GENERALISED KATO CLASSES ON CM ELLIPTIC CURVES OF RANK 2

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ABSTRACT. Let $E/\mathbf{Q}$ be a CM elliptic curve and let $p \geq 5$ be a prime of good ordinary reduction for E. Suppose that $L(E,s)$ vanishes at $s=1$ and has sign $+1$ in its functional equation, so in particular $\mathrm{ord}_{s=1}L(E,s) \geq 2$. In this paper we slightly modify a construction of Darmon–Rotger to define a generalised Kato class $\kappa_p \in \mathrm{Sel}(\mathbf{Q}, V_pE)$, and prove the following rank two analogue of Kolyvagin's result:

$$\kappa_p \neq 0 \quad \Longrightarrow \quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2.$$

Conversely, when $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$ we show that $\kappa_p \neq 0$ *if and only if* the restriction map

$$\mathrm{Sel}(\mathbf{Q}, V_pE) \to E(\mathbf{Q}_p)\hat{\otimes}\mathbf{Q}_p$$

is nonzero. The proof of these results, which extend and strenghten similar results of the author with Hsieh in the non-CM case, exploit a new link between the nonvanishing of generalised Kato classes and a main conjecture in anticyclotomic Iwasawa theory.

1. INTRODUCTION

Let E be an elliptic curve over the rationals. The systematic construction of rational points on E, akin to the construction of Heegner points when $\mathrm{ord}_{s=1}L(E,s) \leq 1$ in the works of Gross–Zagier and Kolyvagin from the 1980s, is a well-known open problem lying behind further progress on the Birch–Swinnerton-Dyer conjecture in situations of higher order of vanishing. As an approximation to this problem, and following the reformulation and vast generalisation of the conjecture by Bloch–Kato, one might also attempt to construct non-torsion Selmer classes in such situations.

Suppose $L(E,s)$ has sign $+1$ in its functional equation and vanishes at $s=1$ (so $\mathrm{ord}_{s=1}L(E,s) \geq 2$). Fix a prime $p \geq 5$, and suppose E has CM by an imaginary quadratic field in which p splits. In this paper, we introduce a *generalised Kato class* $\kappa_p \in \mathrm{Sel}(\mathbf{Q}, V_pE)$ in the p-adic Selmer group fitting into the exact sequence

$$0 \to E(\mathbf{Q}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \to \mathrm{Sel}(\mathbf{Q}, V_pE) \to \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathrm{Ta}_p Ш(E/\mathbf{Q}) \to 0,$$

where $\mathrm{Ta}_p Ш(E/\mathbf{Q})$ is the p-adic Tate module for the Shafarevich–Tate group of E. The class κ_p is obtained from a "twisted variant" of a construction due to Darmon–Rotger [DR16], whence the terminology. One of the main results in this paper is a proof of the following rank two analogue of Kolyvagin's result [Kol88]:

$$\kappa_p \neq 0 \quad \Longrightarrow \quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2.$$

Conversely, still in the setting that $L(E,s)$ vanishes to positive even order at $s=1$, we show that if $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$ then $\kappa_p \neq 0$ *if and only if* the restriction map

$$\mathrm{Sel}(\mathbf{Q}, V_pE) \to E(\mathbf{Q}_p)\hat{\otimes}\mathbf{Q}_p$$

is nonzero, where $E(\mathbf{Q}_p)\hat{\otimes}\mathbf{Q}_p$ is the p-adic completion $\varprojlim_n E(\mathbf{Q}_p)/p^nE(\mathbf{Q}_p)$ tensored with $\mathbf{Q}_p$.

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In the non-CM case, similar results (for the generalised Kato classes κ_p^{DR} introduced in [DR16]) were obtained in a joint work for the author with M.-L. Hsieh [CH22]. The approach introduced in this paper to handle the CM case (a case that is essentially excluded by the approach in *loc. cit.*; for instance, it requires E to have some prime $\ell \neq p$ of multiplicative reduction) also yields a new proof of the original results in the non-CM case (see [Cas23]).

In the rest of this Introduction we explain our results more precisely, and some key ideas behind the proof.

1.1. **Diagonal cycle main conjecture.** Following the work of Darmon–Rotger [DR17], as further systematically developed also by Bertolini–Seveso–Venerucci (see [DR22, BSV22]), attached to a triple of Hida families $(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})$ with tame characters having product $\chi_\varphi\chi_g\chi_h = 1$ one has a *big diagonal class*

$$\kappa(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{H}^1(\mathbf{Q}, \mathbf{V}^\dagger),$$

where $\mathbf{V}^\dagger$ is a self-dual twist of the triple tensor product of the Galois representations associated to $\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h}$. It follows from its geometric construction (interpolating p-adic étale Abel–Jacobi images of generalised Gross–Kudla–Schoen diagonal cycles in p-adic families) that $\kappa(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})$ lands in the *balanced* Selmer group $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger)$.

Denote by $\mathcal{R} = \mathbb{I}_\varphi \hat{\otimes}_{\mathscr{O}} \mathbb{I}_g \hat{\otimes}_{\mathscr{O}} \mathbb{I}_h$ the completed tensor product of the ring of definition of $\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h}$; this is a finite extension of the three-variable Iwasawa algebra $\Lambda\hat{\otimes}_{\mathscr{O}}\Lambda\hat{\otimes}_{\mathscr{O}}\Lambda$, where $\Lambda = \mathscr{O}[\![1 + p\mathbf{Z}_p]\!]$ and $\mathscr{O}$ is a finite extension of $\mathbf{Z}_p$. Under the root number condition

$$\varepsilon_\ell(\mathbf{V}_{\underline{Q}_0}) = +1 \quad \text{for all primes} \quad \ell \mid N_\varphi N_g N_h$$

(see (2.4) for the unexplained notation), the range of specialisations $\underline{Q} \in \mathrm{Spec}(\mathcal{R})(\overline{\mathbf{Q}}_p)$ for which $\kappa(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})$ recovers a generalised Gross–Kudla–Schoen diagonal cycle, is also a range in which the sign of the triple product L-function $L(\mathbf{V}^\dagger_{\underline{Q}}, s)$ is -1.

Motivated by Perrin-Riou's Heegner point main conjecture [PR87] (and more precisely, Howard's extension in the context of big Heegner points [How07]), the following is expected about $\kappa(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})$:

Conjecture A (Big diagonal class main conjecture)**.** *The class $\kappa(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})$ is not $\mathcal{R}$-torsion, the modules $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger)$ and $X^{\mathrm{bal}}(\mathbf{Q}, \mathbf{A}^\dagger)$ both have $\mathcal{R}$-rank one, and*

$$\mathrm{char}_{\mathcal{R}}\big(X^{\mathrm{bal}}(\mathbf{Q}, \mathbf{A}^\dagger)_{\mathrm{tors}}\big) = \mathrm{char}_{\mathcal{R}}\left(\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger)}{\mathcal{R} \cdot \kappa(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})}\right)^2$$

in $\mathcal{R} \otimes \mathbf{Q}_p$, where the subscript tors *denotes the $\mathcal{R}$-torsion submodule.*

Here $X^{\mathrm{bal}}(\mathbf{Q}, \mathbf{A}^\dagger) = \mathrm{Hom}(\mathrm{Sel}(\mathbf{Q}, \mathbf{A}^\dagger), \mathbf{Q}_p/\mathbf{Z}_p)$ is the Pontryagin dual of the balanced Selmer group with coefficients in $\mathbf{A}^\dagger = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{V}^\dagger, \mu_{p^\infty})$ (see §4.2).

When the Hida families $\boldsymbol{g}, \boldsymbol{h}$ are specialised to classical modular forms g, h of weights $l, m \geq 2$ with $l \equiv m \pmod 2$ and $\boldsymbol{\varphi}$ is a CM Hida family, denoting by $\mathbb{V}^\dagger_{\boldsymbol{\varphi}gh}$ the resulting specialisation of $\mathbf{V}^\dagger$, the divisibility

$$\mathrm{char}_{\mathcal{R}_{\boldsymbol{\varphi}gh}}\big(X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^\dagger_{\boldsymbol{\varphi}gh})_{\mathrm{tors}}\big) \supset \mathrm{char}_{\Lambda_{\mathrm{ac}}}\left(\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{\boldsymbol{\varphi}gh})}{\mathcal{R}_{\boldsymbol{\varphi}gh} \cdot \kappa(\boldsymbol{\varphi}, g, h)}\right)^2$$

in $\mathcal{R}_{\boldsymbol{\varphi}gh} \otimes \mathbf{Q}_p$ was proved under some hypotheses in [ACR23b, Thm. 9.10] by constructing an anticyclotomic Euler system (in the sense of Jetchev–Nekovář–Skinner [JNS]) having the specialised big diagonal class $\kappa(\boldsymbol{\varphi}, g, h)$ as its bottom class. On the other hand, closer to the setting of this paper, when $\boldsymbol{\varphi}$ is specialised to a classical modular form φ of even weight $k \geq 2$ and $\boldsymbol{g}, \boldsymbol{h}$ are

both CM Hida families with respect to the same imaginary quadratic field, a similar divisibility is obtained in [CD23, Thm. 3.3.3] by building an anticyclotomic Euler system containing the resulting specialisation of $\kappa(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})$ as its bottom class. However, both of these results are subject to a "big image" hypothesis which in particular for the results of [CD23] excludes the case where φ has CM.

The first main result of this paper is the proof of a two-variable specialisation of Conjecture A in the CM case. We consider the case in which $\varphi = \theta(\lambda_0) \in S_2(\Gamma_0(N_\varphi))$ has CM by an imaginary quadratic field K in which $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits, and

$$\boldsymbol{g} = \boldsymbol{\theta}_{\lambda_1}(S_1), \quad \boldsymbol{h} = \boldsymbol{\theta}_{\lambda_2}(S_2)$$

are both CM Hida families by the same K (see §3.1 for the construction of $\boldsymbol{\theta}_{\lambda_i}(S_i)$). Here λ_0 (resp. λ_1, λ_2) is a Hecke character of K of infinity type $(-1, 0)$ (resp. finite order). We assume the conductor N_φ of φ is prime-to-p, and let N_g, N_h denote the tame conductor of $\boldsymbol{g}, \boldsymbol{h}$, respectively.

Letting $\mathbb{V}^\dagger_{\varphi\boldsymbol{gh}}$ denote the resulting two-variable specialisation of $\mathbf{V}^\dagger$, under mild hypotheses on $\boldsymbol{g}$ and $\boldsymbol{h}$ (see Proposition 4.3.1) it is easy to see that

$$\begin{aligned}\mathbb{V}^\dagger_{\varphi\boldsymbol{gh}} \simeq{}& \mathrm{Ind}_K^{\mathbf{Q}}\big(\lambda_0^{-1}\lambda_1^{-1}\lambda_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big) \oplus \mathrm{Ind}_K^{\mathbf{Q}}\big(\lambda_0^{-1}\lambda_1^{-\mathbf{c}}\lambda_2^{-\mathbf{c}}\Psi_{W_1}^{\mathbf{c}-1}\big)\\ &\oplus \mathrm{Ind}_K^{\mathbf{Q}}\big(\lambda_0^{-1}\lambda_1^{-1}\lambda_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\big) \oplus \mathrm{Ind}_K^{\mathbf{Q}}\big(\lambda_0^{-1}\lambda_1^{-\mathbf{c}}\lambda_2^{-1}\Psi_{W_2}^{\mathbf{c}-1}\big),\end{aligned} \tag{1.1}$$

where $W_1 = (1+S_1)^{1/2}(1+S_2)^{1/2} - 1$ and $W_2 = (1+S_1)^{1/2}(1+S_2)^{-1/2} - 1$ are formal variables parametrising anticyclotomic weight space. By the self-duality condition $\chi_\varphi\chi_g\chi_h = 1$, the Hecke character

$$\lambda := \lambda_0\lambda_1\lambda_2$$

is self-dual, and so are the other three characters appearing in (1.1). Let $\mathrm{sign}(\lambda) \in \{\pm 1\}$ be the sign in the functional equation for the Hecke L-function $L(\lambda^{-1}, s)$ (with center at $s = 0$), and similarly for the other three characters

$$\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}} = \lambda(\lambda_1\lambda_2)^{\mathbf{c}-1}, \quad \lambda_0\lambda_1\lambda_2^{\mathbf{c}} = \lambda\lambda_2^{\mathbf{c}-1}, \quad \lambda_0\lambda_1^{\mathbf{c}}\lambda_2 = \lambda\lambda_1^{\mathbf{c}-1}$$

appearing in (1.1). The theta series φ can be obtained as a weight 2 specialisation of a unique CM Hida family $\boldsymbol{\varphi}$, and we let $\mathcal{R}_{\varphi\boldsymbol{gh}} \simeq \mathscr{O}[\![S_1, S_2]\!]$ be the resulting specialisation of the coefficient ring $\mathcal{R}$. Denote by $\bar\rho_{\boldsymbol{g}}, \bar\rho_{\boldsymbol{h}}$ the residual Galois representations associated to $\boldsymbol{g}, \boldsymbol{h}$, respectively.

Theorem A. *Let the triple*

$$(\varphi, \boldsymbol{g}, \boldsymbol{h}) = (\theta(\lambda_0), \boldsymbol{\theta}_{\lambda_1}(S_1), \boldsymbol{\theta}_{\lambda_2}(S_2))$$

be as above, and suppose that:

(i) $\gcd(N_\varphi, N_g, N_h)$ *is squarefree.*
(ii) $\bar\rho_{\boldsymbol{g}}, \bar\rho_{\boldsymbol{h}}$ *are both absolutely irreducible and p-distinguished.*
(iii) $\mathrm{sign}(\lambda) = \mathrm{sign}(\lambda(\lambda_1\lambda_2)^{\mathbf{c}-1}) = \mathrm{sign}(\lambda\lambda_2^{\mathbf{c}-1}) = \mathrm{sign}(\lambda\lambda_1^{\mathbf{c}-1}) = +1$.

Then $\kappa(\varphi, \boldsymbol{g}, \boldsymbol{h})$ is not $\mathcal{R}_{\varphi\boldsymbol{gh}}$-torsion, the modules $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}})$ and $X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^\dagger_{\varphi\boldsymbol{gh}})$ both have $\mathcal{R}_{\varphi\boldsymbol{gh}}$-rank one, and

$$\mathrm{char}_{\mathcal{R}_{\varphi\boldsymbol{gh}}}\big(X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^\dagger_{\varphi\boldsymbol{gh}})_{\mathrm{tors}}\big) = \mathrm{char}_{\mathcal{R}_{\varphi\boldsymbol{gh}}}\left(\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}})}{\mathcal{R}_{\varphi\boldsymbol{gh}} \cdot \kappa(\varphi, \boldsymbol{g}, \boldsymbol{h})}\right)^2$$

in $\mathcal{R}_{\varphi\boldsymbol{gh}} \otimes \mathbf{Q}_p$. In other words, Conjecture A holds for $(\varphi, \boldsymbol{g}, \boldsymbol{h})$.

A key input in the proof of this result is the relation between Conjecture A and the anticyclotomic main conjecture for Hecke characters. Indeed, in the setting of Theorem A, we show that the balanced Selmer group $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^{\dagger}_{\varphi\boldsymbol{gh}})$ decomposes as

$$\begin{aligned}\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^{\dagger}_{\varphi\boldsymbol{gh}}) \simeq \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda(\lambda_1\lambda_2)^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda\lambda_2^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda\lambda_1^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1}).\end{aligned} \tag{1.2}$$

The Selmer groups in the right-hand side of this decomposition correspond to usual (i.e., Bloch–Kato) anticyclotomic Selmer groups attached to Hecke characters, except for the Selmer group

$$\mathrm{Sel}_{0,\emptyset}(K, T_{\lambda(\lambda_1\lambda_2)^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}),$$

which is obtained from the usual anticyclotomic Selmer group for $T_{\lambda(\lambda_1\lambda_2)^{\mathbf{c}-1}}$ by *reversing* the local conditions at the primes above $\mathfrak{p}$ and $\overline{\mathfrak{p}}$. We exploit the fact that Conjecture A follows from the Iwasawa–Greenberg main conjecture for the p-adic triple product L-function $\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h}) \in \mathcal{R}_{\varphi\boldsymbol{gh}}$ constructed by Hsieh [Hsi21]. Under certain conditions preventing the vanishing of $\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h})$ for sign reasons, this main conjecture predicts that the φ-*unbalanced* Selmer group $X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi\boldsymbol{gh}})$ is $\mathcal{R}_{\varphi\boldsymbol{gh}}$-torsion, with characteristic ideal generated by $\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h})^2$. We prove a decomposition for $X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi\boldsymbol{gh}})$ analogous to (1.2) in which (contrary to the case of the balanced Selmer groups) *all* direct summands agree with classical anticyclotomic Selmer groups for Hecke characters. Together with a parallel factorisation for $\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h})^2$ into a product of four anticyclotomic Katz p-adic L-functions, we thus deduce from the works of Agboola–Howard [AH06] and Arnold [Arn07] (an anticyclotomic specialisation of Rubin's proof of the Iwasawa main conjecture for K [Rub91]) a proof of the Iwasawa–Greenberg main conjecture for $X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi\boldsymbol{gh}})$. (The sign conditions in (iii) of Theorem A are needed at this point, as otherwise our results imply that $X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi\boldsymbol{gh}})$ has $\mathcal{R}_{\varphi\boldsymbol{gh}}$-rank two.) The proof of Theorem A then follows.

