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Non-faithful representations of surface groups into $SL(2, \mathbb{C})$ which kill no simple closed curve

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Abstract We give counterexamples to a version of the simple loop conjecture in which the target group is $PSL(2, \mathbb{C})$. These examples answer a question of Minsky in the negative.

Keywords Simple loop conjecture · Surface group · Hyperbolic geometry

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1 Introduction

The original simple loop conjecture, proved by Gabai in [6], implies that the kernel of any non-injective homomorphism between the fundamental groups of closed orientable surfaces contains an element represented by an essential simple closed curve. It has been conjectured (see Problem 3.96 in Kirby's problem list [16]) that if the target is replaced by the fundamental group of a closed orientable 3-manifold M the same result holds:

Simple Loop Conjecture *Let M be an orientable 3–manifold, and let Σ be a closed orientable surface. The kernel of every non-injective homomorphism from $\pi_1 \Sigma$ to $\pi_1 M$ contains an element represented by an essential simple closed curve on Σ.*

(There are versions of Gabai's theorem and the above conjecture in which Σ and M are allowed to be non-orientable, and an additional *two-sidedness* hypothesis is added. We focus on the orientable case in this paper). Hass proved the simple loop conjecture in case M is Seifert fibered in [9]. Rubinstein and Wang extended Hass's theorem to the case in which M is a graph manifold in [18]. The important case of M hyperbolic is open.

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Minsky further asked [14, Question 5.3] if the same result holds if the target group is $PSL(2, \mathbb{C})$. An affirmative answer would have implied the Simple Loop Conjecture for M hyperbolic. In Proposition 3.4 we give a negative answer to Minsky's question, by finding representations with non-trivial kernel which kill no simple curve. By construction, these counterexamples lift to $SL(2, \mathbb{C})$ (as must any discrete faithful representation of a hyperbolic 3–manifold group, by [3, 3.11], cf. [4]). For these counterexamples, we require genus at least 3.

If the genus is at least 4, we can find such representations with no nontrivial elliptics in their image, so no power of a simple loop is in the kernel.

Theorem 1.1 *Let Σ be a closed orientable surface of genus greater than or equal to* 4. *There is a homomorphism $\theta\colon\ \pi_1\Sigma \to SL(2, \mathbb{C})$ such that*

(1) *θ is not injective.*
(2) *If $\theta(\alpha) = \pm I$ then α is not represented by an essential simple closed curve.*
(3) *If $\theta(\alpha)$ has finite order then $\theta(\alpha) = I$.*

We prove this by a dimension count in the character variety, showing at the same time there are uncountably many conjugacy classes of such homomorphisms. For a group G let $R(G)$ be the set of representations of G into $SL(2, \mathbb{C})$ and $X(G)$ is the set of characters of these homomorphisms. (Both $R(G)$ and $X(G)$ are algebraic sets [3, 1.4.5]. Although $R(G)$ and $X(G)$ need not be irreducible algebraic sets, we follow common usage in referring to them as the *representation variety* and *character variety*, respectively.)

Let Σ be a closed orientable surface of negative Euler characteristic, and let C be a simple closed curve on Σ such that one component of $\Sigma \setminus C$ is a punctured torus. In the remainder of the paper we shall frequently abuse notation by ignoring basepoints and treating C as if it is an element of $\pi_1\Sigma$. Define subsets of $X(\pi_1\Sigma)$ as follows.

$$Z = \{\, x \in X(\pi_1\Sigma) \mid x(C) = 2 \,\}.$$
$$Y = \{\, x \in X(\pi_1\Sigma) \mid \rho(C') = I \text{ for some s.c.c. } C' \text{ and some } \rho \text{ with character } x\}$$
$$E = \{\, x \in Z \mid \exists\, \alpha \in \pi_1\Sigma \quad x(\alpha) \in \mathbb{R} \setminus \{2\} \,\}$$

In the definition of Y, "s.c.c." stands for "essential simple closed curve in Σ". Thus the set Y is the set of characters of representations which kill some essential simple closed curve; the set E contains all characters in Z which are also characters of a representation with elliptics in its image.

We will show:

Theorem 1.2 *If ρ is a representation with character in Z then ρ is not injective. If the genus of Σ is at least* 3 *then Y is a countable union of algebraic sets of complex dimension at most* $\dim_{\mathbb{C}} Z - 1$. *If the genus of Σ is at least* 4, *then E is a countable union of real algebraic sets of real dimension at most* $\dim_{\mathbb{R}} Z - 1$.

Theorem 1.2 implies Theorem 1.1 as follows: Suppose the genus of Σ is at least 4. Theorem 1.2 implies that there is some (necessarily non-injective) representation θ of $\pi_1\Sigma$ whose character x lies in $Z \setminus (Y \cup E)$. Since θ is non-injective, it satisfies condition (1) of Theorem 1.1. Let $\alpha \in \pi_1\Sigma$. Suppose first that α is represented by a simple closed curve. Since $x \notin Y$, we have $\theta(\alpha) \neq I$. Since $x \notin E$, we have $\theta(\alpha) \neq -I$, so condition (2) of Theorem 1.1 holds for θ. Now let $\alpha \in \pi_1\Sigma$ be arbitrary. If $\theta(\alpha)$ has finite order, then $x(\alpha) \in [-2, 2]$. But since $x \notin E$, we must have $x(\alpha) = 2$, and so $\theta(\alpha) = I$. Condition (3) therefore holds for θ, and Theorem (1.1) is established.

