Methods for Evaluating Infinite Series

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March 23, 2010

Geometric Series

The simplest infinite series is the geometric series. Given real (or complex!) numbers a and r,

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{otherwise} \end{cases}$$

The mnemonic for the sum of a geometric series is that it's "the first term divided by one minus the common ratio." You'll see why words are helpful in the examples below.

Example 1. Let's evaluate

$$\sum_{n=2}^{\infty} 3(1/2)^n$$

Be careful; the index doesn't start at $n = 0$. We could do the following manipulations:

$$\sum_{n=2}^{\infty} 3(1/2)^n = \sum_{n=0}^{\infty} 3(1/2)^{n+2} = \sum_{n=0}^{\infty} (3/4)(1/2)^n = \frac{3/4}{1 - 1/2} = 3/2.$$

Alternatively, the mnemonic gives the answer:

$$\sum_{n=2}^{\infty} 3(1/2)^n = \frac{\text{first term}}{1 - \text{ratio}} = \frac{3/4}{1 - 1/2} = 3/2. \qquad \square$$

A useful property of absolutely convergent series is that $\operatorname{Re} \sum = \sum \operatorname{Re}$ and $\operatorname{Im} \sum = \sum \operatorname{Im}$; that is, we can take the real and imaginary parts of a series easily.

Example 2. Let's evaluate

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n}$$

You can use the comparison test to show that this series converges absolutely. Here's how we'll approach the sum:

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = \sum_{n=0}^{\infty} \operatorname{Re} \frac{e^{in\theta}}{2^n} = \sum_{n=0}^{\infty} \operatorname{Re} \left(\frac{e^{i\theta}}{2}\right)^n = \operatorname{Re} \frac{1}{1 - e^{i\theta}/2} = \operatorname{Re} \frac{2}{2 - e^{i\theta}}.$$

To finish the computation, use the identities $\operatorname{Re} z = (z + \overline{z})/2$ and $e^{i\theta} + e^{-i\theta} = 2\cos\theta$.

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = \frac{1}{2 - e^{i\theta}} + \frac{1}{2 - e^{-i\theta}} = \frac{4 - 2\cos\theta}{5 - 4\cos\theta}. \qquad \square$$

Once you know some series, other ones follow easily.

Example 3. Let's evaluate

$$\sum_{n=0}^{\infty} \frac{\cos^2(n\theta)}{2^n}.$$

Let's use the identity $2\cos^2\theta = 1 + \cos 2\theta$. Since all the sums converge absolutely, we can write

$$\sum_{n=0}^{\infty} \frac{\cos^2(n\theta)}{2^n} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1 + \cos 2\theta}{2^n} = \frac{1}{2}\left(\frac{1}{1 - 1/2}\right) + \frac{1}{2}\left(\frac{4 - 2\cos 2\theta}{5 - 4\cos 2\theta}\right) = \frac{7 - 5\cos 2\theta}{5 - 4\cos 2\theta}. \qquad \square$$

Telescoping Sums

Sometimes evaluating the partial sums of a series is easy.

Example 1. Let's evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}.$$

Using partial fractions gives us

$$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2+n} &= \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \lim_{N\to\infty} \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\
&= \lim_{N\to\infty} 1 - \frac{1}{N+1} \\
&= 1.
\end{aligned}$$

□

Notice the way that each term canceled with the previous one. When a sum does this, we say it 'telescopes'. It's important to rely on the definition of an infinite series when trying to telescope a series. This example shows why.

Example 2. Let's evaluate

$$\sum_{n=0}^{\infty} \arctan(n+2) - \arctan n.$$

A handwaving approach might say "the sum clearly telescopes, so the answer is $\arctan(\infty) - \arctan 0 = \pi/2$. But this is wrong! If we're careful we see that

$$\begin{aligned}
\sum_{n=0}^{N} \arctan(n+2) - \arctan n &= \arctan(N+2) + \arctan(N+1) - \arctan 1 - \arctan 0 \\
&= \arctan(N+2) + \arctan(N+1) - \pi/4,
\end{aligned}$$

so the infinite sum is $\pi/2 + \pi/2 - \pi/4 = 3\pi/4$. □

Taylor Series

Occasionally a series can be recognized as a special case of Taylor series.