1.2. **Generalised Kato classes attached to E.** We keep K to be an imaginary quadratic field in which $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits, and now let $E/\mathbf{Q}$ be an elliptic curve with CM by the ring of integers of K. Note that the splitting condition on p implies that E has good ordinary reduction of p. Let ψ_E be the Hecke character of K attached to E, so that

$$L(E, s) = L(\psi_E, s).$$

Suppose the sign is the functional equation of $L(E, s)$ is $w = +1$. In the second part of the paper, we choose Hecke characters $\lambda_0, \lambda_1, \lambda_2$ for K as above satisfying

$$\psi_E = \lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}, \quad L(\psi_E^{-1}\lambda_1^{\mathbf{c}-1}, 0) \cdot L(\psi_E^{-1}\lambda_2^{\mathbf{c}-1}, 0) \cdot L(\psi_E^{-1}(\lambda_1\lambda_2)^{\mathbf{c}-1}, 0) \neq 0, \tag{1.3}$$

whose existence follows from nonvanishing results due to Greenberg and Rohrlich [Gre85, Roh84], and — inspired by a construction of generalised Kato classes due to Darmon–Rotger [DR17, DR16] — we let

$$\kappa_p \in \mathrm{H}^1(\mathbf{Q}, V_pE)$$

be the image of the resulting big diagonal class $\kappa(\varphi, \boldsymbol{g}, \boldsymbol{h})$ (for $(\varphi, \boldsymbol{g}, \boldsymbol{h}) = (\theta(\lambda_0), \boldsymbol{\theta}_{\lambda_1}(S_1), \boldsymbol{\theta}_{\lambda_2}(S_2))$ as above) under the composition

$$\mathrm{H}^1(\mathbf{Q}, \mathbb{V}^{\dagger}_{\varphi\boldsymbol{gh}}) \to \mathrm{H}^1\big(\mathbf{Q}, \mathrm{Ind}_K^{\mathbf{Q}}(T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1})\big) \to \mathrm{H}^1(\mathbf{Q}, \mathrm{Ind}_K^{\mathbf{Q}}(T_{\psi_E})) \simeq \mathrm{H}^1(\mathbf{Q}, T_pE)$$

arising from projection onto the second direct summand in (1.1) and the specialisation at $W_1 = 0$. The construction of κ_p might be seen as a twisted variant of the construction of geneneralised Kato classes in [DR16], which for an elliptic curve $E/\mathbf{Q}$ as above would take $\lambda_0 = \psi_E$ and the finite

order Hecke characters λ_1, λ_2 to be inverses of each other (similarly as in [CH22]). Nevertheless, from the explicit reciprocity law of [DR22] and [BSV22] we deduce the implication

$$L(E,1) = 0 \quad \Longrightarrow \quad \kappa_p \in \mathrm{Sel}(\mathbf{Q}, V_pE).$$

Since we assume that $L(E,s)$ has sign $+1$, the vanishing of $L(E,1)$ implies that $\mathrm{ord}_{s=1}L(E,s) \geq 2$, and so by the Bloch–Kato conjecture [BK90], the Selmer group $\mathrm{Sel}(\mathbf{Q}, V_pE)$ is expected to be at least 2-dimensional. Our next result is consistent with this expectation, and further justifies the view of κ_p as a "rank 2 p-adic regulator".

Theorem B. *Suppose $L(E,s)$ vanishes to positive even order at $s=1$. Let $\lambda_0, \lambda_1, \lambda_2$ be any triple of Hecke characters of K as above satisfying (1.3) and the conditions (i)–(iii) in Theorem A, and let κ_p be the associated generalised Kato class. Then*

$$\kappa_p \neq 0 \quad \Longrightarrow \quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2.$$

Conversely, if $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$ then $\kappa_p \neq 0$ if and only if the restriction map

$$\mathrm{res}_p : \mathrm{Sel}(\mathbf{Q}, V_pE) \to E(\mathbf{Q}_p)\hat{\otimes}\mathbf{Q}_p$$

is nonzero.

The existence of (infinitely many) triples $\lambda_0, \lambda_1, \lambda_2$ satisfying (1.3) and the conditions in Theorem A follows easily from the aforementioned nonvanishing results due to Greenberg and Rorhlich (note that the sign condition (iii) is implied by (1.3)).

In addition to Theorem A, the ingredients in the proof of Theorem B are a version of Mazur's control theorem for the Selmer groups in the decomposition (1.1) and a global duality argument allowing us to relate the rank of $\mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E})$ and $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE)$.

The result of Theorem B is consistent with predictions by Darmon–Rotger [DR16], and it offers some new insights. More precisely, that the conditions $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$ and $\mathrm{res}_p \neq 0$ imply $\kappa_p \neq 0$ is suggested by [*op. cit.*, Conj. 3.12] (in the "rank $(2,0)$ setting" of §4.5.3); a new insight of Theorem B is that the condition $\mathrm{res}_p \neq 0$ is also *necessary* for the nonvanishing of κ_p.

1.3. **Application to rank two Selmer basis.** The construction of $\kappa_p \in \mathrm{Sel}(\mathbf{Q}, V_pE)$ depends on a choice $\lambda_0, \lambda_1, \lambda_2$, but it follows from our results that different choices give rise to the same Selmer class up to scaling. (In fact, whenever nonzero, κ_p generates the one-dimensional subspace $\ker(\mathrm{res}_p) \subset \mathrm{Sel}(\mathbf{Q}, V_pE)$.) It is then natural to ask for a class in the two-dimensional $\mathrm{Sel}(\mathbf{Q}, V_pE)$ complementary to the line spanned by κ_p. Our results also yield an answer to this question under some hypotheses.

Let $\mathcal{W}$ the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$, and let $\mathcal{L}_{\mathfrak{p}} = \mathcal{L}_{\mathfrak{p},\mathfrak{f}} \in \mathcal{W}[\![Z(\mathfrak{f})]\!]$ be the Katz p-adic L-function recalled in Theorem 2.1.1 below, where $Z(\mathfrak{f})$ is the Galois group of the extension $K(E[p^\infty])/K$ and $\mathfrak{f} \subset \mathcal{O}_K$ is the conductor of ψ_E. For $s \in \mathbf{Z}_p$ define

$$L_{\mathfrak{p}}(s) = \mathcal{L}_{\mathfrak{p}}(\psi_E\langle\psi_E\rangle^{s-1}), \quad L^*_{\mathfrak{p}}(s) = \mathcal{L}_{\mathfrak{p}}(\psi_E^{\mathbf{c}}\langle\psi_E^{\mathbf{c}}\rangle^{s-1}),$$

where $\langle - \rangle : \mathbf{Z}_p^\times \to 1 + p\mathbf{Z}_p$ is the projection onto the 1-units.

Theorem C. *Let the hypotheses be as in Theorem B, and assume in addition that $Ш(E/\mathbf{Q})[p^\infty]$ is finite and the following conditions hold:*

$$\mathrm{ord}_{s=1}L_{\mathfrak{p}}(s) = 2, \quad \mathrm{ord}_{s=1}L^*_{\mathfrak{p}}(s) = 1.$$

Then $\mathrm{Sel}(\mathbf{Q}, V_pE)$ is 2-dimensional, with

$$\mathrm{Sel}(\mathbf{Q}, V_pE) = \mathbf{Q}_p\kappa_p \oplus \mathbf{Q}_p x^{(2)}_{\mathfrak{p}},$$

where κ_p is a generalised Kato class and $x_{\mathfrak{p}}^{(2)}$ is a "derived" elliptic unit.

Proof. The assumption that $\mathrm{ord}_{s=1}L_{\mathfrak{p}}(s) = 2$ implies that

$$r := \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) \leq 2$$

by Rubin's proof of the Iwasawa main conjecture for K and the work of Perrin-Riou (see [Rub91, Thm. 4.1] and [PR84, Ch. IV, Thm. 22]). On the other hand, it also implies $r \geq 2$ by the theorem of Coates–Wiles [CW77] and the p-parity conjecture [Guo93]. Therefore $r = 2$, and by [Rub92, Prop. 4.4] the construction of derived elliptic units in [*op. cit.*, §6] yields a class

$$x_{\mathfrak{p}}^{(2)} \in \mathrm{Sel}(K, T_{\mathfrak{p}}E) \simeq \mathrm{Sel}(\mathbf{Q}, T_pE)$$

where $T_{\mathfrak{p}}E$ is the $\mathfrak{p}$-adic Tate module of E. Since by [Rub92, Thm. 9.5(ii)] and our assumptions we have $\mathrm{res}_p(x_{\mathfrak{p}}^{(2)}) \neq 0$, the result follows from Theorem B. □

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2. p-ADIC L-FUNCTIONS

In this section we recall the two p-adic L-functions that will appear in our arguments, one due to Katz [Kat78] attached to Hecke characters of an imaginary quadratic field, and another due to Hsieh [Hsi21] (extending and refining earlier constructions due to Harris–Tilouine [HT01] and Darmon–Rotger [DR14]) attached to triple products of modular forms in Hida families.

Fix a prime $p > 2$ and an imaginary quadratic field K with ring of integers $\mathcal{O}_K$ in which

(spl) $$(p) = \mathfrak{p}\overline{\mathfrak{p}} \text{ splits},$$

with $\mathfrak{p}$ the prime of K above p determined by a fixed embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$.

2.1. **Katz p-adic L-function.** Denote by $D_K < 0$ the discriminant of K, and fix an integral ideal $\mathfrak{C} \subset \mathcal{O}_K$ coprime to p. Let $\mathcal{W}$ be a finite extension of the Witt ring $W(\overline{\mathbf{F}}_p)$, and denote by $Z(\mathfrak{C})$ the Galois group of the ray class field extension $K(\mathfrak{C}p^\infty)/K$.

We say that a Hecke character $\chi : K^\times\backslash\mathbb{A}_K^\times \to \mathbf{C}^\times$ has infinity type (a, b) if $\chi_\infty(z) = z^a\overline{z}^b$. We take *geometric* normalisations for the reciprocity maps of class field theory. If χ has conductor dividing $\mathfrak{C}$, viewing χ as a $\overline{\mathbf{Q}}$-valued character (via a fixed embedding $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}^\times$) defined on the group of fractional ideals of K coprime to $\mathfrak{C}$, with a slight abuse of notation we also denote by χ the character of $Z(\mathfrak{C})$ defined by the rule

$$\chi(\sigma_{\mathfrak{l}}^{-1}) = \iota_p(\chi(\mathfrak{l}))$$

for all primes $\mathfrak{l} \nmid \mathfrak{C}$, where $\sigma_{\mathfrak{l}} \in Z(\mathfrak{C})$ is the Artin symbol of $\mathfrak{l}$.

Theorem 2.1.1. *There exists an element $\mathcal{L}_{\mathfrak{p},\mathfrak{C}} \in \mathcal{W}[\![Z(\mathfrak{C})]\!]$ such that for all Hecke characters χ of conductor dividing $\mathfrak{C}p^\infty$ and infinity type $(k + j, -j)$ with $k \geq 1$, $j \geq 0$, we have*

$$\mathcal{L}_{\mathfrak{p},\mathfrak{C}}(\chi) = \left(\frac{\Omega_p}{\Omega_\infty}\right)^{k+2j} \cdot \Gamma(k + j) \cdot \left(\frac{2\pi}{\sqrt{D_K}}\right)^j \cdot \mathcal{E}_{\mathfrak{p}}(\chi) \cdot L^{(p\mathfrak{C})}(\chi, 0),$$

where $\mathcal{E}_{\mathfrak{p}}(\chi)$ is the modified Euler factor

$$\mathcal{E}_{\mathfrak{p}}(\chi) = \frac{L(0, \chi_{\mathfrak{p}})}{\varepsilon(0, \chi_{\mathfrak{p}}) \cdot L(1, \chi_{\mathfrak{p}}^{-1})},$$

and $L^{(p\mathfrak{C})}(\chi, s)$ is the L-function of χ deprived from the Euler factors at the primes dividing $p\mathfrak{C}$. Moreover, we have the functional equation

$$\mathcal{L}_{\mathfrak{p},\mathfrak{C}}(\chi) = \mathcal{L}_{\mathfrak{p},\overline{\mathfrak{C}}}((\chi^{\mathbf{c}})^{-1}\mathbf{N}^{-1}),$$

where $\chi^{\mathbf{c}}$ is the composition of χ with the action of the non-trivial element $\mathbf{c} \in \mathrm{Aut}(K/\mathbf{Q})$, and the equality is up to a p-adic unit.

Proof. See [Kat78], [HT93]; our formulation follows of the interpolation property follows [Hsi14, Prop. 4.19] most closely. The functional equation is shown in [dS87, Thm. II.6.4]. □

2.2. **Triple product p-adic L-function.** Let $\mathbb{I}$ be a normal domain finite flat over

$$\Lambda := \mathscr{O}[\![1 + p\mathbf{Z}_p]\!],$$

where $\mathscr{O}$ is the ring of integers of a finite extension of $\mathbf{Q}_p$. For a positive integer N with $p \nmid N$ and a Dirichlet character $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathscr{O}^\times$, we denote by $S^o(N, \chi, \mathbb{I}) \subset \mathbb{I}[\![q]\!]$ the space of ordinary $\mathbb{I}$-adic cusp forms of tame level N and branch character χ as defined in [Hsi21, §3.1].

Denote by $\mathfrak{X}_{\mathbb{I}}^+ \subset \mathrm{Spec}\,\mathbb{I}(\overline{\mathbf{Q}}_p)$ the set of *arithmetic points* of $\mathbb{I}$, consisting of the ring homomorphisms $Q : \mathbb{I} \to \overline{\mathbf{Q}}_p$ such that $Q|_{1+p\mathbf{Z}_p}$ is given by $z \mapsto z^{k_Q}\epsilon_Q(z)$ for some $k_Q \in \mathbf{Z}_{\geq 2}$ called the *weight of* Q and $\epsilon_Q(z) \in \mu_{p^\infty}$. As in [Hsi21, §3.1], we say that $\boldsymbol{f} = \sum_{n=1}^\infty a_n(\boldsymbol{f})q^n \in S^o(N, \chi, \mathbb{I})$ is a *primitive Hida family* if for every $Q \in \mathfrak{X}_{\mathbb{I}}^+$ the specialisation $\boldsymbol{f}_Q$ gives the q-expansion of an ordinary p-stabilised newform of weight of k_Q and tame conductor N. Attached to such $\boldsymbol{f}$ we let $\mathfrak{X}_{\mathbb{I}}^{\mathrm{cls}}$ be the set of ring homomorphisms Q as above with $k_Q \in \mathbf{Z}$ such that $\boldsymbol{f}_Q$ is the q-expansion of a classical modular form. Note that $\mathfrak{X}_{\mathbb{I}}^{\mathrm{cls}}$ contains $\mathfrak{X}_{\mathbb{I}}^+$ by Hida's results, but it can also contain points Q — of special relevance for this paper — for which $\boldsymbol{f}_Q$ is a classical *weight one* form.

For $\boldsymbol{f}$ a primitive Hida family, we let

$$\rho_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \mathrm{Aut}_{\mathbb{I}}(V_{\boldsymbol{f}}) \simeq \mathrm{GL}_2(\mathbb{I})$$

denote the associated Galois representation, which here we take to be the *dual* of that in [Hsi21, §3.2]; in particular, the determinant of $\rho_{\boldsymbol{f}}$ is $\chi_{\mathbb{I}} \cdot \varepsilon_{\mathrm{cyc}}$ in the notations of *loc. cit.*, where $\varepsilon_{\mathrm{cyc}}$ is the p-adic cyclotomic character. Note also that the $\rho_{\boldsymbol{f}}$ in *loc. cit.* is valued in $\mathrm{Frac}(\mathbb{I})$, but it is well-known that when the residual representation $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible, one may find a free $\mathbb{I}$-module $V_{\boldsymbol{f}}$ of rank 2 realising the same Galois representation after extension of scalars. There may be different $V_{\boldsymbol{f}}$ giving rise to the same rational $\rho_{\boldsymbol{f}}$ after extension of scalars; for now we take any such $V_{\boldsymbol{f}}$ (assuming $\bar{\rho}_{\boldsymbol{f}}$ to be absolutely irreducible), and later in §4.3 we specify a particular choice well-suited to our applications.