Theorem 6.1 is of independent interest and states that Z is irreducible and thus an affine variety. This suggests a more general study of irreducibility of interesting algebraic subsets of the character variety. The tool used to show Theorem 6.1 is a standard fact from algebraic geometry: a complex affine algebraic set is irreducible if and only if the smooth part is connected, open, and dense, Theorem 8.4. In fact we have been unable to locate this statement we need in the literature which mostly deals with *irreducible* algebraic sets. Therefore we have included a brief appendix, Sect. 8, about algebraic subsets of $\mathbb{C}^n$ which contains the statements we need.

We also provide (Theorem 4.7) a new proof of a theorem of Goldman [7] that the subspace of the character variety of a closed surface consisting of characters of irreducible representations is a manifold.

We have heard from Lars Louder that he also can answer Minsky's question in the negative, using entirely different methods. His examples at the same time show that there are two-dimensional hyperbolic *limit groups* which are not surface groups, but are homomorphic images of surface groups under maps which kill no simple closed curve. Whereas the representations used in our paper always have nontrivial parabolics in their image, it is possible to find faithful representations of Louder's groups with all-loxodromic (but indiscrete) image.[1]

1.1 Conventions and outline

The algebraic geometry needed for this paper is discussed in the appendix Sect. 8. By definition $SL(2,\mathbb{C}) \subset \mathbb{C}^4$ is an affine algebraic subset and the group operations are regular maps. Suppose G is a group generated by the finite subset $\mathcal{S} \subset G$. For simplicity we assume $\mathcal{S}$ is not empty and closed under taking inverses. Then $R(G;\mathcal{S})$ is the affine algebraic subset of $\prod_{\mathcal{S}} SL(2,\mathbb{C})$ that satisfies the relations in G and is called a *representation variety* for G. If $\mathcal{S}'$ is another finite generating set then there is a regular isomorphism $R(G;\mathcal{S}) \to R(G;\mathcal{S}')$. We will be sloppy and refer to *the* representation variety $R(G)$ for some choice $R(G;\mathcal{S})$, even though it is not well defined and might be reducible. Observe there is a natural bijection $R(G) \longrightarrow Hom(G, SL(2,\mathbb{C}))$.

The *character* of $\rho \in Hom(G, SL(2,\mathbb{C}))$ is $\chi_\rho = \mathrm{tr} \circ \rho : G \longrightarrow \mathbb{C}$. Let $\mathcal{S}^+$ denote the set of words of length at most $s = |\mathcal{S}|$ in the elements of $\mathcal{S}$. Then $X(G;\mathcal{S}) \subset \mathbb{C}^{\mathcal{S}^+}$ is the set of all $\chi_\rho|\mathcal{S}^+$. It is an affine algebraic set [3]. The *character variety* $X(G)$ means some choice of $X(G;\mathcal{S})$ and might be reducible. Using the *trace relation* $\mathrm{tr}AB + \mathrm{tr}A^{-1}B = \mathrm{tr}A \cdot \mathrm{tr}B$ for $A, B \in SL(2,\mathbb{C})$ it follows that the trace of every element of G is a polynomial in the traces of elements of $\mathcal{S}^+$ thus $X(G)$ is well defined up to regular isomorphism.

The commutator $[\alpha,\beta]$ denotes always $\alpha\beta\alpha^{-1}\beta^{-1}$, whether α and β are group elements or matrices. Unless explicitly noted otherwise, topological statements about varieties are with respect to the classical (not Zariski) topology.

Here is an outline of the paper. In Sect. 2 we study the character variety of a free product of surface groups. This is used (Lemma 2.6) to show that if Σ has genus at least 3, then the set of characters of representations which kill a given simple closed curve has codimension at least 2 in the character variety of Σ. The set Z has codimension 1 (see Lemma 3.3), so $Z \setminus Y$ is nonempty.

In Sect. 3 we recall (Lemma 3.1) that a representation into $SL(2,\mathbb{C})$ of the free group of rank two generated by α and β which sends $[\alpha,\beta]$ to an element of trace $+2$ is reducible, thus has solvable image, and is therefore not injective. This result is well known [3] and is

[1] Since this paper was submitted, Louder's preprint [11] has appeared, as has Calegari's preprint [2] applying stable commutator length to Minsky's question. Even more recent work can be found in [1,12].

in contrast to the fact there are injective homomorphisms for which the trace is -2. It it is deduced that Z is composed entirely of characters of non-injective representations. Since $Z \setminus Y$ is nonempty, the answer to Minsky's question is no (Proposition 3.4). In this section, the genus of Σ is assumed to be at least 3.

In Sect. 4, we show (Lemma 4.6) that a representation of a surface is irreducible if and only if it contains a punctured torus such that the restriction of the representation to this punctured torus is irreducible. Then we use this Lemma to give a new proof of Goldman's theorem that the characters of irreducible representations are smooth points of the character variety of a surface.

In Sect. 5, we prove several results about lifting deformations of characters to deformations of representations of surface groups, and extending such deformations from proper subsurfaces. These results are mostly used for the main technical result in Sect. 6.