Example 1. Let's evaluate

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

This looks a lot like the series for e^x. With a little adjustment:

$$\sum_{n=1}^{\infty} \frac{2^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} - 1 = e^2 - 1.$$ □

Sometimes a series looks similar enough to a known Taylor series that derivatives and integrals might save the day.

Example 2. Let's evaluate

$$\sum_{n=0}^{\infty} \frac{n}{3^n}.$$

Now this is more interesting than the last example. Let's define a power series:

$$f(x) = \sum_{n=0}^{\infty} n x^n,$$

where we want $f(1/3)$. The major trick is seeing the derivative:

$$f(x) = x \sum_{n=0}^{\infty} n x^{n-1} = x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = x \frac{d}{dx} \frac{1}{1-x} = \frac{x}{(1-x)^2}.$$

So set $x = 1/3$ to get $f(1/3) = 3/4$. □

This trick is sneaky enough to merit another example.

Example 3. Let's evaluate

$$\sum_{n=0}^{\infty} \frac{n+1}{n!}.$$

Let's try writing another power series; define

$$f(x) = \sum_{n=0}^{\infty} \frac{(n+1)}{n!} x^n.$$

Now it looks like a derivative:

$$f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1} = \frac{d}{dx} x \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \frac{d}{dx} \left(x e^x \right) = x e^x + e^x.$$

Set $x = 1$ to get $f(1) = 2e$. □

Fourier Methods

We only need two things from Fourier analysis to evaluate sums: The convergence theorem and Parseval's identity. Let's take these in turn. Given a piecewise smooth function f continuous on the interval $[-L, L]$, define for nonnegative integers n,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n\pi x/L)\, dx \qquad \text{and} \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(n\pi x/L)\, dx.$$

Then for all x (strictly) between $-L$ and L,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L).$$

And for $x = \pm L$,

$$\frac{1}{2} \left(\lim_{x \to -L+} f(x) + \lim_{x \to L-} f(x) \right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L).$$

Using this theorem, a clever mathematician can recognize Fourier expansions hiding within his sums.

Example 1. Let's evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

All of the previous techniques fail here, but suppose we remember the Fourier series

$$x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right), \qquad -\pi \le x \le \pi.$$

Setting $x = 0$ gives us

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

so the sum is $-\pi^2/12$. $\square$

Honestly, this technique requires a repository of Fourier series. The other theorem which comes in handy is Parseval's identity: with the same definitions as before *but specializing* $L = \pi$, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Here's the main example.

Example 2. Let's evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Again, using

$$x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right), \qquad -\pi \le x \le \pi,$$

we have $a_0 = 2\pi^2/3$, and $a_n = 4(-1)^n/n^2$ for all other n. Each b_n is zero. Therefore

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{(2\pi^2/3)^2}{2} + \sum_{n=1}^{\infty} \frac{16}{n^4}.$$

Evaluating the integral and rearranging gives

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$ $\square$

Miscellaneous

Here are some examples of series that don't quite fit into the other categories.

Example 1. Let's evaluate the double sum

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{3^{m+n}}.$$

We can treat this like a double integral:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{3^{m+n}} = \sum_{m=0}^{\infty}\left(\frac{1}{3^m}\sum_{n=0}^{\infty}\frac{1}{3^n}\right) = \left(\sum_{n=0}^{\infty}\frac{1}{3^n}\right)\left(\sum_{m=0}^{\infty}\frac{1}{3^m}\right) = \left(\frac{1}{1-1/3}\right)^2 = \frac{9}{4}.$$ $\square$

Speaking of double integrals, here's an example where interchanging sums can be fruitful.

Example 2. Let's evaluate

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{1}{3^{m+n}}.$$

While no means impossible, this sum is a bit messy. It's nicer if we interchange the order first (this is allowed because all terms are positive). To see how to change the order, just think about how it's done for integrals.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{1}{3^{m+n}} = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \frac{1}{3^{m+n}} = \sum_{m=0}^{\infty}\frac{1}{3^m}\sum_{n=m}^{\infty}\frac{1}{3^n} = \sum_{m=0}^{\infty}\frac{1}{3^m}\left(\frac{1/3^m}{1-1/3}\right) = \frac{3}{2}\sum_{m=0}^{\infty}\frac{1}{9^m} = \frac{27}{16}.$$ $\square$