By [Wil88, Thm. 2.2.2], restricted to $G_{\mathbf{Q}_p}$ the Galois representation $V_{\boldsymbol{f}}$ fits into a short exact sequence

$$0 \to V_{\boldsymbol{f}}^+ \to V_{\boldsymbol{f}} \to V_{\boldsymbol{f}}^- \to 0,$$

where the quotient $V_{\boldsymbol{f}}^-$ is free of rank one over $\mathbb{I}$, with the $G_{\mathbf{Q}_p}$-action given by the unramified character sending an arithmetic Frobenius σ_p to $a_p(\boldsymbol{f})$.

Let

$$(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h}) \in S^o(N_\varphi, \chi_\varphi, \mathbb{I}_\varphi) \times S^o(N_g, \chi_g, \mathbb{I}_g) \times S^o(N_h, \chi_h, \mathbb{I}_h)$$

be a triple of primitive Hida families with

$$\chi_\varphi\chi_g\chi_h = \omega^{2a} \text{ for some } a \in \mathbf{Z}, \tag{2.1}$$

where ω is the Teichmüller character. Put

$$\mathcal{R} = \mathbb{I}_\varphi \hat{\otimes}_{\mathscr{O}} \mathbb{I}_g \hat{\otimes}_{\mathscr{O}} \mathbb{I}_h,$$

which is a finite extension of the three-variable Iwasawa algebra $\Lambda\hat{\otimes}_{\mathscr{O}}\Lambda\hat{\otimes}_{\mathscr{O}}\Lambda$, and let

$$\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{\varphi}} := \{(Q_0,Q_1,Q_2)\in \mathfrak{X}_{\mathbb{I}_{\boldsymbol{\varphi}}}^{+}\times \mathfrak{X}_{\mathbb{I}_{\boldsymbol{g}}}^{\mathrm{cls}}\times \mathfrak{X}_{\mathbb{I}_{\boldsymbol{h}}}^{\mathrm{cls}} \\ : k_{Q_0}\geq k_{Q_1}+k_{Q_2} \text{ and } k_{Q_0}\equiv k_{Q_1}+k_{Q_2} \pmod 2\}$$

be the weight space for $\mathcal{R}$ in the so-called *$\boldsymbol{\varphi}$-unbalanced range*.

Let $\mathbf{V}=V_{\boldsymbol{\varphi}}\hat{\otimes}_{\mathscr{O}}V_{\boldsymbol{g}}\hat{\otimes}_{\mathscr{O}}V_{\boldsymbol{h}}$ be the triple tensor product Galois representation attached to $(\boldsymbol{\varphi},\boldsymbol{g},\boldsymbol{h})$, and writing $\det\mathbf{V}=\mathcal{X}^2\varepsilon_{\mathrm{cyc}}$ (as is possible by (2.1)), set

$$\mathbf{V}^{\dagger} := \mathbf{V}\otimes\mathcal{X}^{-1}, \tag{2.2}$$

which is a self-dual twist of $\mathbf{V}$. Define the rank four $G_{\mathbf{Q}_p}$-invariant subspace $\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^{\dagger})\subset\mathbf{V}^{\dagger}$ by

$$\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^{\dagger}) := V_{\boldsymbol{\varphi}}^{+}\hat{\otimes}_{\mathscr{O}}V_{\boldsymbol{g}}\hat{\otimes}_{\mathscr{O}}V_{\boldsymbol{h}}\otimes\mathcal{X}^{-1}. \tag{2.3}$$

For every $\underline{Q}=(Q_0,Q_1,Q_2)\in\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{\varphi}}$ we denote by $\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}_{\underline{Q}}^{\dagger})\subset\mathbf{V}_{\underline{Q}}^{\dagger}$ the corresponding specialisation. Finally, for every rational prime ℓ denote by $\varepsilon_{\ell}(\mathbf{V}_{\underline{Q}}^{\dagger})$ the epsilon factor attached to the restriction of $\mathbf{V}_{\underline{Q}}^{\dagger}$ to $G_{\mathbf{Q}_\ell}$ as in [Tat79, p. 21], and assume that

$$\text{for some } \underline{Q}\in\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{\varphi}}, \text{ we have } \varepsilon_{\ell}(\mathbf{V}_{\underline{Q}}^{\dagger})=+1 \text{ for all primes } \ell\mid N_{\boldsymbol{\varphi}}N_{\boldsymbol{g}}N_{\boldsymbol{h}}. \tag{2.4}$$

As explained in [Hsi21, §1.2], it is known that condition (2.4) is independent of $\underline{Q}$, and it implies that the sign in the functional equation for the triple product L-function

$$L(\mathbf{V}_{\underline{Q}}^{\dagger},s)$$

(relating its values at s and $-s$) is $+1$ for all $\underline{Q}\in\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{\varphi}}$.

For the next statement, we refer the reader to §2.3 for a review of the congruence ideal associated with a primitive Hida family.

Theorem 2.2.1. *Let $(\boldsymbol{\varphi},\boldsymbol{g},\boldsymbol{h})$ be a triple of primitive Hida families as above satisfying conditions (2.1) and (2.4). Assume in addition that:*

- $\gcd(N_\varphi,N_g,N_h)$ *is square-free,*
- *the residual representation $\bar{\rho}_{\boldsymbol{\varphi}}$ is absolutely irreducible and p-distinguished,*

and fix a generator $\eta_{\boldsymbol{\varphi}}$ of the congruence ideal of $\boldsymbol{\varphi}$. Then there exists a unique element

$$\mathscr{L}_p^{\boldsymbol{\varphi}}(\boldsymbol{\varphi},\boldsymbol{g},\boldsymbol{h})\in\mathcal{R}$$

such that for all $\underline{Q}=(Q_0,Q_1,Q_2)\in\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{\varphi}}$ of weight (k_0,k_1,k_2) with $\epsilon_{Q_0}=1$ we have

$$(\mathscr{L}_p^{\boldsymbol{\varphi}}(\boldsymbol{\varphi},\boldsymbol{g},\boldsymbol{h})(\underline{Q}))^2=\Gamma_{\mathbf{V}_{\underline{Q}}^{\dagger}}(0)\cdot\frac{L(\mathbf{V}_{\underline{Q}}^{\dagger},0)}{(\sqrt{-1})^{2k_0}\cdot\Omega_{\boldsymbol{\varphi}_{Q_0}}^2}\cdot\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}_{\underline{Q}}^{\dagger}))\cdot\prod_{\ell\in\Sigma_{\mathrm{exc}}}(1+\ell^{-1})^2,$$

where:

- $\Gamma_{\mathbf{V}_{\underline{Q}}^{\dagger}}(0)=\Gamma_{\mathbf{C}}(c_{\underline{Q}})\Gamma_{\mathbf{C}}(c_{\underline{Q}}+2-k_1-k_2)\Gamma_{\mathbf{C}}(c_{\underline{Q}}+1-k_1)\Gamma_{\mathbf{C}}(c_{\underline{Q}}+1-k_2)$, *with*

$$c_{\underline{Q}}=(k_0+k_1+k_2-2)/2$$

and $\Gamma_{\mathbf{C}}(s)=2(2\pi)^{-s}\Gamma(s)$;

- $\Omega_{\boldsymbol{\varphi}_{Q_0}}$ *is the canonical period*

$$\Omega_{\boldsymbol{\varphi}_{Q_0}} := (-2\sqrt{-1})^{k_0+1} \cdot \frac{\|\boldsymbol{\varphi}^\circ_{Q_0}\|^2_{\Gamma_0(N_\varphi)}}{\imath_p(\eta_{\boldsymbol{\varphi}_{Q_0}})} \cdot \Big(1 - \frac{\chi'_\varphi(p)p^{k_0-1}}{\alpha^2_{Q_0}}\Big)\Big(1 - \frac{\chi'_\varphi(p)p^{k_0-2}}{\alpha^2_{Q_0}}\Big),$$

with $\boldsymbol{\varphi}^\circ_{Q_0} \in S_{k_0}(N_\varphi)$ *the newform of conductor* N_φ *associated with* $\boldsymbol{\varphi}_{Q_0}$*,* χ'_φ *the prime-to-*p *part of* χ_φ*, and* α_{Q_0} *the specialisation of* $a_p(\boldsymbol{\varphi}) \in \mathbb{I}^\times_\varphi$ *at* Q_0*;*

- $\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^\dagger_{\underline{Q}}))$ *is the modified* p*-Euler factor*

$$\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^\dagger_{\underline{Q}})) := \frac{L_p(\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^\dagger_{\underline{Q}}), 0)}{\varepsilon_p(\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^\dagger_{\underline{Q}})) \cdot L_p(\mathbf{V}^\dagger_{\underline{Q}}/\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^\dagger_{\underline{Q}}), 0)} \cdot \frac{1}{L_p(\mathbf{V}^\dagger_{\underline{Q}}, 0)},$$

and Σ_{exc} *is an explicitly defined subset of the prime factors of* $N_\varphi N_g N_h$*,* [Hsi21, p. 416].

Proof. This is [Hsi21, Thm. A]. □

Remark 2.2.2. For simplicity, we have stated the interpolation property of $\mathscr{L}_p^{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})$ only for $\underline{Q}$ with $\epsilon_{Q_0} = 1$, as this will suffice for our purposes; see [Hsi21, Thm. A] for the interpolation property for all $\underline{Q} \in \mathfrak{X}^{\boldsymbol{\varphi}}_{\mathcal{R}}$.

2.3. **Congruence ideal.** Let $\boldsymbol{f} \in S^o(N_f, \chi_f, \mathbb{I})$ be a primitive Hida family defined over $\mathbb{I}$. Associated with $\boldsymbol{f}$ there is a $\mathbb{I}$-algebra homomorphism

$$\lambda_{\boldsymbol{f}} : \mathbb{T}(N_f, \mathbb{I}) \to \mathbb{I}$$

where $\mathbb{T}(N_f, \mathbb{I})$ is the Hecke algebra acting on $\oplus_\chi S^o(N_f, \chi, \mathbb{I})$, with χ running over the characters of $(\mathbf{Z}/pN_f\mathbf{Z})^\times$. Let $\mathbb{T}_{\mathfrak{m}}$ be the local component of $\mathbb{T}(N_f, \mathbb{I})$ through which $\lambda_{\boldsymbol{f}}$ factors, and following [Hid88] define the *congruence ideal* $C(\boldsymbol{f})$ of $\boldsymbol{f}$ by

$$C(\boldsymbol{f}) := \lambda_{\boldsymbol{f}}(\mathrm{Ann}_{\mathbb{T}_{\mathfrak{m}}}(\ker \lambda_{\boldsymbol{f}})) \subset \mathbb{I}.$$

When the residual representation $\bar\rho_{\boldsymbol{f}}$ is absolutely irreducible and p-distinguished, it follows from the results of [Wil95] and [Hid88] that $C(\boldsymbol{f})$ is generated by a nonzero element $\eta_{\boldsymbol{f}} \in \mathbb{I}$.

3. Factorisation of p-adic triple product L-function

In this section we relate the triple product p-adic L-function attached to triples of Hida families with CM by K to a product of anticyclotomic Katz p-adic L-functions.

3.1. **Hida families with CM.** We review the construction of CM Hida families following the exposition in [Hsi21, §8.1]. Since it will suffice for our purposes, we assume that the class number $h_K = |\mathrm{Pic}(\mathcal{O}_K)|$ of K is coprime to p. Let K_∞ be the unique $\mathbf{Z}_p^2$-extension of K, and denote by $K_{\mathfrak{p}^\infty}$ the maximal subfield of K_∞ unramified outside $\mathfrak{p}$. Put

$$\Gamma_\infty := \mathrm{Gal}(K_\infty/K) \simeq \mathbf{Z}_p^2, \qquad \Gamma_{\mathfrak{p}^\infty} := \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K) \simeq \mathbf{Z}_p.$$

Recall that for every ideal $\mathfrak{C} \subset \mathcal{O}_K$ we let $K(\mathfrak{C})$ be the ray class field of K of conductor $\mathfrak{C}$. Thus $K_{\mathfrak{p}^\infty}$ is also the maximal $\mathbf{Z}_p$-extension inside $K(\mathfrak{p}^\infty)$. By our assumption on h_K, the restriction of the Artin map to $K_{\mathfrak{p}}^\times$ induces an isomorphism $1 + p\mathbf{Z}_p \simeq \Gamma_{\mathfrak{p}^\infty}$, where we identified $\mathbf{Z}_p^\times$ and $\mathcal{O}_{K_{\mathfrak{p}}}^\times$ by the embedding $\imath_p$. Denote by $\gamma_{\mathfrak{p}}$ the topological generator of $\Gamma_{\mathfrak{p}^\infty}$ corresponding to $1+p$ under this isomorphism, and for each variable S let $\Psi_S : \Gamma_\infty \to \mathbf{Z}_p[\![S]\!]^\times$ be the universal character given by

$$\Psi_S(\sigma) = (1+S)^{l(\sigma)},$$

where $l(\sigma) \in \mathbf{Z}_p$ is such that $\sigma|_{K_{\mathfrak{p}^\infty}} = \gamma_{\mathfrak{p}}^{l(\sigma)}$. Fix $\mathfrak{C}$ prime to p, and for any finite order character $\psi : G_K \to \mathscr{O}^\times$ of tame conductor $\mathfrak{C}$ put

$$\boldsymbol{\theta}_\psi(S)(q) = \sum_{(\mathfrak{a},\mathfrak{p}\mathfrak{C})=1} \psi(\sigma_\mathfrak{a})\Psi_S^{-1}(\sigma_\mathfrak{a})q^{\mathbf{N}(\mathfrak{a})} \in \mathscr{O}[\![S]\!][\![q]\!],$$

where $\sigma_\mathfrak{a} \in \mathrm{Gal}(K(\mathfrak{c}\mathfrak{p}^\infty)/K)$ is the Artin symbol. Then $\boldsymbol{\theta}_\psi(S)$ is a primitive Hida family (called the primitive *CM Hida family* with branch character ψ) defined over $\mathscr{O}[\![S]\!]$ of level $D_K\mathbf{N}(\mathfrak{C})$ and tame character $(\psi \circ \mathscr{V})\eta_{K/\mathbf{Q}}\omega^{-1}$, where

$$\mathscr{V} : G_{\mathbf{Q}}^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$$

is the transfer map and $\eta_{K/\mathbf{Q}}$ is the quadratic character associated to $K/\mathbf{Q}$.

3.2. **Congruence ideal of CM Hida families.** Let $\boldsymbol{\varphi}$ be a primitive CM Hida family defined over $\mathscr{O}[\![S]\!]$. In this section we recall the characterisation of the congruence ideal of $\boldsymbol{\varphi}$ that follows from its relation with the anticyclotomic main conjecture for the underlying imaginary quadratic field K.

Let $N_\varphi = D_K\mathbf{N}(\mathfrak{C})$ be the tame conductor of $\boldsymbol{\varphi}$, and assume that the ideal $\mathfrak{C} \subset \mathcal{O}_K$ is coprime to p and fixed under the action of complex conjugation, so $\mathbf{c}$ acts on $Z(\mathfrak{C}) = \mathrm{Gal}(K(\mathfrak{C}p^\infty)/K)$. Let $\Delta_\mathfrak{C}$ be the torsion subgroup and $Z(\mathfrak{C})$, and put $\Gamma_K := Z(\mathfrak{C})/\Delta_\mathfrak{C} \simeq \mathbf{Z}_p^2$, which is identified with the Galois group of the unique $\mathbf{Z}_p^2$-extension K_∞/K. Fix a decomposition

$$Z(\mathfrak{C}) \simeq \Delta_\mathfrak{C} \times \Gamma_K, \tag{3.1}$$

and note that if $p \nmid h_K$, as we assume in the following, then $\Delta_\mathfrak{C}$ has order prime to p.