In Sect. 6, we show that Z is irreducible (Theorem 6.1). This is the most technical part of the paper.

Finally in Sect. 7 we show how the irreducibility of Z implies that E is a countable union of positive codimension subsets of Z, and complete the proof of Theorem 1.2.

2 Free products

If G and H are groups and $G * H$ their free product, the representation variety $R(G * H)$ can be canonically identified with $R(G) \times R(H)$. We recall the following standard fact.

Lemma 2.1 *Let A, B be affine algebraic sets, and let $X = A \times B$. The irreducible components of X are the products of irreducible components of A and B.*

Proof Suppose that $A = \bigcup_i A_i$ and $B = \bigcup_j B_j$ are the canonical decompositions of A and B into irreducibles. For each i, j, the set $A_i \times B_j \subset X$ is a variety [19, p. 35]. So we can write X as a union of irreducibles

$$X = \bigcup_{i,j} A_i \times B_j. \tag{1}$$

One checks easily that $A_i \times B_j \subseteq A_{i'} \times B_{j'}$ implies that $i = i'$ and $j = j'$ so the expression (1) is irredundant. Such an irredundant expression is unique [19, p. 34], so every irreducible component of X appears. □

The irreducible components of $R(G*H)$ are therefore products of irreducible components of $R(G)$ with irreducible components of $R(H)$.

Definition 2.2 We say a representation $\rho\colon\ G \to SL(2, \mathbb{C})$ is *noncentral* if its image does not lie in the center $\{\pm I\}$. A representation is *reducible* if there is a proper invariant subspace for the action on $\mathbb{C}^2$. It is *irreducible* if it is not reducible.

Lemma 2.3 *Let C be a component of $X(G*H)$, so that C is the image of $A \times B \subseteq R(G*H)$, where A is an irreducible component of $R(G)$ and B is an irreducible component of $R(H)$. Suppose that A and B each contain a noncentral representation. Then*

$$\dim_{\mathbb{C}}(C) = \dim_{\mathbb{C}}(A) + \dim_{\mathbb{C}}(B) - 3.$$

Proof We first show that C is not composed entirely of characters of reducible representations.

Claim 2.3.1 *$A \times B$ contains some irreducible representation of $G * H$.*

Proof Indeed, let $\rho_A\colon\ A \to SL(2, \mathbb{C})$ and $\rho_B\colon\ B \to SL(2, \mathbb{C})$ be the noncentral representations in A and B. If either representation is irreducible or if ρ_A and ρ_B have disjoint fixed point sets at infinity, then $\rho = (\rho_A, \rho_B)$ is irreducible. If ρ_A and ρ_B have the same fixed point sets, we may conjugate ρ_B so its fixed point set is disjoint from that of ρ_A. □

Given the claim, the lemma follows immediately from [3, 1.5.3]. □

The following result follows from a more general result of Rapinchuk–Benyash-Krivetz–Chernousov [17, Theorem 3].

Proposition 2.4 *If Σ is a surface of genus $g \geq 2$ then $R(\pi_1\Sigma)$ is an irreducible variety of complex dimension $6g-3$. Moreover $X(\pi_1\Sigma)$ is an irreducible variety of complex dimension $6g-6$.*

Remark 2.5 It should be possible prove Proposition 2.4 with the method we use below to show Z is irreducible.

Lemma 2.6 *If Σ is a closed orientable surface of genus $g \geq 3$, and $\alpha \in \pi_1(\Sigma)$ is represented by a simple closed curve, then the complex codimension of $X(\pi_1\Sigma/\langle\langle\alpha\rangle\rangle)$in $X(\pi_1\Sigma)$ is at least 2. In other words*

$$\dim_{\mathbb{C}}(X(\pi_1\Sigma/\langle\langle\alpha\rangle\rangle)) \leq \dim_{\mathbb{C}}(X(\pi_1\Sigma)) - 2.$$

Proof Let X_α be the character variety of $\pi_1(\Sigma)/\langle\langle\alpha\rangle\rangle$.

There are three cases to consider. Suppose first that α is represented by a non-separating curve. It follows that X_α is the character variety of $\mathbb{Z} * S$, where S is the fundamental group of the closed orientable surface of genus $g-1$. The representation variety of $\mathbb{Z}$ is 3–dimensional, and the representation variety of S is $(6g-9)$–dimensional, by Proposition 2.4. Lemma 2.3 then implies that

$$\dim_{\mathbb{C}} X_\alpha = 6g - 9 + 3 - 3 = 6g - 9 = \dim_{\mathbb{C}}(X(\pi_1(\Sigma))) - 3.$$

We next suppose that α separates Σ into a surface of genus 1 and one of genus $g-1$. Then X_α is the representation variety of $(\mathbb{Z} \oplus \mathbb{Z}) * S$, where S is again the fundamental group of the closed orientable surface of genus $g-1$. The representation variety of $\mathbb{Z} \oplus \mathbb{Z}$ is 4–dimensional, so Lemma 2.3 implies

$$\dim_{\mathbb{C}} X_\alpha = 6g - 9 + 4 - 3 = 6g - 8 = \dim_{\mathbb{C}}(X(\pi_1(\Sigma))) - 2.$$

