbf{Q}_p,T)}\cdot\prod_{\ell|N}\frac{\left[\mathrm{H}^1_f(\mathbf{Q}_\ell,T):\mathrm{H}^1_{\mathrm{unr}}(\mathbf{Q}_\ell,T)\right]}{\#\mathrm{H}^2(\mathbf{Q}_\ell,T)},\tag{5.18}$$

where $\mathrm{H}^1_{/f}(\mathbf{Q}_p,T)$ denotes the quotient $\mathrm{H}^1(\mathbf{Q}_p,T)$ by $\mathrm{H}^1_f(\mathbf{Q}_p,T)$. Note that Poitou–Tate duality also gives

$$0\longrightarrow\mathrm{Sel}(\mathbf{Q},A)\longrightarrow\mathfrak{Sel}(\mathbf{Q},A)\longrightarrow\bigoplus_{\ell|N}\mathrm{H}^1_{\mathrm{unr}}(\mathbf{Q}_\ell,A)\longrightarrow\mathrm{Sel}(\mathbf{Q},T)^\vee,$$

where we used that $\mathrm{H}^1_{\mathrm{unr}}(\mathbf{Q}_\ell,A)=0$ for all primes $\ell|N$.

By [Kat04, Thm. 14.2], the nonvanishing of $L(f,k/2)$ implies $\#\mathrm{Sel}(\mathbf{Q},A)<\infty$, and so $\mathrm{Sel}(\mathbf{Q},T)=0$. Thus

$$\#\mathfrak{Sel}(\mathbf{Q},A)=\#\mathrm{Sel}(\mathbf{Q},A)\cdot\prod_{\ell|N}\#\mathrm{H}^1_{\mathrm{unr}}(\mathbf{Q}_\ell,A).\tag{5.19}$$

As explained in [CS26, §2.2], as a consequence of [Kat04, Thm. 12.4] there is a canonical isomorphism

$$Q(\Lambda) \otimes_\Lambda \det_\Lambda^{-1} \mathbf{R}\Gamma(G_{\mathbf{Q},S}, \mathbf{T}) \cong Q(\Lambda) \otimes_\Lambda \mathrm{H}^1(G_{\mathbf{Q},S}, \mathbf{T}), \tag{5.20}$$

where $Q(\Lambda)$ denotes the field of fractions of Λ. Moreover, as explained in [CS26, Prop. 2.2.3] and the references therein (see esp. §2.3.2 and Proposition 3.10 in [KS24]), Conjecture 2.19 for $\eta = \omega^{1-k/2}$ is then equivalent[10] to the following: The inverse image of $\mathbf{z}_f^{\{N\}}$ under the canonical isomorphism (5.20) is a Λ-basis

$$\mathfrak{z}_{\mathbf{Q}_\infty} \in \det_\Lambda^{-1} \mathbf{R}\Gamma(G_{\mathbf{Q},S}, \mathbf{T}). \tag{5.21}$$

Thus letting $\nu : \Lambda \to \mathscr{O}$ be the augmentation map, by Theorem A the image of $\mathfrak{z}_{\mathbf{Q}}$ of (5.21) under the isomorphism

$$\det_\Lambda^{-1} \mathbf{R}\Gamma(G_{\mathbf{Q},S}, \mathbf{T}) \otimes_{\Lambda,\nu} \mathscr{O} \cong \det_{\mathscr{O}}^{-1} \mathbf{R}\Gamma(G_{\mathbf{Q},S}, T)$$

from [FK06, Prop. 1.6.5(3)] gives an $\mathscr{O}$-basis $\mathfrak{z}_{\mathbf{Q}} \in \det_{\mathscr{O}}^{-1} \mathbf{R}\Gamma(G_{\mathbf{Q},S}, T)$; equivalently, the image $\mathbf{z}_{f,0}^{\{N\}}$ of (5.17) under the natural projection $\mathrm{H}^1(G_{\mathbf{Q},S}, T) \to \mathrm{H}^1(G_{\mathbf{Q},S}, T)$ satisfies

$$\#\mathrm{H}^2(G_{\mathbf{Q},S}, T) = \left[\mathrm{H}^1(G_{\mathbf{Q},S}, T) \colon \mathbf{z}_{f,0}^{\{N\}}\right]. \tag{5.22}$$

(cf. [BKS24, Prop. 2.6]).