Let $Z(\mathfrak{C})^+$ be the maximal subgroup of $Z(\mathfrak{C})$ fixed by the action of $\mathbf{c}$. Put $Z(\mathfrak{C})^- = Z(\mathfrak{C})/Z(\mathfrak{C})^+$, and denote by $\pi : Z(\mathfrak{C}) \to Z(\mathfrak{C})^-$ the natural projection. We also have a decomposition (which we fix to be compatible with (3.1) under π)

$$Z(\mathfrak{C})^- \simeq \Delta_\mathfrak{C}^- \times \Gamma^-,$$

where $\Delta_\mathfrak{C}^-$ is the torsion subgroup of $\Delta_\mathfrak{C}$ and Γ^- is the eigenspace of Γ_K where $\mathbf{c}$ acts as inversion.

The residual representation $\bar\rho_{\boldsymbol{\varphi}}$ is of the form

$$\bar\rho_{\boldsymbol{\varphi}} \simeq \mathrm{Ind}_K^{\mathbf{Q}}\bar\psi$$

for some Galois character $\bar\psi : G_K \to \mathbb{F}^\times$, where $\mathbb{F}$ is the residue field of $\mathscr{O}$. Put $\bar\psi^- := \bar\psi^{\mathbf{c}-1}$, and upon enlarging $\mathscr{O}$ if necessary, denote by $\psi^- : G_K \to \mathscr{O}^\times$ the Teichmüller lift of $\bar\psi^-$. We shall call ψ^- the anticyclotomic *branch character* corresponding to the CM Hida family $\boldsymbol{\varphi}$. Since ψ^- has order prime to p, its composition with π factors through $\Delta_\mathfrak{C}$, but its prime-to-p conductor may be a proper divisor of $\mathfrak{C}$. Let $\mathfrak{c}$ be the prime-to-p conductor of ψ^- viewed as a character of $\Delta_\mathfrak{C}$ via π (i.e., $\mathfrak{c}$ is the maximal divisor of $\mathfrak{C}$ such that $\chi\pi$ factors though the quotient map $\Delta_\mathfrak{C} \twoheadrightarrow \Delta_\mathfrak{c}$), put $\Gamma_K^{\mathbf{c}-1} = \{\gamma^{\mathbf{c}-1} : \gamma \in \Gamma_K\}$, and denote by $\mathcal{L}_{\mathfrak{p},\psi^-}^{(\mathbf{c}-1)} \in \mathcal{W}[\![\Gamma_K^{\mathbf{c}-1}]\!]$ the image of the Katz p-adc L-function $\mathcal{L}_{\mathfrak{p},\mathfrak{c}}$ under the composition

$$\mathcal{W}[\![Z(\mathfrak{c})]\!] \to \mathcal{W}[\![\Gamma_K]\!] \to \mathcal{W}[\![\Gamma_K^{\mathbf{c}-1}]\!],$$

where the first arrow is the projection defined by ψ^- (viewed as a primitive character on $\Delta_\mathfrak{c}$) and the second arrow is given by $\gamma \mapsto \gamma^{\mathbf{c}-1}$ for $\gamma \in \Gamma_K$. Since p is odd, the map $\gamma^{\mathbf{c}-1} \mapsto (\gamma|_{K_{\mathfrak{p}^\infty}})^{1/2}$ yields an isomorphism $\Gamma_K^{\mathbf{c}-1} \simeq \Gamma_{\mathfrak{p}^\infty}$. Upon choosing a topological generator $\gamma_\mathfrak{p} \in \Gamma_{\mathfrak{p}^\infty}$ as in §3.1, we shall thus view $\mathcal{L}_{\mathfrak{p},\psi^-}^{(\mathbf{c}-1)}$ as an element in $\mathcal{W}[\![S]\!]$.

Lemma 3.2.1. *Let $\boldsymbol{\varphi}$ be a primitive CM Hida family with corresponding to the anticyclotomic branch character ψ^-. Suppose that $\bar{\psi}^-|_{G_{K_\mathfrak{p}}} \neq 1$, where $G_{K_\mathfrak{p}} \subset G_K$ is a decomposition group at $\mathfrak{p}$. Then the congruence ideal $C(\boldsymbol{\varphi})$ is generated by $\mathcal{L}_{\mathfrak{p},\psi^-}^{(\mathfrak{c}-1)}$ in $\mathcal{W}[\![S]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.*

Proof. The assumption on $\bar{\psi}^-$ implies that $\bar{\rho}_{\boldsymbol{\varphi}}$ is absolutely irreducible and p-distinguished, and so the congruence ideal $C(\boldsymbol{\varphi}) \subset \mathscr{O}[\![S]\!]$ is known to be principal. Letting $\eta_{\boldsymbol{\varphi}} \in \mathcal{O}[\![S]\!]$ be a generator, from [HT93, Thm. I] and [HT94, Thm. 0.3] we have the divisibilities

$$\mathcal{L}_{\mathfrak{p},\psi^-}^{(\mathfrak{c}-1)} \mid \eta_{\boldsymbol{\varphi}} \mid \mathcal{F}_{\mathfrak{p},\psi^-}^{(\mathfrak{c}-1)} \tag{3.2}$$

in $\mathcal{W}[\![S]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ and $\mathscr{O}[\![S]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, respectively, where $\mathcal{F}_{\mathfrak{p},\psi^-}^{(\mathfrak{c}-1)} \in \mathscr{O}[\![S]\!]$ is a characteristic power series for the Pontyagin dual of the ψ^--isotypic component of $\mathrm{Gal}(M_\infty^-/K_{\mathfrak{c},\infty}^-)$, the Galois group of the maximal pro-p abelian extension of $K_{\mathfrak{c},\infty}^- := K(\mathfrak{c}p^\infty)^{Z(\mathfrak{c})^+}$ unramified outside $\mathfrak{p}$. (For the second divisibility in (3.2), note that the local non-triviality of $\bar{\psi}^-$ implies that $\mathrm{ord}_P(\mathcal{L}_{\mathfrak{p},\psi^-}^{(\mathfrak{c}-1)}) = 0$ for all "trivial zero primes" $P \subset \mathcal{W}[\![S]\!]$ in the sense of [HT94].) Since Rubin's proof [Rub91] of the two-variable main conjecture for K yields the divisibility $(\mathcal{F}_{\mathfrak{p},\psi^-}^{(\mathfrak{c}-1)}) \supset (\mathcal{L}_{\mathfrak{p},\psi^-}^{(\mathfrak{c}-1)})$ (see e.g. [SU14, Prop. 3.9] for the descent argument), together with (3.2) the result follows. □

Remark 3.2.2. As explained in [ACR23a, §4.4], under some further hypotheses on the branch character ψ^-, it follows from Hida's proof [Hid06a] on the anticyclotomic main conjecture for K, refining the results of [HT93, HT94], that $C(\boldsymbol{\varphi})$ is *integrally* generated by $h_K \cdot \mathcal{L}_{\mathfrak{p},\psi^-}^{(\mathfrak{c}-1)}$, where h_K is the class number of K.

3.3. **Proof of the factorisation.** Now we let

$$(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{\theta}_{\xi_0}(S_0), \boldsymbol{\theta}_{\lambda_1}(S_1), \boldsymbol{\theta}_{\lambda_2}(S_2))$$

be a triple of primitive CM Hida families satisfying conditions (2.1) and (2.4). Here $\xi_0, \lambda_1, \lambda_2$ are finite characters of K, and we denote by $\mathfrak{C}_i$ $(i = 0, 1, 2)$ their conductors, which we assume to be stable under the action of complex conjugation. Suppose $\varphi = \theta(\lambda_0)$ is the weight 2 newform of conductor N_φ associated to the specialisation of $\boldsymbol{\varphi}$ at $Q_0 \in \mathfrak{X}^+_{\mathscr{O}[\![S_0]\!]}$ (so in particular $\epsilon_{Q_0} = 1$). Let

$$\mathcal{R} \simeq \mathscr{O}[\![S_0, S_1, S_2]\!] \to \mathscr{O}[\![S_1, S_2]\!] \tag{3.3}$$

be the specialisation map at Q_0, and denote by

$$\mathscr{L}_p^\varphi(\varphi, \boldsymbol{g}, \boldsymbol{h})(S_1, S_2) \in \mathscr{O}[\![S_1, S_2]\!]$$

the image of the triple product p-adic L-function $\mathscr{L}_p^{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h}) \in \mathcal{R}$ of Theorem 2.2.1 under this map. On the other hand, for χ a Hecke character of K of prime-to-p conductor $\mathfrak{C}$, we denote by $\mathcal{L}_{\mathfrak{p},\chi}^-$ the image of $\mathrm{Tw}_{\chi^{-1}}(\mathcal{L}_{\mathfrak{p},\mathfrak{C}})$ under the natural projection $\mathcal{W}[\![Z(\mathfrak{C})]\!] \to \mathcal{W}[\![\Gamma^-]\!]$, where

$$\mathrm{Tw}_{\chi^{-1}} : \mathcal{W}[\![Z(\mathfrak{C}')]\!] \to \mathcal{W}[\![Z(\mathfrak{C}')]\!]$$

is the $\mathcal{W}$-linear isomorphism given by $\gamma \mapsto \chi^{-1}(\gamma)\gamma$ for $\gamma \in Z(\mathfrak{C})$. As usual, upon choosing a topological generator $\gamma^- \in \Gamma^-$, we shall identify $\mathcal{W}[\![\Gamma^-]\!]$ with the power series ring $\mathcal{W}[\![W]\!]$ via $\gamma^- \mapsto 1 + W$. Finally, denote by $\lambda \mapsto \lambda^\iota$ the involution of $\mathcal{W}[\![\Gamma^-]\!]$ given by $\gamma \mapsto \gamma^{-1}$ for $\gamma \in \Gamma^-$.

Proposition 3.3.1. *Let $(\varphi, \boldsymbol{g}, \boldsymbol{h}) = (\theta(\lambda_0), \boldsymbol{\theta}_{\lambda_1}(S_1), \boldsymbol{\theta}_{\lambda_2}(S_2))$ be a CM triple as above, and suppose in addition that:*

- *$\gcd(N_\varphi, N_g, N_h)$ is squarefree.*
- *The anticyclotomic branch character of $\boldsymbol{\varphi}$ satisfies $\bar{\psi}^-|_{G_{K_\mathfrak{p}}} \neq 1$.*

Put

$$W_1 = (1+S_1)^{1/2}(1+S_2)^{1/2} - 1, \quad W_2 = (1+S_1)^{1/2}(1+S_2)^{-1/2} - 1.$$

Then we have the factorisation

$$\begin{aligned}\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h})^2(S_1, S_2) = u \cdot \mathcal{L}^-_{\mathfrak{p},\lambda_0\lambda_1\lambda_2}(W_1)^{\iota} \cdot \mathcal{L}^-_{\mathfrak{p},\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}}(W_1) \\ \times \mathcal{L}^-_{\mathfrak{p},\lambda_0\lambda_1\lambda_2^{\mathbf{c}}}(W_2)^{\iota} \cdot \mathcal{L}^-_{\mathfrak{p},\lambda_0\lambda_1^{\mathbf{c}}\lambda_2}(W_2),\end{aligned}$$

where $u \in \mathcal{W}[1/p]^{\times}$.

Proof. For $i = 1, 2$, let ζ_i be a primitive p^{n_i}-th root of unity with $n_i > 0$, and put $Q_1 = \zeta_1\zeta_2 - 1$, $Q_2 = \zeta_1\zeta_2^{-1} - 1$, so the specialisations $\boldsymbol{g}_{Q_1}, \boldsymbol{h}_{Q_2}$ are both CM forms of weight 1. Let $\epsilon_i : \Gamma^- \to \mu_{p^\infty}$ be the finite order character given by $\epsilon_i(\gamma^-) = \zeta_i$ and put $\underline{Q} = (Q_0, Q_1, Q_2)$. Noting that for the above triple $(\varphi, \boldsymbol{g}, \boldsymbol{h})$ the character $\mathcal{X}$ in (2.2) is given by

$$\mathcal{X} = (\Psi_{S_1}^{1/2}\Psi_{S_2}^{1/2} \circ \mathscr{V}) : G_{\mathbf{Q}} \to \mathbf{Z}_p[\![S_1, S_2]\!]^{\times},$$

we find the decomposition of p-adic representations

$$\begin{aligned}\mathbf{V}_{\underline{Q}}^{\dagger} \otimes \mathbf{Q}_p = \mathrm{Ind}_K^{\mathbf{Q}}(\lambda_0^{-1}\lambda_1^{-1}\lambda_2^{-1}\epsilon_1^{-1}) \oplus \mathrm{Ind}_K^{\mathbf{Q}}(\lambda_0^{-1}\lambda_1^{-\mathbf{c}}\lambda_2^{-\mathbf{c}}\epsilon_1) \\ \oplus \mathrm{Ind}_K^{\mathbf{Q}}(\lambda_0^{-1}\lambda_1^{-1}\lambda_2^{-\mathbf{c}}\epsilon_2^{-1}) \oplus \mathrm{Ind}_K^{\mathbf{Q}}(\lambda_0^{-1}\lambda_1^{-\mathbf{c}}\lambda_2^{-1}\epsilon_2);\end{aligned}$$

$$\begin{aligned}\mathscr{F}_p^{\varphi}(\mathbf{V}_{\underline{Q}}^{\dagger}) \otimes \mathbf{Q}_p = \lambda_{0,\mathfrak{p}}^{-1}\lambda_{1,\mathfrak{p}}^{-1}\lambda_{2,\mathfrak{p}}^{-1}\epsilon_{1,\mathfrak{p}}^{-1} \oplus \lambda_{0,\mathfrak{p}}^{-1}\lambda_{1,\mathfrak{p}}^{-\mathbf{c}}\lambda_{2,\mathfrak{p}}^{-\mathbf{c}}\epsilon_{1,\mathfrak{p}} \\ \oplus \lambda_{0,\mathfrak{p}}^{-1}\lambda_{1,\mathfrak{p}}^{-1}\lambda_{2,\mathfrak{p}}^{-\mathbf{c}}\epsilon_{2,\mathfrak{p}}^{-1} \oplus \lambda_{0,\mathfrak{p}}^{-1}\lambda_{1,\mathfrak{p}}^{-\mathbf{c}}\lambda_{2,\mathfrak{p}}^{-1}\epsilon_{2,\mathfrak{p}}.\end{aligned}$$

Thus the terms appearing in the interpolation formula of Theorem 2.2.1 become:

$$\begin{aligned}\Gamma_{\mathbf{V}_{\underline{Q}}^{\dagger}}(0) \cdot L(\mathbf{V}_{\underline{Q}}^{\dagger}, 0) = \pi^{-4} \cdot L(\lambda_0^{-1}\lambda_1^{-1}\lambda_2^{-1}\epsilon_1^{-1}, 0) \cdot L(\lambda_0^{-1}\lambda_1^{-\mathbf{c}}\lambda_2^{-\mathbf{c}}\epsilon_1, 0) \\ \times L(\lambda_0^{-1}\lambda_1^{-1}\lambda_2^{-\mathbf{c}}\epsilon_2^{-1}, 0) \cdot L(\lambda_0^{-1}\lambda_1^{-\mathbf{c}}\lambda_2^{-1}\epsilon_2, 0);\end{aligned} \tag{3.4}$$

$$\begin{aligned}\mathcal{E}_p(\mathscr{F}_p^{\varphi}(\mathbf{V}_{\underline{Q}}^{\dagger})) = \mathcal{E}_{\mathfrak{p}}(\lambda_0^{-1}\lambda_1^{-1}\lambda_2^{-1}\epsilon_1^{-1}) \cdot \mathcal{E}_{\mathfrak{p}}(\lambda_0^{-1}\lambda_1^{-\mathbf{c}}\lambda_2^{-\mathbf{c}}\epsilon_1) \\ \times \mathcal{E}_{\mathfrak{p}}(\lambda_0^{-1}\lambda_1^{-1}\lambda_2^{-\mathbf{c}}\epsilon_2^{-1}) \cdot \mathcal{E}_{\mathfrak{p}}(\lambda_0^{-1}\lambda_1^{-\mathbf{c}}\lambda_2^{-1}\epsilon_2);\end{aligned} \tag{3.5}$$

$$\Omega_{\boldsymbol{\varphi}_{Q_0}} = (-2\sqrt{-1})^3 \cdot \frac{\|\varphi\|^2_{\Gamma_0(N_\varphi)}}{\imath_p(\eta_{\boldsymbol{\varphi}_{Q_0}})} \cdot \left(1 - \frac{\lambda_0(\mathfrak{p})}{\lambda_0(\bar{\mathfrak{p}})}\right)\left(1 - \frac{\lambda_0(\mathfrak{p})}{p\lambda_0(\bar{\mathfrak{p}})}\right). \tag{3.6}$$