Finally, we suppose that α separates Σ into two surfaces of genus g_1 and g_2, both of which are at least 2. Again applying Proposition 2.4 and Lemma 2.3 gives

$$\dim_{\mathbb{C}} X_\alpha = 6g_1 - 3 + 6g_2 - 3 - 3 = 6g - 9 = \dim_{\mathbb{C}}(X(\pi_1(\Sigma))) - 3.$$

□

Corollary 2.7 *Let Σ be a closed orientable surface of genus at least 3. Let Y be the subset of $X(\pi_1\Sigma)$ consisting of characters of representations which kill some essential simple closed curve in Σ. Then Y is a countable union of subvarieties of complex codimension at least 2.*

3 Non-faithful representations which kill no simple loop

In this section we combine the analysis in the last section with a lemma of Culler–Shalen to show that the answer to Minsky's question is "no."

3.1 Trace 2 and reducibility

Recall that a representation $\rho : G \to SL(2,\mathbb{C})$ is *reducible* if there is a proper invariant subspace for the action on $\mathbb{C}^2$. This is equivalent to there being a common eigenvector, and to the representation being conjugate to an upper triangular one. The following is well known (see for example [3, 1.5.5]).

Lemma 3.1 *Suppose that ρ is a representation into $SL(2,\mathbb{C})$ of a free group of rank 2 generated by α and β. Then ρ is reducible if and only if* $\operatorname{trace}(\rho[\alpha,\beta]) = +2$.

Proof The only if direction is an easy computation. For the other direction we assume $\operatorname{trace}(\rho[\alpha,\beta]) = +2$. Set $A = \rho(\alpha)$, $B = \rho(\beta)$. The result is clear if $A = \pm I$, so we assume $A \neq \pm I$. First assume that A is not parabolic. Then after a conjugacy we may assume that A fixes 0 and ∞ so that

$$A = \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}, \text{ and } B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A computation shows that

$$\operatorname{trace}(ABA^{-1}B^{-1}) - 2 = -bc(x - x^{-1})^2.$$

This must equal 0. Since $A \neq \pm I$ we get $x \neq \pm 1$ hence $bc = 0$. Thus the image of ρ is either upper or lower triangular; this gives the result in case A is not parabolic. In case A is parabolic we may conjugate A and B so that

$$A = \pm\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A computation shows that

$$\operatorname{trace}(ABA^{-1}B^{-1}) - 2 = c^2x^2.$$

If this quantity is 0 then we must have $c = 0$ since $A \neq \pm I$. Thus A and C are both upper triangular and the result follows. This completes the proof. □

Corollary 3.2 *Suppose that ρ is a representation of the fundamental group, G, of a surface of negative Euler characteristic and that α, β do not generate a cyclic subgroup of G. If* $\operatorname{trace}(\rho([\alpha,\beta])) = 2$ *then ρ is not injective.*

Proof The subgroup $\langle\alpha,\beta\rangle$ of G is free of rank two. On the other hand, by Lemma 3.1 the image is an upper triangular group of 2×2 matrices, hence two-step solvable. In particular (writing x^y for $y^{-1}xy$), the element $\left[\left[\alpha,\alpha^{\beta}\right],\left[\alpha^{\beta^2},\alpha^{\beta^3}\right]\right]$ is in the kernel of ρ. □

3.2 $Z \setminus Y$ is nonempty

In this subsection, as in the introduction, we fix a closed orientable surface Σ of genus $g \geq 3$. We moreover fix choices of α, β in $\pi_1\Sigma$ that are represented by two simple closed curves which intersect once transversally, so that their commutator $C = [\alpha,\beta]$ is also simple. With this notation, we let Z, Y, and E be the sets defined in the introduction. In particular Z is the subset of $X(\pi_1\Sigma)$ consisting of those characters x such that $x([\alpha,\beta]) = +2$, and $Y \subset X(\pi_1\Sigma)$ is composed of characters of representations killing at least one simple closed curve.

Lemma 3.3 *The set Z has complex codimension* 1 *in* $X(\pi_1\Sigma)$.

Proof The regular function $f(x) = x([\alpha, \beta]) - 2$ on $X(\pi_1\Sigma)$ vanishes at the character of the trivial representation, so $Z \subset X(\pi_1\Sigma)$ is nonempty. Since $f(x) \neq 0$ when x is the character of a Fuchsian representation, f is not identically zero on $X(\pi_1\Sigma)$. Since $X(\pi_1\Sigma)$ is irreducible (Proposition 2.4), the set Z has complex codimension 1 in $X(\pi_1\Sigma)$. □

Corollary 2.7 states that Y has complex codimension at least 2 in $X(\pi_1\Sigma)$. Combined with Lemma 3.3 and Corollary 3.2 we obtain the following, which already gives a negative answer to Minsky's question.

Proposition 3.4 *The set $Z \setminus Y$ is not empty. Every representation whose character is in this set is not faithful and kills no simple closed curve.*

In Sect. 7 we show that $Z \setminus Y$ contains characters of representations without elliptics, assuming the genus of Σ is at least 4.