On the other hand, the same computation as in [Kat04, Prop. 14.21] (cf. (2.11)) gives

$$\frac{\left[\mathrm{H}^1_{/f}(\mathbf{Q}_p, T) : \mathbf{z}_0^{\{N\}}\right]}{\#\mathrm{H}^2(\mathbf{Q}_p, T)} = \left[\mathscr{O} \colon \frac{L_{\{N\}}(f, k/2)}{(-2\pi i)^{k/2} \cdot \Omega_{\omega_f}^{\pm}}\right] = \mathrm{Eul}_N(f, k/2) \cdot \left[\mathscr{O} \colon \frac{L(f, k/2)}{(-2\pi i)^{k/2} \cdot \Omega_{\omega_f}^{\pm}}\right], \tag{5.23}$$

where $\pm = (-1)^{k/2-1}$. Hence combining (5.18), (5.19), (5.22), (5.23) and noting the equalities

$$\#\mathrm{H}^2(\mathbf{Q}_\ell, T) = \#\mathrm{H}^0(\mathbf{Q}_\ell, A) = \mathrm{Eul}_\ell(f, k/2) \cdot \left[\mathrm{H}^1_f(\mathbf{Q}_\ell, T) \colon \mathrm{H}^1_{\mathrm{unr}}(\mathbf{Q}_\ell, T)\right]$$

for every prime $\ell | N$, this concludes the proof of Theorem B.

5.6.2. *p-part of the Birch–Swinnerton-Dyer formula.* After a period comparison, Corollary C is just a special case of Theorem B. More precisely, letting $f \in S_2(\Gamma_0(N))$ be the newform associated to E by modularity [BCDT01], the p-regularity hypothesis is known to hold by [CE98] and by Theorem B the nonvanishing of $L(E,1) = L(f,1)$ gives $\#E(\mathbf{Q}) < \infty$,

$$\#\mathrm{Sel}(\mathbf{Q}, T_f^\vee) = \#\mathrm{Sel}_{p^\infty}(E/\mathbf{Q}) = \#\text{Ш}(E/\mathbf{Q})[p^\infty] < \infty,$$

and

$$\left[\mathbf{Z}_p \colon \frac{L(E,1)}{(-2\pi i) \cdot \Omega_{\omega_f}^{\pm}}\right] = \#\text{Ш}(E/\mathbf{Q})[p^\infty] \cdot \prod_{\ell | N} c_\ell^{(p)}(E),$$

where $c_\ell^{(p)}(E)$ denotes the p-part of the Tamagawa number of E at ℓ (which agrees with $\mathrm{Tam}_\ell(T_f(1)) = \#\mathrm{H}^1_f(\mathbf{Q}_\ell, T_f(1))$ by e.g. the discussion in [Gre99, p. 74]). It remains to show that the period $(-2\pi i) \cdot \Omega_{\omega_f}^+$ agrees up to a p-adic unit with the positive Néron period Ω_E of E. As shown in [GV00, Prop. 3.1], we have the equality up to a p-adic unit

$$(-2\pi i) \cdot \Omega_{\omega_f}^+ \sim_p \Omega_{E^{\mathrm{opt}}},$$

where $E^{\mathrm{opt}}/\mathbf{Q}$ is the optimal curve in the sense of [Ste89, Prop. 1.4] in the $\mathbf{Q}$-isogeny class associated to f (note that the p-ordinarity assumption in [GV00, Prop. 3.1] is not needed for the argument, but only for the definition of $\delta_f^{\pm}$ in *op. cit.* which for our application can be taken to $\delta_{\mathcal{O}}^{\pm}$ from §2.4.2). Since the irreducibility of $\bar{\rho}_f \cong E[p]$ implies that $\Omega_{E^{\mathrm{opt}}}$ is a p-adic unit multiple of Ω_E, this concludes the proof of Corollary C.

[10] For $k > 2$, using the twisting [Rub00, Lem. VI.1.2] with $\rho = \langle \varepsilon \rangle^{1-k/2} : \Gamma \to \mathbf{Z}_p^\times$

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