On the other hand, from Hida's formula for the adjoint L-value [HT93, Thm. 7.1] and Dirichlet's class number formula we obtain

$$\|\varphi\|^2_{\Gamma_0(N_\varphi)} = \frac{D_K^2}{2^4\pi^3} \cdot \frac{2\pi h_K}{w_K\sqrt{D_K}} \cdot L(\lambda_0\lambda_0^{-\mathbf{c}}, 1).$$

Noting that $L(\lambda_0\lambda_0^{-\mathbf{c}},1)=L(\lambda_0^{\mathbf{c}}\lambda_0^{-1},1)=L(\lambda_0^{\mathbf{c}}\lambda_0^{-1}\mathbf{N}^{-1},0)$ and that $\lambda_0^{\mathbf{c}}\lambda_0^{-1}\mathbf{N}^{-1}$ has infinity type $(2,0)$, and letting $\mathfrak{c}$ denote the prime-to-$\mathfrak{p}$ conductor of $\lambda_0^{\mathbf{c}-1}$, the interpolation property in Theorem 2.1.1 can thus be rewritten as

$$\begin{aligned}\mathcal{L}_{\mathfrak{p},\mathfrak{c}}(\lambda_0^{\mathbf{c}}\lambda_0^{-1}\mathbf{N}^{-1}) &= \left(\frac{\Omega_p}{\Omega_\infty}\right)^2\cdot\frac{2^3\pi^2}{\sqrt{D_K}^3}\\ &\quad\times\left(1-\frac{\lambda_0(\mathfrak{p})}{\lambda_0(\bar{\mathfrak{p}})}\right)\left(1-\frac{\lambda_0(\mathfrak{p})}{p\lambda_0(\bar{\mathfrak{p}})}\right)\cdot\frac{w_K}{h_K}\cdot\|\varphi\|^2_{\Gamma_0(N_\varphi)}\\ &= -\left(\frac{\Omega_p}{\Omega_\infty}\right)^2\cdot\frac{\pi^2\sqrt{-1}}{\sqrt{D_K}^3}\cdot\Omega_{\varphi_{Q_0}}\cdot\eta_{\varphi_{Q_0}}\cdot\frac{w_K}{h_K},\end{aligned}$$

using (3.6) for the second equality. By the functional equation for Katz's p-adic L-function (see Theorem 2.1.1) together with Lemma 3.2.1, this shows that

$$\frac{1}{\Omega^2_{\varphi_{Q_0}}} = u'\left(\frac{\Omega_p}{\Omega_\infty}\right)^4\cdot\frac{\pi^4}{D_K^3}, \tag{3.7}$$

where $u'\in\mathcal{W}[1/p]^\times$ is independent of Q_1,Q_2. Thus substituting (3.4), (3.5), (3.6) and (3.7) into the interpolation formula for $\mathscr{L}_p^{\varphi}(\varphi,\boldsymbol{g},\boldsymbol{h})$ we thus arrive at

$$\begin{aligned}\mathscr{L}_p^{\varphi}(\varphi,\boldsymbol{g},\boldsymbol{h})^2(\zeta_1\zeta_2-1,\zeta_1\zeta_2^{-1}-1) &= u'D_K^{-3}\cdot\mathcal{L}^-_{\mathfrak{p},\lambda_0\lambda_1\lambda_2}(\zeta_1^{-1}-1)\cdot\mathcal{L}^-_{\mathfrak{p},\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}}(\zeta_1-1)\\ &\quad\times\mathcal{L}^-_{\mathfrak{p},\lambda_0\lambda_1\lambda_2^{\mathbf{c}}}(\zeta_2^{-1}-1)\cdot\mathcal{L}^-_{\mathfrak{p},\lambda_0\lambda_1^{\mathbf{c}}\lambda_2}(\zeta_2-1),\end{aligned}$$

for all non-trivial p-power roots of unity ζ_1,ζ_2, and the result follows. □

4. Selmer group decompositions

In this section we define different Selmer groups attached to Hecke characters and triple products of modular forms, and for one of the latter Selmer groups, we prove a decomposition mirroring the factorisation in Proposition 3.3.1.

4.1. **Selmer groups for Hecke characters.** Let ν be a Hecke character of K with values in the ring of integers $\mathscr{O}$ of a finite extension Φ of $\mathbf{Q}_p$. Denote by $\mathscr{O}_\nu$ the free $\mathscr{O}$-module of rank 1 on which G_K acts via ν^{-1} and put

$$T_\nu=\mathscr{O}_\nu,\quad V_\nu=T_\nu\otimes_{\mathscr{O}}\Phi,\quad A_\nu=V_\nu/T_\nu=T_\nu\otimes_{\mathscr{O}}(\Phi/\mathscr{O}).$$

Let Σ be any finite set of places of K containing ∞ and the primes dividing p or the conductor of ν, and for any finite extension F/K denote by $G_{F,\Sigma}$ the Galois group of the maximal extension of F unramified outside the places above Σ.

Definition 4.1.1. Let F/K be a finite extension, and for v a prime of F above p put

$$\mathrm{H}^1_\emptyset(F_v,V_\nu)=\mathrm{H}^1(F_v,V_\nu),\quad \mathrm{H}^1_0(F_v,V_\nu)=\{0\}.$$

For $(\mathcal{L}_{\mathfrak{p}},\mathcal{L}_{\bar{\mathfrak{p}}})\in\{\emptyset,0\}^{\oplus 2}$ define the Selmer group $\mathrm{Sel}_{\mathcal{L}_{\mathfrak{p}},\mathcal{L}_{\bar{\mathfrak{p}}}}(F,V_\nu)$ by

$$\begin{aligned}\mathrm{Sel}_{\mathcal{L}_{\mathfrak{p}},\mathcal{L}_{\bar{\mathfrak{p}}}}(F,V_\nu)=\ker\Bigg(\mathrm{H}^1(G_{F,\Sigma},V_\nu)\to\prod_{v|\mathfrak{p}}\frac{\mathrm{H}^1(F_v,V_\nu)}{\mathrm{H}^1_{\mathcal{L}_{\mathfrak{p}}}(F_v,V_\nu)}\times\prod_{v|\bar{\mathfrak{p}}}\frac{\mathrm{H}^1(F_v,V_\nu)}{\mathrm{H}^1_{\mathcal{L}_{\bar{\mathfrak{p}}}}(F_v,V_\nu)}\\ \times\prod_{v\in\Sigma,v\nmid p\infty}\mathrm{H}^1(F_v,V_\nu)\Bigg).\end{aligned}$$

Remark 4.1.2. In particular, if ν has infinity type $(-1,0)$, the Selmer group

$$\mathrm{Sel}_{\emptyset,0}(F,V_\nu)=\ker\Bigg(\mathrm{H}^1(G_{F,\Sigma},V_\nu)\to\prod_{v|\overline{\mathfrak{p}}}\mathrm{H}^1(F_v,V_\nu)\times\prod_{v\in\Sigma,v\nmid p\infty}\mathrm{H}^1(F_v,V_\nu)\Bigg)$$

agrees with the Bloch–Kato Selmer group of V_ν (see e.g. [AH06, §1.1] or [Arn07, §1.2]). (Therefore in that case $\mathrm{Sel}_{0,\emptyset}(F,V_{\nu^{\mathbf{c}}})$ agrees with the Bloch–Kato Selmer group of $V_{\nu^{\mathbf{c}}}$.)

For $?\in\{\emptyset,0\}$, the local condition $\mathrm{H}^1_?(F_v,T_\nu)$ and $\mathrm{H}^1_?(F_v,A_\nu)$ are defined from the above by propagation, and using these we define $\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(F,T_\nu)$ and $\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(F,A_\nu)$ by the same recipe as before. Then, letting K_∞/K denote the anticyclotomic $\mathbf{Z}_p$-extension of K, we set

$$\begin{aligned}\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K_\infty,T_\nu)&=\varprojlim_n\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K_n,T_\nu),\\ \mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K_\infty,A_\nu)&=\varinjlim_n\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K_n,A_\nu).\end{aligned}\tag{4.1}$$

Let $\Lambda=\mathscr{O}[\![\Gamma^-]\!]\simeq\mathscr{O}[\![W]\!]$ be the anticyclotomic Iwasawa algebra, and denote by $\Psi_W:G_K\to\Lambda^\times$ the universal character given $g\mapsto[g|_{K_\infty}]$, where $\gamma\mapsto[\gamma]$ is the inclusion of Γ^- intro Λ^- as a group-like element. Then by Shapiro's lemma we have isomorphisms

$$\mathrm{H}^1(K,T_\nu\otimes\Psi_W^{-1})\simeq\varprojlim_n\mathrm{H}^1(K_n,T_\nu),\quad \mathrm{H}^1(K,A_\nu\otimes\Psi_W)\simeq\varinjlim_n\mathrm{H}^1(K_n,A_\nu).$$

In the following we let $\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K,T_\nu\otimes\Psi_W^{-1})$ and $\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K,A_\nu\otimes\Psi_W)$ be the Selmer groups corresponding to (4.1) under these isomorphism, and let

$$X_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K,A_\nu\otimes\Psi_W)=\mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K,A_\nu\otimes\Psi_W),\mathbf{Q}_p/\mathbf{Z}_p)$$

denote the Pontryagin dual of $\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K,A_\nu\otimes\Psi_W)$.

We conclude this section by recalling the following result due to Agboola–Howard [AH06] and Arnold [Arn07] on the anticyclotomic Iwasawa main conjecture for K.

Theorem 4.1.3. *Let ν be a Hecke character of K of infinity type $(-1,0)$ satisfying that $\nu^{\mathbf{c}}=\overline{\nu}$, where $\overline{\nu}$ is the complex conjugate of ν, and suppose that ν has sign $+1$ in its functional equation. Then $\mathrm{Sel}_{\emptyset,0}(K,T_\nu\otimes\Psi_W^{-1})=0$, $X_{0,\emptyset}(K,A_{\nu^{\mathbf{c}}}\otimes\Psi_W)$ is Λ-torsion, and*

$$\mathrm{char}_\Lambda\big(X_{0,\emptyset}(K,A_{\nu^{\mathbf{c}}}\otimes\Psi_W)\big)=\big(\mathcal{L}^-_{\mathfrak{p},\nu}(W)\big)$$

as ideals in $\Lambda_\mathcal{W}\otimes\mathbf{Q}_p$.

Proof. In the case where ν is the Hecke character associated with an elliptic curve $E/\mathbf{Q}$ with CM by K, this is [AH06, Thm. 2.4.17]. The general case is given in [Arn07, Thm. 2.1]. □

4.2. **Selmer groups for triple products.** Let $(\boldsymbol{\varphi},\boldsymbol{g},\boldsymbol{h})$ be a triple of primitive Hida families as in §2.2 satisfying (2.1), and recall the self-dual twist $\mathbf{V}^\dagger=V_{\boldsymbol{\varphi}}\otimes V_{\boldsymbol{g}}\otimes V_{\boldsymbol{h}}\otimes\mathcal{X}^{-1}$ of the tensor product of their associated big Galois representations.

Definition 4.2.1. Put

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)=\mathscr{F}_p^2(\mathbf{V}^\dagger):=\big(V_{\boldsymbol{\varphi}}^+\otimes V_{\boldsymbol{g}}^+\otimes V_{\boldsymbol{h}}+V_{\boldsymbol{\varphi}}^+\otimes V_{\boldsymbol{g}}\otimes V_{\boldsymbol{h}}^+ + V_{\boldsymbol{\varphi}}\otimes V_{\boldsymbol{g}}^+\otimes V_{\boldsymbol{h}}^+\big)\otimes\mathcal{X}^{-1},$$

and define the *balanced local condition* $\mathrm{H}^1_{\mathrm{bal}}(\mathbf{Q}_p,\mathbf{V}^\dagger)$ by

$$\mathrm{H}^1_{\mathrm{bal}}(\mathbf{Q}_p,\mathbf{V}^\dagger):=\mathrm{im}\big(\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger))\to\mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}^\dagger)\big).$$

Similarly, put $\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^\dagger) := \big(V_{\boldsymbol{\varphi}}^+ \otimes V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}}\big) \otimes \mathcal{X}^{-1}$ and define the *$\boldsymbol{\varphi}$-unbalanced local condition* $\mathrm{H}^1_{\boldsymbol{\varphi}}(\mathbf{Q}_p, \mathbf{V}^\dagger)$ by

$$\mathrm{H}^1_{\boldsymbol{\varphi}}(\mathbf{Q}_p, \mathbf{V}^\dagger) := \mathrm{im}\big(\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^\dagger)) \to \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger)\big).$$

It is easy to see that the maps appearing in these definitions are injective, and in the following we shall use this to identify $\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{V}^\dagger)$ with $\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^?(\mathbf{V}^\dagger))$ for $? \in \{\mathrm{bal}, \boldsymbol{\varphi}\}$.

Definition 4.2.2. Let S be a finite set of primes containing ∞ and the primes dividing $pN_\varphi N_g N_h$, and let $G_{\mathbf{Q},S}$ be the Galois group of the maximal extension of $\mathbf{Q}$ unramified outside S. For every $? \in \{\mathrm{bal}, \boldsymbol{\varphi}\}$, define the Selmer group $\mathrm{Sel}^?(\mathbf{Q}, \mathbf{V}^\dagger)$ by

$$\mathrm{Sel}^?(\mathbf{Q}, \mathbf{V}^\dagger) := \ker\left\{ \mathrm{H}^1(G_{\mathbf{Q},S}, \mathbf{V}^\dagger) \to \frac{\mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger)}{\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{V}^\dagger)} \times \prod_{v \neq p} \mathrm{H}^1(\mathbf{Q}_v^{\mathrm{nr}}, \mathbf{V}^\dagger) \right\}.$$

We call $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger)$ (resp. $\mathrm{Sel}^{\boldsymbol{\varphi}}(\mathbf{Q}, \mathbf{V}^\dagger)$) the *balanced* (resp. *$\boldsymbol{\varphi}$-unbalanced*) Selmer group.

Let $\mathbf{A}^\dagger = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{V}^\dagger, \mu_{p^\infty})$ and for $? \in \{\mathrm{bal}, \boldsymbol{\varphi}\}$ define $\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{A}^\dagger) \subset \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)$ to be the orthogonal complement of $\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{V}^\dagger)$ under the local Tate duality

$$\mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger) \times \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

Similarly as above, we then define the balanced and $\boldsymbol{\varphi}$-unbalanced Selmer groups with coefficients in $\mathbf{A}^\dagger$ by

$$\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger) := \ker\left\{ \mathrm{H}^1(G_{\mathbf{Q},S}, \mathbf{A}^\dagger) \to \frac{\mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)}{\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{A}^\dagger)} \times \prod_{v \neq p} \mathrm{H}^1(\mathbf{Q}_v^{\mathrm{nr}}, \mathbf{A}^\dagger) \right\},$$

and let $X^?(\mathbf{Q}, \mathbf{A}^\dagger) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger), \mathbf{Q}_p/\mathbf{Z}_p)$ be the Pontryagin dual of $\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger)$.

4.3. **Proof of the decompositions.** For a primitive Hida family $\boldsymbol{f}$, from now on we let $V_{\boldsymbol{f}}$ be the realisation of $\rho_{\boldsymbol{f}}$ arising from the p-adic élale cohomology of the p-tower of modular curves of tame level N_f as in the work of Ohta [Oht99, Oht00] (see also [KLZ17, §7.2], whose conventions we adopt). We shall exploit the following fact.

Proposition 4.3.1. *Let $\boldsymbol{\varphi} = \boldsymbol{\theta}_\xi(S)$ be a primitive CM Hida family associated to the finite order character ξ. If $\bar{\xi}^-|_{G_{K_\mathfrak{p}}} \neq 1$, where $G_{K_\mathfrak{p}} \subset G_K$ is a decomposition group at $\mathfrak{p}$, then*

$$V_{\boldsymbol{f}} \simeq \mathrm{Ind}_K^{\mathbf{Q}}(T_\xi \otimes \Psi_S)$$

as $\mathscr{O}[\![S]\!][G_{\mathbf{Q}}]$-modules.