4 Smooth points of character varieties: A theorem of Goldman

In this section we show that the character of an irreducible representation of a (possibly punctured) surface group into $SL(2, \mathbb{C})$ is a smooth point of the character variety. Although the character variety is not necessarily an irreducible algebraic set, the natural notions of *smooth point* still coincide; see the Appendix, Lemma 8.2. We will use the following lemma to show that irreducibility of a free group representation is detected by a rank-two free factor of a particular form.

Lemma 4.1 (detecting irreducibility) *Suppose $\mathcal{S} \subset SL(2, \mathbb{C})$ generates a group Γ which has no common fixed point in $\hat{\mathbb{C}}$. Then there is $C \in \mathcal{S}$ such that* $\mathrm{tr}([C, D]) \neq 2$ *and either $D \in \mathcal{S}$ or there are $A \neq B \in \mathcal{S} \setminus \{C\}$ and $D = A \cdot B \cdot A$.*

Proof Without loss we may assume $\mathcal{S}$ does not contain $\pm I$, thus every element of $\mathcal{S}$ has at most 2 fixed points. If $C \in \mathcal{S}$ has a unique fixed point $z \in \hat{\mathbb{C}}$ then since Γ has no common fixed point there is some $D \in \mathcal{S}$ such that D does not fix z and C, D have the required property. So we reduce to the case that every element of $\mathcal{S}$ fixes exactly two points in $\hat{\mathbb{C}}$.

We regard two elements of $\mathcal{S}$ as equivalent if they have the same fixed points. If there are two elements in $\mathcal{S}$ with no fixed point in common then we are done. Thus we may assume every pair of equivalence classes has one fixed point in common. Since there is no point fixed by every element of $\mathcal{S}$ the only remaining case is that there are exactly three equivalence classes from which we choose representatives A, B, C and points $a, b, c \in \hat{\mathbb{C}}$ such that A fixes b, c and B fixes c, a and C fixes a, b.

We first claim that at least one of AB, BC, or CA does not have order 2 in $PSL(2, \mathbb{C})$. Note that a matrix in $SL(2, \mathbb{C})$ represents an element of order 2 in $PSL(2, \mathbb{C})$ if and only if its trace is zero. We conjugate so that $a = 1$, $b = 0$ and $c = \infty$. Then

$$A = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \quad B = \begin{pmatrix} q & q^{-1} - q \\ 0 & q^{-1} \end{pmatrix} \quad C = \begin{pmatrix} r & 0 \\ r - r^{-1} & r^{-1} \end{pmatrix}$$

and $p, q, r \notin \{-1, 0, 1\}$. Assuming that AB, BC, and CA are all order 2 in $PSL(2, \mathbb{C})$, we discover by computation that

$$p^2 = -1/q^2, \quad q^2 = -r^2, \quad \text{and} \quad r^2 = -1/p^2.$$

We deduce that $p^2 = -p^2$, and so $p = 0$, a contradiction.

We can cyclically permute A, B, and C, if necessary, so that AB does not have order 2 in $PSL(2, \mathbb{C})$.

Finally, we argue that if AB does not have order 2 in $PSL(2, \mathbb{C})$, then C and $D = ABA$ have no fixed point in common and therefore give the required pair of elements. We compute

$$ABA = \begin{pmatrix} p^2 q & q^{-1} - q \\ 0 & p^{-2}q^{-1} \end{pmatrix}.$$

From this one see that ABA does not fix 0 and that it fixes 1 if and only if

$$\begin{aligned} & p^2 q + q^{-1} - q = p^{-2}q^{-1} \\ & \iff q(p^2 - 1) + q^{-1}(1 - p^{-2}) = 0 \\ & \iff q(p^2 - 1)(1 + p^{-2}q^{-2}) = 0. \end{aligned}$$

By assumption $q \neq 0$ and $p \neq \pm 1$. It follows that ABA and C have a fixed point in common, namely $z = 1$, if and only if $1 + p^{-2}q^{-2} = 0$. This is equivalent to the condition that $tr(AB) = 0$, which does not hold since AB does not have order 2 in $PSL(2, \mathbb{C})$. This contradiction implies that 1 is not fixed by ABA. □

Suppose $\mathcal{F}$ is a free group of rank $k \geq 2$ and $\mathcal{S} = (\alpha, \beta, \gamma_3, \ldots, \gamma_k)$ is an ordered free generating set. Given a representation $\rho \in R(\mathcal{F})$ define

$$A = \rho(\alpha), \quad B = \rho(\beta) \quad \text{and } C_i = \rho(\gamma_i).$$

The representation ρ is called $\mathcal{S}$*-good* if

$$A = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 \\ c & 1/b \end{pmatrix}, \quad b \neq 0, \pm 1, \quad tr[A, B] \neq 2 \tag{2}$$

and the $\mathcal{S}$*-good representation variety* $R_{\mathcal{S}}(\mathcal{F}) \subset R(\mathcal{F})$ is the set of all such. Note that $\mathrm{tr}[A, B] = abc - ab^{-1}c + c^2 + b^2 + b^{-2}$ so we may identify $R_{\mathcal{S}}(\mathcal{F})$ with the smooth manifold

$$\left\{ (a, b, c, M_3, \ldots, M_k) \in \mathbb{C}^3 \times (SL(2, \mathbb{C}))^{k-2} \;\middle|\; \begin{array}{c} b \notin \{0, \pm 1\} \\ abc - ab^{-1}c + c^2 + b^2 + b^{-2} \neq 2 \end{array} \right\}.$$

Observe that if (e_1, e_2) is the standard ordered basis of $\mathbb{C}^2$ and ρ is $\mathcal{S}$–good, then e_2 is an eigenvector of B that is not an eigenvector of A and $e_1 = Ae_2$. Conversely, if $\rho \in R(\mathcal{F})$ and $\mathrm{tr}([A, B]) \neq 2$ and $\mathrm{tr}(B) \neq \pm 2$ then ρ is irreducible by 3.1 and B has two distinct eigenvectors. Since ρ is irreducible, each eigenvector e_2 of B is not an eigenvector of A. Thus there are *two* distinct choices of ordered basis (Ae_2, e_2) and therefore at least two distinct conjugates of ρ that are in $R_{\mathcal{S}}(\mathcal{F})$.

Lemma 4.2 *If $\mathcal{F}$ is a finitely generated free group of rank $k \geq 2$ and $\rho \in R(\mathcal{F})$ is irreducible then there is an ordered basis $(\alpha, \beta, \gamma_3, \ldots, \gamma_k)$ of $\mathcal{F}$ and a conjugate ρ' of ρ which is $\mathcal{S}$-good.*

Proof By 4.1 there is an ordered basis $\mathcal{S} = (\alpha', \beta', \gamma_3, \ldots, \gamma_k)$ of $\mathcal{F}$ such that $\mathrm{tr}(\rho[\alpha', \beta']) \neq 2$. Then ρ restricted to the subgroup $\mathcal{G} \subset \mathcal{F}$ generated by α', β' is irreducible and it follows that there is another free basis (α, β) of $\mathcal{G}$ that $\mathrm{tr}(\rho\beta) \neq \pm 2$. By 3.1 it follows that $\mathrm{tr}(\rho[\alpha, \beta]) \neq 2$ since $\rho|\mathcal{G}$ is irreducible. By the above remarks ρ is conjugate to $\rho' \in R_{\mathcal{S}}(\mathcal{F})$. □

The map $X : R(\mathcal{F}) \longrightarrow X(\mathcal{F})$ which sends a representation to its character is smooth, in fact regular. The restriction of this map to $R_{\mathcal{S}}(\mathcal{F})$ is a smooth map denoted $X_{\mathcal{S}} : R_{\mathcal{S}}(\mathcal{F}) \longrightarrow X(\mathcal{F})$. By Lemma 4.2 the image of $R_{\mathcal{S}}$ is the *open* subset $X_{\mathcal{S}}(\mathcal{F})$ of $X(\mathcal{F})$ of all characters x with $x(\beta) \neq \pm 2$ and $x([\alpha, \beta]) \neq 2$. By the remark before 4.2 $X_{\mathcal{S}}$ is at least $2:1$. We show that $R_{\mathcal{S}}$ is a 2–fold cover of $X_{\mathcal{S}}$, using the following lemma about traces of 2×2 matrices, which can be proved by an easy calculation:

Lemma 4.3 *Let $A, B \in SL(2, \mathbb{C})$. If $\mathrm{tr}(ABA^{-1}B^{-1}) \neq 2$, then the linear map $\theta_{A,B}\colon M_2(\mathbb{C}) \to \mathbb{C}^4$ given by*

$$\theta_{A,B}(M) = (\mathrm{tr}(M), \mathrm{tr}(AM), \mathrm{tr}(BM), \mathrm{tr}(ABM))$$

is an isomorphism of vector spaces. Moreover $\psi : [SL(2, \mathbb{C})]^2 \times \mathbb{C}^3 \longrightarrow M_2(\mathbb{C})$ given by $\psi(A, B, \mathbf{z}) = \theta^{-1}_{A,B}(\mathbf{z})$ is smooth.

Lemma 4.4 *$X_{\mathcal{S}} : R_{\mathcal{S}}(\mathcal{F}) \longrightarrow X_{\mathcal{S}}(\mathcal{F})$ is 2-fold covering space and a local diffeomorphism. The image is an open subset of $X(\mathcal{F})$.*

Proof Throughout this proof we use the notation as in the discussion before Lemma 4.2, so

$$tr(A) = a \quad tr(B) = b + b^{-1} \quad tr(AB) = ab + c \tag{3}$$

The map

$$f : \mathbb{C} \times (\mathbb{C} \setminus \{0, \pm 1\}) \times \mathbb{C} \longrightarrow \mathbb{C} \times (\mathbb{C} \setminus \{\pm 2\}) \times \mathbb{C}$$

given by

$$f(a, b, c) = (a, b + b^{-1}, ab + c)$$

is a 2–fold covering and a local diffeomorphism.

It follows that for any $\rho \in R_{\mathcal{S}}(\mathcal{F})$ that $X_{\mathcal{S}}(\rho)$ determines a, b, c and hence (A, B) up to two possibilities. Moreover it follows from Lemma 4.3 that $X_{\mathcal{S}}(\rho)$ and a choice for (A, B) determines each C_i and thus ρ completely. Combining this with the fact $X_{\mathcal{S}}$ is at least $2:1$ shows $X_{\mathcal{S}}$ is everywhere $2:1$ onto its image.