Proof. This follows from a slight extension of the isomorphism in [LLZ15, Cor. 5.2.5]. See [BL18, §3.2.3] for details. □

Suppose now that $(\varphi, \boldsymbol{g}, \boldsymbol{h})$ is a CM triple as in §3.3, with $\varphi = \theta(\lambda_0) \in S_2(\Gamma_0(N_\varphi))$ a newform obtained as the specialisation of $\boldsymbol{\varphi} = \boldsymbol{\theta}_{\xi_0}(S_0)$ at $Q_0 \in \mathfrak{X}^+_{\mathscr{O}[\![S_0]\!]}$, and let $\mathbb{V}^\dagger_{\boldsymbol{\varphi gh}}$ be the image of $\mathbf{V}^\dagger$ under the specialisation map (3.3) at Q_0. Then from Proposition 4.3.1 we obtain

$$\mathbb{V}^\dagger_{\boldsymbol{\varphi gh}} \simeq \mathrm{Ind}_K^{\mathbf{Q}}(T_{\lambda_0}) \otimes \mathrm{Ind}_K^{\mathbf{Q}}(T_{\lambda_1} \otimes \Psi_{S_1}) \otimes \mathrm{Ind}_K^{\mathbf{Q}}(T_{\lambda_2} \otimes \Psi_{S_2}) \otimes \mathcal{X}^{-1},$$

where $\mathcal{X} = (\Psi_{S_1}^{1/2}\Psi_{S_2}^{1/2} \circ \mathscr{V}) : G_{\mathbf{Q}} \to \mathbf{Z}_p[\![S_1, S_2]\!]^\times$, and so an immediate computation shows that

$$\begin{aligned} \mathbb{V}^\dagger_{\boldsymbol{\varphi gh}} \simeq\ & \mathrm{Ind}_K^{\mathbf{Q}}(T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Ind}_K^{\mathbf{Q}}(T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ & \oplus \mathrm{Ind}_K^{\mathbf{Q}}(T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus \mathrm{Ind}_K^{\mathbf{Q}}(T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}), \end{aligned} \tag{4.2}$$

where W_1, W_2 are as in Proposition 3.3.1. Therefore,

$$\begin{aligned}\mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}}) \simeq \mathrm{H}^1(K, T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{H}^1(K, T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ \oplus \mathrm{H}^1(K, T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus \mathrm{H}^1(K, T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1})\end{aligned} \tag{4.3}$$

by Shapiro's lemma.

Proposition 4.3.2. *Under (4.3), the balanced Selmer group decomposes as*

$$\begin{aligned}\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}}) \simeq \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}),\end{aligned}$$

and the φ-unbalanced Selmer group decomposes as

$$\begin{aligned}\mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}}) \simeq \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}).\end{aligned}$$

The proof will follow easily from the following.

Lemma 4.3.3. *Under (4.3), the Selmer group $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}})$ corresponds to the submodule of*

$$\mathrm{H}^1(K, (T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \oplus (T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}))$$

consisting of unramified-outside-p classes x with $\mathrm{res}_v(x)$ belonging to

$$\begin{cases} \mathrm{H}^1(K_{\mathfrak{p}}, (T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1})) & \text{if } v = \mathfrak{p}, \\ \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) & \text{if } v = \overline{\mathfrak{p}}, \end{cases}$$

and the Selmer group $\mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}})$ similarly corresponds to the submodule consisting of unramified-outside-p classes x with $\mathrm{res}_{\overline{\mathfrak{p}}}(x) = 0$ (and no condition at $\mathfrak{p}$).

Proof. Using (4.2) we see that the balanced local condition is given by

$$\begin{aligned}\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger_{\varphi\boldsymbol{gh}}) = (T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda_0^{\mathbf{c}}\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \\ \oplus (T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}),\end{aligned}$$

from where we obtain

$$\begin{aligned}\mathscr{F}_{\mathfrak{p}}^{\mathrm{bal}}(\mathbb{V}^\dagger_{\varphi\boldsymbol{gh}}) &= (T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}), \\ \mathscr{F}_{\overline{\mathfrak{p}}}^{\mathrm{bal}}(\mathbb{V}^\dagger_{\varphi\boldsymbol{gh}}) &= T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1},\end{aligned} \tag{4.4}$$

yielding the stated descriptions of $\mathrm{Sel}^{\mathrm{bal}}(K, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}})$. Similarly, we see that the φ-unbalanced local condition is given by

$$\begin{aligned}\mathscr{F}_p^{\varphi}(\mathbb{V}^\dagger_{\varphi\boldsymbol{gh}}) = (T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ \oplus (T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}),\end{aligned}$$

and this immediately yields the stated description of $\mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}})$. □

Proof of Proposition 4.3.2. Put

$$\begin{aligned}\widetilde{\mathbb{V}}^\dagger_{\varphi\boldsymbol{gh}} := (T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ \oplus (T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}),\end{aligned} \tag{4.5}$$

so we have

$$\mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}}) \simeq \mathrm{H}^1(K, \widetilde{\mathbb{V}}^\dagger_{\varphi\boldsymbol{gh}}). \tag{4.6}$$

Denoting by $\mathrm{Sel}^{\{p\}}(K, \widetilde{\mathbb{V}}^\dagger_{\varphi\boldsymbol{gh}})$ the submodule of $\mathrm{H}^1(K, \widetilde{\mathbb{V}}^\dagger_{\varphi\boldsymbol{gh}})$ consisting of unramified-outside-p classes, it follows from Lemma 4.3.3 that under the above isomorphism the balanced Selmer group $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}})$ corresponds to the kernel of the restriction map from $\mathrm{Sel}^{\{p\}}(K, \widetilde{\mathbb{V}}^\dagger_{\varphi\boldsymbol{gh}})$ to

$$\frac{\mathrm{H}^1(K_{\mathfrak{p}}, \widetilde{\mathbb{V}}^\dagger_{\varphi\boldsymbol{gh}})}{\mathrm{H}^1(K_{\mathfrak{p}}, (T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}))} \times \frac{\mathrm{H}^1(K_{\overline{\mathfrak{p}}}, \widetilde{\mathbb{V}}^\dagger_{\varphi\boldsymbol{gh}})}{\mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1})},$$

and this kernel is isomorphic to

$$\begin{aligned}\mathrm{Sel}_{\emptyset,0}(K, T_{\lambda_0\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda_0\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}) \\ \times \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}).\end{aligned}$$

This shows the result for $\mathrm{Sel}^{\mathrm{bal}}(K, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}})$, and the case of $\mathrm{Sel}^{\varphi}(K, \mathbb{V}^\dagger_{\varphi\boldsymbol{gh}})$ follows from Lemma 4.3.3 in the same manner. □

As a consequence we also obtain the following decomposition for the Selmer groups with coefficients in $\mathbb{A}^\dagger_{\varphi\boldsymbol{gh}} = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbb{V}^\dagger_{\varphi\boldsymbol{gh}}, \mu_{p^\infty})$, mirroring in the case of $\mathrm{Sel}^{\varphi}(K, \mathbb{A}^\dagger_{\varphi\boldsymbol{gh}})$ the factorisation of p-adic L-functions in Proposition 3.3.1.

Corollary 4.3.4. *The balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^\dagger_{\varphi\boldsymbol{gh}})$ *decomposes as*

$$\begin{aligned}\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^\dagger_{\varphi\boldsymbol{gh}}) \simeq \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda_0^{\mathbf{c}}\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{\emptyset,0}(K, A_{\lambda_0^{\mathbf{c}}\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \\ \oplus \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda_0^{\mathbf{c}}\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda_0^{\mathbf{c}}\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}),\end{aligned}$$

and the φ-unbalanced Selmer group $\mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbb{A}^\dagger_{\varphi\boldsymbol{gh}})$ *decomposes as*

$$\begin{aligned}\mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbb{A}^\dagger_{\varphi\boldsymbol{gh}}) \simeq \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda_0^{\mathbf{c}}\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda_0^{\mathbf{c}}\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \\ \oplus \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda_0^{\mathbf{c}}\lambda_1^{\mathbf{c}}\lambda_2} \otimes \Psi_{W_2}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda_0^{\mathbf{c}}\lambda_1\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}).\end{aligned}$$

Proof. This is immediate from Proposition 4.3.2 and local Tate duality. □

Remark 4.3.5. Note that the decompositions of $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^\dagger_{\varphi\boldsymbol{gh}})$ and $\mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbb{A}^\dagger_{\varphi\boldsymbol{gh}})$ in Corollary 4.3.4 only differ in their second direct summand: the term $\mathrm{Sel}_{\emptyset,0}(K, A_{\lambda_0^{\mathbf{c}}\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}})$ in the former is replaced by $\mathrm{Sel}_{0,\emptyset}(K, A_{\lambda_0^{\mathbf{c}}\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}})$ in the latter (i.e., the local conditions at the primes above p are reversed). From the description in §4.1, it follows that $\mathrm{Sel}_{0,\emptyset}(K, A_{\lambda_0^{\mathbf{c}}\lambda_1\lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}})$ corresponds to a Bloch–Kato Selmer group for $\lambda_0^{\mathbf{c}}\lambda_1\lambda_2$ over the anticyclotomic $\mathbf{Z}_p$-extension of K.

5. Diagonal cycle main conjecture

In this section we give the proof of a two-variable variant of the diagonal cycle main conjecture formulated in [ACR23b] specialised to the CM case.

5.1. **Big diagonal classes.** Let $(\varphi, \boldsymbol{g}, \boldsymbol{h})$ be a CM triple as in §3.3, with $\varphi = \theta(\lambda_0) \in S_2(\Gamma_0(N_\varphi))$ the newform corresponding to the specialisation of $\boldsymbol{\varphi} = \boldsymbol{\theta}_{\xi_0}(S_0)$ at $Q_0 \in \mathfrak{X}^+_{\mathscr{O}[\![S_0]\!]}$. Denote by

$$\kappa(\varphi, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi \boldsymbol{gh}}) \tag{5.1}$$

the specialisation at Q_0 of the big diagonal class $\kappa(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})$ constructed in [BSV22, §8.1].

More precisely, letting $N = \mathrm{lcm}(N_\varphi, N_g, N_h)$, the construction in *loc. cit.* produces a class in the cohomology of a representation $\mathbb{V}^\dagger_{\varphi \boldsymbol{gh}}(N)$ non-canonically isomorphic to finitely many copies of $\mathbb{V}^\dagger_{\varphi \boldsymbol{gh}}$. Our class (5.1) is obtained by taking the image under the map $\mathbb{V}^\dagger_{\varphi \boldsymbol{gh}}(N) \to \mathbb{V}^\dagger_{\varphi \boldsymbol{gh}}$ associated to the level-N test vectors for the triple $(\varphi, \boldsymbol{g}, \boldsymbol{h})$ furnished by [Hsi21, Thm. A].

It follows from [BSV22, Cor. 8.2] that $\kappa(\varphi, \boldsymbol{gh})$ lands in $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi \boldsymbol{gh}})$, and therefore viewing this class in $\mathrm{H}^1(K, \widetilde{\mathbb{V}}^\dagger_{\varphi \boldsymbol{gh}})$ via (4.6), by Proposition 4.3.2 this implies that

$$\mathrm{res}_{\overline{\mathfrak{p}}}(\kappa(\varphi, \boldsymbol{g}, \boldsymbol{h})) \in \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda_0 \lambda_1^{\mathfrak{c}} \lambda_2^{\mathfrak{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}),$$

where $W_1 = (1+S_1)^{1/2}(1+S_2)^{1/2} - 1$.

Put $u_i = 1 + p$ $(i = 1, 2)$ and for any $\mathscr{O}[\![S_1, S_2]\!]$-module M and integers $k_1, k_2 \geq 1$ of the same parity, denote by M_{k_1,k_2} the specialisation of M at $S_i = u_i^{k_i - 1} - 1$. Let Φ be the field of fractions of $\mathscr{O}$. Then one easily checks that there are isomorphisms

$$\begin{cases} \log_{\overline{\mathfrak{p}}} : \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda_0 \lambda_1^{\mathfrak{c}} \lambda_2^{\mathfrak{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1})_{k_1,k_2} \otimes \mathbf{Q}_p \to \Phi & \text{if } k_1 + k_2 \geq 3, \\ \exp^*_{\overline{\mathfrak{p}}} : \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda_0 \lambda_1^{\mathfrak{c}} \lambda_2^{\mathfrak{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1})_{k_1,k_2} \otimes \mathbf{Q}_p \to \Phi & \text{if } k_1 + k_2 = 2, \end{cases} \tag{5.2}$$

given by the Bloch–Kato logarithm and dual exponential maps, respectively.

Theorem 5.1.1 (Explicit reciprocity law)**.** *Let $(\varphi, \boldsymbol{g}, \boldsymbol{h}) = (\theta(\lambda_0), \boldsymbol{\theta}_{\lambda_1}(S_1)\boldsymbol{\theta}_{\lambda_2}(S_2))$ be a CM triple as above, with $\mathcal{R}_{\varphi \boldsymbol{gh}} = \mathscr{O}[\![S_1, S_2]\!]$. There is an injective $\mathcal{R}_{\varphi \boldsymbol{gh}}$-module homomorphism*

$$\mathrm{Log}^\varphi : \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda_0 \lambda_1^{\mathfrak{c}} \lambda_2^{\mathfrak{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \to \mathcal{R}_{\varphi \boldsymbol{gh}}$$

with pseudo-null cokernel satisfying for any $\mathfrak{Z} \in \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda_0 \lambda_1^{\mathfrak{c}} \lambda_2^{\mathfrak{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1})$ the interpolation property

$$\mathrm{Log}^\varphi(\mathfrak{Z})_{k_1,k_2} = c_{k_1,k_2} \times \begin{cases} \log_{\overline{\mathfrak{p}}}(\mathfrak{Z}_{k_1,k_2}) & \text{if } k_1 + k_2 \geq 3, \\ \exp_{\overline{\mathfrak{p}}}(\mathfrak{Z}_{k_1,k_2}) & \text{if } k_1 + k_2 = 2, \end{cases}$$

where c_{k_1,k_2} is an explicit nonzero constant, and such that

$$\mathrm{Log}^\varphi\big(\mathrm{res}_{\overline{\mathfrak{p}}}(\kappa(\varphi, \boldsymbol{g}, \boldsymbol{h}))\big) = \mathscr{L}_p^\varphi(\varphi, \boldsymbol{g}, \boldsymbol{h}).$$

Proof. Let $\mathscr{F}_p^3(\mathbf{V}^\dagger)$ be the rank one subspace of $\mathscr{F}_p^2(\mathbf{V}^\dagger) = \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)$ given by

$$\mathscr{F}_p^3(\mathbf{V}^\dagger) = V_{\boldsymbol{\varphi}}^+ \hat{\otimes}_{\mathscr{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathscr{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1},$$

Then one immediately finds that $\mathscr{F}_p^2(\mathbf{V}^\dagger)/\mathscr{F}_p^3(\mathbf{V}^\dagger)$ naturally contains the representation

$$\mathbf{V}_{\boldsymbol{\varphi}}^{\boldsymbol{gh}} := (V_{\boldsymbol{\varphi}}/V_{\boldsymbol{\varphi}}^+) \hat{\otimes}_{\mathscr{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathscr{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1} \tag{5.3}$$

as a direct summand. Letting $\mathscr{F}_p^3(\mathbb{V}^\dagger_{\varphi \boldsymbol{gh}})$ be the specialisation of $\mathscr{F}_p^3(\mathbf{V}^\dagger)$, and likewise for $\mathbb{V}_{\boldsymbol{\varphi}}^{\boldsymbol{gh}}$, in terms of the description given in the proof of Lemma 4.3.3, we find that

$$\begin{cases} \mathscr{F}_{\mathfrak{p}}^3(\mathbb{V}^\dagger_{\varphi \boldsymbol{gh}}) = T_{\lambda_0 \lambda_1 \lambda_2} \otimes \Psi_{W_1}^{1-\mathbf{c}}, \\ \mathscr{F}_{\overline{\mathfrak{p}}}^3(\mathbb{V}^\dagger_{\varphi \boldsymbol{gh}}) = 0, \end{cases}$$

and that

$$\mathbb{V}_{\varphi}^{\boldsymbol{gh}} \simeq \mathscr{F}_{\bar{\mathfrak{p}}}^{\mathrm{bal}}(\mathbb{V}_{\varphi\boldsymbol{gh}}^{\dagger}) \simeq T_{\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}$$

The construction of the map Log^{φ} thus follows from specialising at Q_0 the three-variable p-adic regulator map of [BSV22, §7.1] similarly as in the proof of [ACR23b, Prop. 7.3], and the stated explicit reciprocity law is then a consequence of [BSV22, Thm. A] (see also [DR22, Thm. 10]). □

As explained in [ACR23b, §7.3], the following result can be seen as the equivalence between two formulations ('without p-adic L-functions' and 'with p-adic L-functions', respectively) of the Iwasawa–Greenberg main conjecture [Gre94] for $\mathbb{V}_{\varphi\boldsymbol{gh}}^{\dagger}$.