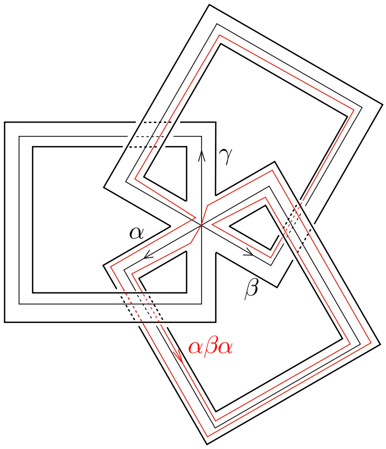
The character variety $X(\mathcal{F})$ is a subset of some affine space $\mathbb{C}^n$ but is not in general a manifold. Recall that a function defined on an subset of affine space is *smooth* if there is some extension to a open neighborhood which is smooth. The local inverse of $X_{\mathcal{S}}$ is smooth because f is a local diffeomorphism and the map ψ of 4.3 is smooth. □

The next result is an immediate consequence of Lemma 4.4 and provides a local section of the character map $X : R(\mathcal{F}) \longrightarrow X(\mathcal{F})$ defined on a neighborhood of the character of an irreducible representation. The image of this section is an open set in some $\mathcal{S}$-good representation variety.

Theorem 4.5 *Suppose $\mathcal{F}$ is a finitely generated free group of rank at least 2 and $x_0 \in X(\mathcal{F})$ is the character of an irreducible representation. Then x_0 is a smooth point of the character variety $X(\mathcal{F})$. Moreover there is a neighborhood $U \subset X(\mathcal{F})$ of x_0 and free generating set $\mathcal{S}$ of $\mathcal{F}$ and an open set $V \subset R_{\mathcal{S}}(\mathcal{F})$ such that $X_V : V \longrightarrow U$ is a diffeomorphism.*

Lemma 4.6 (irreducibility is detected by punctured tori) *Suppose Σ is an orientable surface of genus $g \geq 2$ and $\rho \in R(\pi_1 \Sigma)$ is irreducible. Then there is a once punctured torus $T \subset \Sigma$ such that $\rho|\pi_1 T$ is irreducible. The boundary of T is an essential simple closed curve C and* $\mathrm{trace}(\rho(C)) \neq 2$.

Fig. 1 A twice punctured torus



Proof By 3.1 trace$(\rho(C)) \neq 2$ iff $\rho|\pi_1 T$ is irreducible. The surface Σ can be obtained by suitably identifying opposite sides of a regular polygon, P, with $4g$ sides. Let p be the center of P.

Label the sides of ∂P in cyclic order as $a_1, \ldots, a_{2g}, b_1, \ldots, b_{2g}$ so that a_i is identified to b_i reversing orientation. Let α_i be the loop in Σ based at p which meets ∂P once transversally in the interior of a_i and is represented by a straight line segment in P, oriented toward a_i. We also use α_i for the corresponding element of $\pi_1(\Sigma, p)$. Then $\mathcal{S} = \{\alpha_1, \ldots, \alpha_{2g}\}$ generates $\pi_1\Sigma$ and every pair of distinct α_i intersect once transversally at p.

Apply 4.1 to produce elements γ, δ so that $\mathrm{tr}(\rho[\gamma, \delta]) \neq 2$ and either $\{\gamma, \delta\} \subseteq \mathcal{S}$ or $\gamma \in \mathcal{S}$ and $\delta = \alpha\beta\alpha$ for some $\{\alpha, \beta\} \subseteq \mathcal{S}$. In the first case we may take T to be a regular neighborhood of $\gamma \cup \delta$. In the second case, a regular neighborhood of $\alpha \cup \beta \cup \gamma$ is a twice punctured torus Q, whose fundamental group is free on the generators $\mathcal{S}' = \{\alpha, \beta, \gamma\}$. After permuting these element and replacing some of them by their inverses if necessary, we may assume there is an order 3 automorphism of Q acting as a 3–cycle on $\mathcal{S}'$ (see Fig. 1). However we might no longer have $\delta = \alpha\beta\alpha$.

So we apply 4.1 *again* to $\Gamma = \pi_1(Q)$, with free basis $\mathcal{S}'$. After another cyclic permutation we may now assume $\delta = \alpha\beta\alpha$ or $\beta\alpha\beta$. Figure 1 shows γ and $\alpha\beta\alpha$ are represented by simple closed curves also called γ, δ which intersect once transversely at p. It follows that the interior of a regular neighborhood of $\gamma \cup \delta$ is a punctured torus T with the required property. The same reasoning works when $\delta = \beta\alpha\beta$. $\square$

Theorem 4.7 *Suppose Σ is a closed orientable surface of genus $g \geq 2$ and $x_0 \in X(\pi_1\Sigma)$ is the character of an irreducible representation ρ_0. Then x_0 is a smooth point of $X(\pi_1\Sigma)$.*

Proof This is in Goldman [7], but not formally stated there. The idea is to construct a diffeomorphism from a neighborhood of x_0 in the character variety to a smooth submanifold in the representation variety. This diffeomorphism is a local section of the character map (which is locally a submersion) as in Lemma 4.4.