Proposition 5.1.2. *The following statements (1)-(2) are equivalent:*

(1) $\kappa(\varphi, \boldsymbol{g}, \boldsymbol{h})$ *is not* $\mathcal{R}_{\varphi\boldsymbol{gh}}$*-torsion,*

$$\mathrm{rank}_{\mathcal{R}_{\varphi\boldsymbol{gh}}}\big(\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}_{\varphi\boldsymbol{gh}}^{\dagger})\big) = \mathrm{rank}_{\mathcal{R}_{\varphi\boldsymbol{gh}}}\big(X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}_{\varphi\boldsymbol{gh}}^{\dagger})\big) = 1,$$

and the following equality holds in $\mathcal{R}_{\varphi\boldsymbol{gh}} \otimes \mathbf{Q}_p$*:*

$$\mathrm{char}_{\mathcal{R}_{\varphi\boldsymbol{gh}}}\big(X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}_{\varphi\boldsymbol{gh}}^{\dagger})_{\mathrm{tors}}\big) = \mathrm{char}_{\mathcal{R}_{\varphi\boldsymbol{gh}}}\left(\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}_{\varphi\boldsymbol{gh}}^{\dagger})}{(\kappa(\varphi, \boldsymbol{g}, \boldsymbol{h}))}\right)^2.$$

(2) $\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h})$ *is nonzero,*

$$\mathrm{rank}_{\mathcal{R}_{\varphi\boldsymbol{gh}}}\big(\mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbb{V}_{\varphi\boldsymbol{gh}}^{\dagger}) = \mathrm{rank}_{\mathcal{R}_{\varphi\boldsymbol{gh}}}\big(X^{\varphi}(\mathbf{Q}, \mathbb{A}_{\varphi\boldsymbol{gh}}^{\dagger})\big) = 0,$$

and the following equality holds in $\mathcal{R}_{\varphi\boldsymbol{gh}} \otimes \mathbf{Q}_p$*:*

$$\mathrm{char}_{\mathcal{R}_{\varphi\boldsymbol{gh}}}\big(X^{\varphi}(\mathbf{Q}, \mathbb{A}_{\varphi\boldsymbol{gh}}^{\dagger})\big) = \big(\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h})\big)^2.$$

Proof. This can be deduced from Theorem 5.1.1 and global duality in the same way as [ACR23b, Thm. 7.15]. See [Lai22] for the details in the stated level of generality. □

5.2. **Proof of Theorem A.** We can now conclude the proof of our main first main result towards the diagonal cycle main conjecture (Conjecture A). Let K be an imaginary quadratic field in which the prime $p \geq 5$ splits: $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Let λ_0 (resp. λ_1, λ_2) be a Heche character of K of infinity type $(-1, 0)$ (resp. finite order). For $i = 0, 1, 2$, let $\mathfrak{C}_i$ be the conductor of λ_i and suppose that $\lambda_i^{\mathbf{c}} = \overline{\lambda_i}$ (so in particular $\overline{\mathfrak{C}_i} = \mathfrak{C}_i$). Then $L(\lambda_0^{-1}\lambda_1^{-1}\lambda_2^{-1}, 0)$ is self-dual, and we let $\mathrm{sign}(\lambda_0\lambda_1\lambda_2) \in \{\pm 1\}$ be the sign in its functional equation (with center at $s = 0$). Being anticyclotomic twists of $\lambda_0\lambda_1\lambda_2$, the same applies to the characters $\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}$, $\lambda_0\lambda_1\lambda_2^{\mathbf{c}}$, and $\lambda_0\lambda_1^{\mathbf{c}}\lambda_2$. We then let

$$(\varphi, \boldsymbol{g}, \boldsymbol{h}) = (\theta(\lambda_0), \boldsymbol{\theta}_{\lambda_1}(S_1), \boldsymbol{\theta}_{\lambda_2}(S_2))$$

be the triple consisting of the newform $\varphi = \theta(\lambda_0) \in S_2(\Gamma_0(N_\varphi))$, where $N_\varphi = D_K\mathbf{N}(\mathfrak{C}_0)$, and the primitive CM Hida families associated to λ_1 and λ_2, respectively.

Theorem 5.2.1. *Suppose that* $p \nmid \mathfrak{C}_i$ *for* $i = 0, 1, 2$ *and that:*

(i) $\gcd(N_\varphi, N_g, N_h)$ *is squarefree.*
(ii) $\bar{\rho}_{\boldsymbol{g}}$ *and* $\bar{\rho}_{\boldsymbol{h}}$ *are both absolutely irreducible and* p*-distinguished.*
(iii) $\mathrm{sign}(\lambda_0\lambda_1\lambda_2) = \mathrm{sign}(\lambda_0\lambda_1^{\mathbf{c}}\lambda_2^{\mathbf{c}}) = \mathrm{sign}(\lambda_0\lambda_1\lambda_2^{\mathbf{c}}) = \mathrm{sign}(\lambda_0\lambda_1^{\mathbf{c}}\lambda_2) = +1.$

Then class $\kappa(\varphi, \boldsymbol{g}, \boldsymbol{h})$ *is not* $\mathcal{R}_{\varphi\boldsymbol{gh}}$*-torsion, the modules* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}_{\varphi\boldsymbol{gh}}^{\dagger})$ *and* $X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}_{\varphi\boldsymbol{gh}}^{\dagger})$ *both have* $\mathcal{R}_{\varphi\boldsymbol{gh}}$*-rank one, and the following equality holds in* $\mathcal{R}_{\varphi\boldsymbol{gh}} \otimes \mathbf{Q}_p$*:*

$$\mathrm{char}_{\mathcal{R}_{\varphi\boldsymbol{gh}}}\big(X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}_{\varphi\boldsymbol{gh}}^{\dagger})_{\mathrm{tors}}\big) = \mathrm{char}_{\mathcal{R}_{\varphi\boldsymbol{gh}}}\left(\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}_{\varphi\boldsymbol{gh}}^{\dagger})}{\mathcal{R}_{\varphi\boldsymbol{gh}} \cdot \kappa(\varphi, \boldsymbol{g}, \boldsymbol{h})}\right)^2.$$

In other words, Conjecture A specialised to the triple $(\varphi, \boldsymbol{g}, \boldsymbol{h})$ holds.

Proof. By Greenberg's nonvanishing results [Gre83, Gre85] (see [AH06, Cor. 2.1.5] and [Arn07, Prop. 2.3]), our sign assumptions imply that the four anticyclotomic Katz p-adic L-functions in the factorisation of Proposition 3.3.1 are nonzero, and so we deduce that $\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h})$ is also nonzero. (Note that the condition that $\bar{\rho}_{\varphi}$ be absolutely irreducible and p-distinguished as needed in Proposition 3.3.1 is automatic in the case $\varphi = \theta(\lambda_0) \in S_2(\Gamma_0(N_{\varphi}))$; see e.g. [KLZ17, Rem. 7.2.7].) Moreover, Theorem 4.1.3 implies that each of the direct summands in the decomposition of $X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi \boldsymbol{gh}})$ given in Proposition 4.3.2 is torsion, with characteristic ideal generated by a corresponding anticyclotomic Katz p-adic L-function. Therefore $X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi \boldsymbol{gh}})$ is $\mathcal{R}_{\varphi \boldsymbol{gh}}$-torsion, with

$$\mathrm{char}_{\mathcal{R}_{\varphi \boldsymbol{gh}}}\big(X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi \boldsymbol{gh}})\big) = \big(\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h})\big)^2$$

in $\mathcal{R}_{\varphi \boldsymbol{gh}} \otimes \mathbf{Q}_p$. Together with the equivalence of Proposition 5.1.2, this concludes the proof. □

6. Generalised Kato classes

In this section we define generalised Kato classes attached to CM elliptic curves $E/\mathbf{Q}$, following a slight modification of the construction due to Darmon–Rotger [DR16]. Then we prove our main result on the nonvashing of these classes in situations of rank 2.

6.1. **Construction of the classes.** Let $E/\mathbf{Q}$ be an elliptic curve with CM by the maximal order $\mathcal{O}_K$. Let $p \geq 5$ be a prime of good *ordinary* reduction for E; in particular, (spl) holds. Let ψ_E be the Grossencharacter of K associated to E by the theory of complex multiplication, so we have

$$L(E, s) = L(\psi_E, s).$$

Note that $L(\psi_E, s) = L(\psi_E^{\mathbf{c}}, s) = L(\psi_E^{-1}, s-1)$, since $\psi_E \psi_E^{\mathbf{c}} = \mathbf{N}$.

Let ℓ_1, ℓ_2 be distinct primes split in K with $(\ell_1 \ell_2, pN_E) = 1$, where N_E is the conductor of E. For $i = 1, 2$, let ϕ_i be a ring class character of conductor $\ell_i^{m_i}\mathcal{O}_K$ for some $m_i > 0$.

Proposition 6.1.1. *Suppose E has root number $+1$. Then there exist infinitely many ring class characters ϕ_1, ϕ_2 as above with*

$$\phi_1|_{G_{K_{\mathfrak{p}}}}, \phi_2|_{G_{K_{\mathfrak{p}}}} \neq 1,$$

where $G_{K_{\mathfrak{p}}} \subset G_K$ is a decomposition group at $\mathfrak{p}$, and such that

$$L(\psi_E^{-1}\phi_1^{-1}, 0) \cdot L(\psi_E^{-1}\phi_2^{-1}, 0) \cdot L(\psi_E^{-1}\phi_1^{-1}\phi_2^{-1}, 0) \neq 0.$$

Proof. Since the primes ℓ_1 and ℓ_2 both split in K, for any ring class characters ϕ_1, ϕ_2 as above the signs in the functional equations for $L(\psi_E^{-1}\phi_1^{-1}, s), L(\psi_E^{-1}\phi_2^{-1}, s), L(\psi_E^{-1}\phi_1^{-1}\phi_2^{-1}, s)$ are the same as the sign of $L(E, s)$. Thus if E has root number $+1$, the nonvanishing results of [Gre85] and [Roh84] imply that, as ϕ_1, ϕ_2 vary, only finitely many of the values $L(\psi_E^{-1}\phi_1^{-1}, 0) \cdot L(\psi_E^{-1}\phi_2^{-1}, 0) \cdot L(\psi_E^{-1}\phi_1^{-1}\phi_2^{-1}, 0)$ are zero, whence the result. □

We now fix a pair of ring class characters ϕ_1, ϕ_2 as above, and writing

$$\phi_i = \lambda_i^{1-\mathbf{c}}$$

with λ_i a ray class character modulo $\ell^{m_i}\mathcal{O}_K$ (see [Hid06b, Lem. 5.31], for example), we consider the CM triple

$$(\varphi, \boldsymbol{g}, \boldsymbol{h}) = (\theta(\psi_E \lambda_1^{-\mathbf{c}} \lambda_2^{-\mathbf{c}}), \boldsymbol{\theta}_{\lambda_1}(S_1), \boldsymbol{\theta}_{\lambda_2}(S_2)). \tag{6.1}$$

The following definition might be seen as twisted variant of the *generalised Kato classes* introduced by Darmon–Rotger [DR16] (whose construction in the setting conjecturally of relevance for

rank two elliptic curves over $\mathbf{Q}$, namely the "adjoint case"studied in [*op. cit.*,§4.3], would suggest taking $\varphi = \theta(\psi_E)$, rather than φ as above).

Definition 6.1.2 (Generalised Kato class)**.** Denote by $\mathbb{V}^\dagger_{\varphi\underline{\boldsymbol{gh}}}$ the restriction of $\mathbb{V}^\dagger_{\varphi\boldsymbol{gh}}$ to $S_1 = S_2$, let $\kappa(\varphi, \underline{\boldsymbol{gh}}) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\underline{\boldsymbol{gh}}})$ be the corresponding projection of (5.1), and put $S = S_1$. We define κ_p to be the image of $\kappa(\varphi, \underline{\boldsymbol{gh}})$ under the composition

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\underline{\boldsymbol{gh}}}) \to \mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E} \otimes \Psi_S^{\mathbf{c}^{-1}}) \to \mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E}),$$

where the first arrow is given by the projection onto the second direct summand from the decomposition of $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{\varphi\underline{\boldsymbol{gh}}})$ from Proposition 4.3.2 (with $\lambda_0 := \psi_E \lambda_1^{-\mathbf{c}} \lambda_2^{-\mathbf{c}}$), and the second arrow is induced by the multiplication by S on $T_{\psi_E} \otimes \Psi_S^{\mathbf{c}^{-1}}$.

Using $\mathrm{H}^1(K, T_{\psi_E}) \otimes \mathbf{Q}_p \simeq \mathrm{H}^1(\mathbf{Q}, V_pE)$, we shall also view κ_p in the latter group.

Lemma 6.1.3. *Suppose $L(E,1) = 0$. Then $\kappa_p \in \mathrm{Sel}(\mathbf{Q}, V_pE)$.*

Proof. Viewing κ_p in $\mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E})$, it suffices to show that $\mathrm{res}_{\overline{\mathfrak{p}}}(\kappa_p) = 0$. The assumption implies that $L(\psi_E^{-1}, 0) = 0$, and so from Theorem 2.1.1, Proposition 3.3.1, and Theorem 5.1.1, we obtain

$$\begin{aligned} L(E,1) = 0 \quaD\Longrightarrow\quad \mathscr{L}^-_{\psi_E}(0) = 0 \quad&\Longrightarrow\quad \mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h})(0,0) = 0 \\ &\Longrightarrow\quad \exp^*_{\overline{\mathfrak{p}}}(\kappa(\varphi, \boldsymbol{g}, \boldsymbol{h})_{1,1}) = 0, \end{aligned}$$

where, as in Theorem 5.1.1, $\kappa(\varphi, \boldsymbol{g}, \boldsymbol{h})_{k_1,k_2}$ denotes the specialisation of $\kappa(\varphi, \boldsymbol{g}, \boldsymbol{h})$ at $S_i = u^{k_i - 1} - 1$. Since for $k_1 + k_2 = 2$ the Bloch–Kato dual exponential (5.2) is injective, this gives the result. □

6.2. **A control theorem.** Let

$$\Lambda = \mathscr{O}[\![S]\!].$$

For any Λ-module M, we $M_{/S}$ for the cokernel of multiplication by S, i.e., $M_{/S} := M/SM$. The following is a variant of Mazur's control theorem for the "reversed" Selmer groups of Remark 4.3.5.

Proposition 6.2.1. *There are natural isomorphisms*

$$\begin{aligned} r^* : \mathrm{Sel}_{\emptyset,0}(K, A_{\psi_E^{\mathbf{c}}}) &\to \mathrm{Sel}_{\emptyset,0}(K, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})[S], \\ r : \mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E} \otimes \Psi_S^{\mathbf{c}^{-1}})_{/S} &\to \mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E}), \end{aligned}$$

induced by multiplication by S.

Proof. The map r^* fits into the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Sel}_{\emptyset,0}(K, A_{\psi_E^{\mathbf{c}}}) & \longrightarrow & \mathrm{Sel}^{\{p\}}(K, A_{\psi_E^{\mathbf{c}}}) & \longrightarrow & \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, A_{\psi_E^{\mathbf{c}}}) \\ & & \downarrow{\scriptstyle r^*} & & \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle t^*} \\ 0 & \longrightarrow & \mathrm{Sel}_{\emptyset,0}(K, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})[S] & \longrightarrow & \mathrm{Sel}^{\{p\}}(K, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})[S] & \longrightarrow & \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})[S]. \end{array}$$

It follows from Lubin–Tate theory that $K_{\overline{\mathfrak{p}}}(E[\overline{\mathfrak{p}}])$ has degree $p-1$ over $K_{\overline{\mathfrak{p}}}$ (see e.g. [dS87, Ch. I]), and therefore $\mathrm{H}^0(K_{\infty,\overline{\mathfrak{p}}}, E[\overline{\mathfrak{p}}^\infty]) = 0$ since Γ^- is pro-p. By inflation-restriction, it follows that the map t^* is an isomorphism.

The map s^* fits into the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Sel}^{\{p\}}(K, A_{\psi_E^{\mathbf{c}}}) & \longrightarrow & \mathrm{H}^1(K_\Sigma/K, A_{\psi_E^{\mathbf{c}}}) & \longrightarrow & \bigoplus_{w\in\Sigma, w\nmid p} \mathrm{H}^1(K_w, A_{\psi_E^{\mathbf{c}}}) \\ & & \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle v^*} \\ 0 & \longrightarrow & \mathrm{Sel}^{\{p\}}(K, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})[S] & \longrightarrow & \mathrm{H}^1(K_\Sigma/K, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})[S] & \longrightarrow & \bigoplus_{w\in\Sigma, w\nmid p} \mathrm{H}^1(K_w, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})[S]. \end{array}$$

The vanishing of $\mathrm{H}^0(K_{\infty,\bar{\mathfrak{p}}}, E[\bar{\mathfrak{p}}^\infty])$ implies that u^* is an isomorphism, and as shown in the proof of [dS87, Lem. IV.3.5] as a consequence of [Maz72, Cor. 4.4], the map v^* is injective.

Therefore, by the Snake Lemma applied to the above two diagrams we conclude that $\ker(r^*) = \ker(s^*) = 0$ and that $\mathrm{coker}(r^*) = \mathrm{coker}(s^*) = \ker(v^*) = 0$. This gives the result for r^*, and the case of r is shown in the same manner. □

Finally, for the proof of our second main result we shall use the following relation between the rank of the "reversed" Selmer group for ψ_E and that of its usual Selmer group.

Lemma 6.2.2. *We have*

$$\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E}) = \begin{cases} \mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \mathrm{Sel}(K, T_{\psi_E}) - 1 & \text{if } \mathrm{loc}_p \neq 0, \\ \mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \mathrm{Sel}(K, T_{\psi_E}) + 1 & \text{if } \mathrm{loc}_p = 0, \end{cases}$$

where $\mathrm{loc}_p : \mathrm{Sel}(\mathbf{Q}, V_p E) \to E(\mathbf{Q}_p)\hat{\otimes}\mathbf{Q}_p$ *the restriction map at* p.

Proof. By global duality we have the exact sequence

$$0 \to \mathrm{Sel}(K, T_{\psi_E}) \to \mathrm{Sel}_{\mathrm{rel}}(K, T_{\psi_E}) \xrightarrow{\alpha} \prod_{v|p} \frac{\mathrm{H}^1(K_v, T_{\psi_E})}{E(K_v)\otimes \mathcal{O}_{K,\mathfrak{p}}} \xrightarrow{\beta^\vee} \mathrm{Sel}(K, A_{\psi_E^{\mathbf{c}}})^\vee, \tag{6.2}$$

where the last arrow corresponds (by Tate's local duality) to the Pontryagin dual of the restriction map

$$\beta : \mathrm{Sel}(K, A_{\psi_E^{\mathbf{c}}}) \to \prod_{v|p} E(K_v) \otimes (K_{\bar{\mathfrak{p}}}/\mathcal{O}_{K,\bar{\mathfrak{p}}}).$$

Note that the target of the map β has $\mathcal{O}_{K,\mathfrak{p}}$-rank one, and from the action of complex conjugation and the isomorphism $\mathrm{Sel}(K, A_{\psi_E}) \simeq \mathrm{Sel}_{p^\infty}(E/\mathbf{Q})$ we see that $\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \mathrm{im}(\beta) = \dim_{\mathbf{Q}_p} \mathrm{im}(\mathrm{loc}_p)$.

Suppose first that $\mathrm{loc}_p \neq 0$, so by the above remarks the map β has finite cokernel. By (6.2), it follows that α has finite image, and therefore

$$\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \mathrm{Sel}_{\mathrm{rel}}(K, T_{\psi_E}) = \mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \mathrm{Sel}(K, T_{\psi_E}).$$

In particular, this implies that

$$\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E}) = \mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \ker\big(\mathrm{loc}_{\mathfrak{p}} : \mathrm{Sel}(K, T_{\psi_E}) \to E(K_{\mathfrak{p}}) \otimes \mathcal{O}_{K,\mathfrak{p}}\big),$$

yielding the result in this case. On the other hand, if $\mathrm{loc}_p = 0$ then from (6.2) it follows that

$$\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \mathrm{Sel}_{\mathrm{rel}}(K, T_{\psi_E}) = \mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \mathrm{Sel}(K, T_{\psi_E}) + 1.$$

Since $\mathrm{H}^1(K_{\mathfrak{p}}, T_{\psi_E})/E(K_{\mathfrak{p}}) \otimes \mathcal{O}_{K,\mathfrak{p}}$ is torsion, the modules $\mathrm{Sel}_{\mathrm{rel}}(K, T_{\psi_E})$ and $\mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E})$ have the same rank, so this concludes the proof. □

6.3. **Proof of Theorem B.** We now prove our main result on the nonvanishing of generalised Kato classes $\kappa_p \in \mathrm{Sel}(\mathbf{Q}, V_pE)$ in situations where $\mathrm{ord}_{s=1}L(E,s) \geq 2$.

Recall that we let $E/\mathbf{Q}$ be an elliptic curve with CM by an imaginary quadratic field K in which the prime $p \geq 5$ splits as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$. Suppose E has root number $+1$, and let $\kappa_p \in \mathrm{H}^1(\mathbf{Q}, V_pE)$ be the generalised Kato class attached to a pair of ring class characters $\phi_1 = \lambda_1^{1-\mathbf{c}}, \phi_2 = \lambda_2^{1-\mathbf{c}}$ as in Proposition 6.1.1; then, by Lemma 6.1.3, $\kappa_p \in \mathrm{Sel}(\mathbf{Q}, V_pE)$ as long as $L(E,1) = 0$.

Theorem 6.3.1. *Suppose $L(E,s)$ vanishes to positive even order at $s=1$. Then*

$$\kappa_p \neq 0 \quad \Longrightarrow \quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2.$$

Conversely, if $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$ then $\kappa_p \neq 0$ if and only if the restriction map

$$\mathrm{res}_p : \mathrm{Sel}(\mathbf{Q}, V_pE) \to E(\mathbf{Q}_p)\hat{\otimes}\mathbf{Q}_p$$

is nonzero.

Proof. Put $S = S_1$. Restricted to $S_1 = S_2$, the factorisation in Proposition 3.3.1 (in which then $W_1 = S$ and $W_2 = 0$) reads as the equality

$$\mathscr{L}_p^{\varphi}(\varphi, \underline{\boldsymbol{gh}})^2(S) = \mathcal{L}^-_{\mathfrak{p},\psi_E\phi_1\phi_2}(S)^{\iota} \cdot \mathcal{L}^-_{\mathfrak{p},\psi_E}(S) \cdot \mathcal{L}^-_{\mathfrak{p},\psi_E\phi_1}(0) \cdot \mathcal{L}^-_{\mathfrak{p},\psi_E\phi_2}(0)$$

up to a multiplication by a unit $u \in \mathcal{W}[1/p]^{\times}$. On the other hand, the decomposition in Corollary 4.3.4 for the unbalanced Selmer group becomes

$$\begin{aligned} X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi\underline{\boldsymbol{gh}}}) \simeq\ & X_{0,\emptyset}(K, A_{\psi_E^{\mathbf{c}}\phi_1^{\mathbf{c}}\phi_2^{\mathbf{c}}} \otimes \Psi_S^{\mathbf{c}-1}) \oplus X_{0,\emptyset}(K, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}}) \\ & \oplus X_{0,\emptyset}(K, A_{\psi_E^{\mathbf{c}}\phi_1^{\mathbf{c}}} \otimes \Psi_S^{\mathbf{c}-1})_{/S} \oplus X_{0,\emptyset}(K, A_{\psi_E^{\mathbf{c}}\phi_2^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})_{/S}. \end{aligned}$$

Noting that the involution ι gives $X_{0,\emptyset}(K, A_{\psi_E^{\mathbf{c}}\phi_1^{\mathbf{c}}\phi_2^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})^{\iota} = X_{0,\emptyset}(K, A_{\psi_E^{\mathbf{c}}\phi_1^{\mathbf{c}}\phi_2^{\mathbf{c}}} \otimes \Psi_S^{\mathbf{c}-1})$, from Theorem 4.1.3 we deduce that $X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi\underline{\boldsymbol{gh}}})$ is Λ-torsion with

$$\mathrm{char}_{\Lambda}\big(X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi\underline{\boldsymbol{gh}}})\big) = \big(\mathscr{L}_p^{\varphi}(\varphi, \underline{\boldsymbol{gh}})\big)^2$$

as ideals in $\Lambda_{\mathcal{W}} \otimes \mathbf{Q}_p$. By Proposition 5.1.2, it follows that

$$\mathrm{rank}_{\Lambda}\big(X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi\underline{\boldsymbol{gh}}})\big) = \mathrm{rank}_{\Lambda}\big(\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^{\dagger}_{\varphi\underline{\boldsymbol{gh}}})\big) = 1, \tag{6.3}$$

and

$$\mathrm{char}_{\Lambda}\big(X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi\underline{\boldsymbol{gh}}})_{\mathrm{tors}}\big) = \mathrm{char}_{\Lambda}\left(\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^{\dagger}_{\varphi\underline{\boldsymbol{gh}}})}{\Lambda \cdot \kappa(\varphi, \underline{\boldsymbol{gh}})}\right)^2 \tag{6.4}$$

as ideals in $\Lambda_{\mathcal{W}} \otimes \mathbf{Q}_p$. Now, the implications

$$\begin{aligned} L(\psi_E^{-1}\phi_1^{-1}\phi_2^{-1}, 0) \neq 0 \quad &\Longrightarrow \quad \mathcal{L}^-_{\mathfrak{p},\psi_E\phi_1\phi_2}(0) \neq 0 \\ &\Longrightarrow \quad \Big|X_{0,\emptyset}(K, A_{\psi_E^{\mathbf{c}}\phi_1^{\mathbf{c}}\phi_2^{\mathbf{c}}} \otimes \Psi_S^{\mathbf{c}-1})_{/S}\Big| < \infty. \end{aligned}$$

follow from the interpolation property of $\mathcal{L}^-_{\mathfrak{p},\psi_E\phi_1\phi_2}$ and the combination of Theorem 4.1.3 and Mazur's control theorem, respectively. The nonvanishing of $L(\psi_E^{-1}\phi_1^{-1}, 0)$ and $L(\psi_E^{-1}\phi_2^{-1}, 0)$ similarly implies

$$\Big|X_{0,\emptyset}(K, A_{\phi_E^{\mathbf{c}}\phi_1^{\mathbf{c}}} \otimes \Psi_S^{\mathbf{c}-1})_{/S}\Big| < \infty, \quad \Big|X_{0,\emptyset}(K, A_{\phi_E^{\mathbf{c}}\phi_2^{\mathbf{c}}} \otimes \Psi_S^{\mathbf{c}-1})_{/S}\Big| < \infty,$$

and so from (6.3) and the balanced Selmer group decomposition from Corollary 4.3.4:

$$X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^{\dagger}_{\varphi\underline{\boldsymbol{gh}}}) \simeq X_{0,\emptyset}(K, A_{\psi_E^{\mathbf{c}}\phi_1^{\mathbf{c}}\phi_2^{\mathbf{c}}} \otimes \Psi_S^{\mathbf{c}-1}) \oplus X_{\emptyset,0}(K, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}}) \\ \oplus X_{0,\emptyset}(K, A_{\psi_E^{\mathbf{c}}\phi_1^{\mathbf{c}}} \otimes \Psi_S^{\mathbf{c}-1})_{/S} \oplus X_{0,\emptyset}(K, A_{\psi_E^{\mathbf{c}}\phi_2^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})_{/S},$$

we deduce that

$$\mathrm{rank}_{\Lambda}\big(X_{\emptyset,0}(K, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})\big) = 1. \tag{6.5}$$

On the other hand, by the control theorem of Proposition 6.2.1 the $\mathcal{O}_{K,\mathfrak{p}}$-rank of the specialisation $X_{\emptyset,0}(\mathbf{Q}, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})_{/S}$ satisfies

$$\begin{aligned} \mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(X_{\emptyset,0}(K, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})_{/S}\big) &= \mathrm{corank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(\mathrm{Sel}_{\emptyset,0}(K, A_{\psi_E^{\mathbf{c}}})\big) \\ &= \mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(\mathrm{Sel}_{\emptyset,0}(K, T_{\psi_E^{\mathbf{c}}})\big) \\ &= \mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(\mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E})\big), \end{aligned} \tag{6.6}$$

using the isomorphism $\mathrm{Sel}_{\emptyset,0}(K, T_{\psi_E^{\mathbf{c}}}) \simeq \mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E})$ given by the action of complex conjugation for the last equality.

Denote by $\kappa_E(\varphi, \underline{\boldsymbol{gh}}) \in \mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E} \otimes \Psi_S^{\mathbf{c}-1})$ the projection of $\kappa(\varphi, \underline{\boldsymbol{gh}})$ onto the second direct summand in the balanced Selmer groups decomposition from Proposition 4.3.2:

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^{\dagger}_{\varphi\underline{\boldsymbol{gh}}}) \simeq \mathrm{Sel}_{\emptyset,0}(K, T_{\psi_E\phi_1\phi_2} \otimes \Psi_S^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E} \otimes \Psi_S^{\mathbf{c}-1}) \\ \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\psi_E\phi_1} \otimes \Psi_S^{1-\mathbf{c}})_{/S} \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\psi_E\phi_2} \otimes \Psi_S^{\mathbf{c}-1})_{/S}.$$

Since our assumptions imply that $\psi_E\phi_1, \psi_E\phi_2, \psi_E\phi_1\phi_2$ have all root number $+1$, by Theorem 4.1.3 we have

$$\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^{\dagger}_{\varphi\underline{\boldsymbol{gh}}})}{\Lambda \cdot \kappa(\varphi, \underline{\boldsymbol{gh}})} \simeq \frac{\mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E} \otimes \Psi_S^{\mathbf{c}-1})}{\Lambda \cdot \kappa_E(\varphi, \underline{\boldsymbol{gh}})}.$$

Together with Theorem 5.2.1 (note that the choice of ϕ_1, ϕ_2 in Proposition 6.1.1 guarantee that the associated primtitive CM Hida families $\boldsymbol{g}, \boldsymbol{h}$ satisfy the conditions in that result), this gives that $\mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E} \otimes \Psi_S^{\mathbf{c}-1})$ has Λ-rank one, and we have the following equality of ideals in $\Lambda_{\mathcal{W}} \otimes \mathbf{Q}_p$:

$$\mathrm{char}_{\Lambda}\big(X_{\emptyset,0}(K, A_{\psi_E^{\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})_{\mathrm{tors}}\big) = \mathcal{L}^{-}_{\mathfrak{p},\psi_E^{\mathbf{c}}\phi_1^{\mathbf{c}}\phi_2^{\mathbf{c}}}(S)^{\iota} \cdot \mathrm{char}_{\Lambda}\big(\mathfrak{Z}_E\big)^2, \tag{6.7}$$

where $\mathfrak{Z}_E = \mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E} \otimes \Psi_S^{\mathbf{c}-1})/\Lambda \cdot \kappa_E(\varphi, \underline{\boldsymbol{gh}})$. From (6.5), (6.6), and (6.7), we thus see that

$$\begin{aligned} \mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(\mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E})\big) &= 1 - \mathrm{ord}_S\big(\mathcal{L}^{-}_{\mathfrak{p},\psi_E\phi_1\phi_2}(S)^{\iota}\big) + 2\,\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big((\mathfrak{Z}_E)_{/S}\big) \\ &= 1 + 2\,\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big((\mathfrak{Z}_E)_{/S}\big), \end{aligned}$$

using Theorem 2.1.1 and the nonvanishing of $L(\psi_E^{-1}\phi_1^{-1}\phi_2^{-1}, 0)$ for the last equality.

Since by construction the injection

$$\mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E} \otimes \Psi_S^{\mathbf{c}-1})_{/S} \to \mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E})$$

of Proposition 6.2.1 sends $\kappa_E(\varphi, \underline{\boldsymbol{gh}})$ mod S into κ_p, we conclude that

$$\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(\mathrm{Sel}_{0,\emptyset}(K, T_{\psi_E})\big) = 1 \quad \Longleftrightarrow \quad \kappa_p \neq 0. \tag{6.8}$$

The first claim in the Theorem now follows from (6.8) and Lemma 6.2.2, noting that by the work of Rubin [Rub91] proving the Iwasawa main conjecture for K, the vanishing of $L(E, 1)$ implies the non-triviality of $\mathrm{Sel}(\mathbf{Q}, V_pE)$. Similarly, the last claim in the Theorem is a direct consequence of (6.8) and Lemma 6.2.2. □

Remark 6.3.2. The result of Theorem 6.3.1 confirms expectations suggested by the conjectures of Darmon–Rotger [DR16] (see esp. Conjecture 3.12 in *op. cit.* as specialised to the "rank $(2,0)$ setting" in §4.5.3), and further shows that the nonvanishing of the restriction map res_p is *necessary* for the implication $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2 \Longrightarrow \kappa_p \neq 0$ to hold.

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