By Lemma 4.6, there is an embedded punctured torus $T \subset \Sigma$ so that $\rho_0|\pi_1(T)$ is irreducible. Thus

$$x_0(\partial T) = tr(\rho_0(\partial T)) \neq 2.$$

We can choose free generators α_1 and β_1 for $\pi_1(T)$ so that $x_0(\beta_1) \neq \pm 2$ and so the loop represented by $[\alpha_1, \beta_1] \simeq \partial T$ is simple in Σ. We can then choose simple loops $\alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$ in Σ so that

$$w = \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1$$

is the defining relation for $\pi_1(\Sigma)$. Let $\mathcal{F} = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \rangle$ be the free group on these generators, so the surjection $\mathcal{F} \to \pi_1 \Sigma$ induces an inclusion

$$R(\pi_1 \Sigma) \subset R(\mathcal{F}).$$

Precisely, if $c\colon\ R(\mathcal{F}) \to SL(2, \mathbb{C})$ is given by $c(\rho) = \rho(w)$, we have $R(\pi_1 \Sigma) = c^{-1}(I)$. Similarly $X(\pi_1 \Sigma) \subset X(\mathcal{F})$ is the set of characters of representations in $R(\pi_1 \Sigma)$. Using the ordered basis $\mathcal{S} = (\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ of $\mathcal{F}$ define

$$R_{\mathcal{S}}(\pi_1 \Sigma) = R(\pi_1 \Sigma) \cap R_{\mathcal{S}}(\mathcal{F}) \qquad \textit{and} \qquad X_{\mathcal{S}}(\pi_1 \Sigma) = X(\pi_1 \Sigma) \cap X_{\mathcal{S}}(\mathcal{F})$$

Since $X_{\mathcal{S}}(\mathcal{F})$ is open in $X(\mathcal{F})$ it follows that $X_{\mathcal{S}}(\pi_1 \Sigma)$ is open in $X(\pi_1 \Sigma)$. Now $x_0(\beta_1) \neq \pm 2$ and $x_0([\alpha_1, \beta_1]) \neq 2$ and it follows that ρ_0 can be conjugated into $R_{\mathcal{S}}(\pi_1 \Sigma)$ thus $x_0 \in X_{\mathcal{S}}(\pi_1 \Sigma)$. We replace ρ_0 by this conjugate so that $\rho_0 \in R_{\mathcal{S}}(\pi_1 \Sigma)$. The map $X_{\mathcal{S}}$ from Lemma 4.4 restricts to a smooth map

$$X_{\mathcal{S},\Sigma} : R_{\mathcal{S}}(\pi_1 \Sigma) \longrightarrow X_{\mathcal{S}}(\pi_1 \Sigma)$$

This restriction is still a $2:1$ cover and so has a smooth local inverse near x_0. A small open neighborhood of x_0 in $X(\pi_1 \Sigma)$ is contained in $X_{\mathcal{S}}(\pi_1 \Sigma)$. The proof is completed below by showing that ρ_0 is a manifold point in $R_{\mathcal{S}}(\pi_1 \Sigma)$.

Let $c_{\mathcal{S}} : R_{\mathcal{S}}(\mathcal{F}) \longrightarrow SL(2, \mathbb{C})$ be the restriction of c. It suffices to show this is a submersion, as then $c_{\mathcal{S}}^{-1}(I)$ is a smooth submanifold of $R_{\mathcal{S}}$. Define

$$g : \mathbb{C} \times (\mathbb{C} \setminus \{0, \pm 1\}) \times \mathbb{C} \longrightarrow SL(2, \mathbb{C})$$

by

$$\begin{aligned} g(a, b, c) &= \left[\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ c & 1/b \end{pmatrix} \right] \\ &= \begin{pmatrix} b^{-2} - ab^{-1}c + abc + c^2 & a - ab^2 - bc \\ -bc & b^2 \end{pmatrix} = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}. \end{aligned}$$

We show that g is a submersion. It then follows that $c_{\mathcal{S}}$ is a submersion.

Away from $r_1 = 0$, the entries (r_1, r_2, r_3) form a system of local coordinates on $SL(2, \mathbb{C})$; away from $r_2 = 0$, the entries (r_1, r_2, r_4) form a system of local coordinates. Any point in $SL(2, \mathbb{C})$ is in at least one of these coordinate patches. In the first, we have

$$\det \left(\frac{\partial(r_1, r_2, r_3)}{\partial(a, b, c)} \right) = 2(b^{-2} - 1)(1 - abc + ab^3c + b^2c^2) = 2b^2(b^{-2} - 1)r_1. \tag{4}$$

Since $b \neq 0, \pm 1$, the quantity in (4) is nonzero. In the second patch, we have

$$\det \left(\frac{\partial(r_1, r_2, r_4)}{\partial(a, b, c)} \right) = 2(1 - b^2)(-a + ab^2 + bc) = -2(1 - b^2)r_2. \tag{5}$$

Again, since $b \neq \pm 1$, the quantity in (5) is nonzero. It follows that dg has rank three everywhere so g is a submersion. □

An alternate proof can be based on a result of [10] that the conjugation action of $PSL(2, \mathbb{C})$ on the space of irreducible representations is proper and free.

5 Deforming representations of surfaces

Our proof in Sect. 6 that Z is irreducible works by defining an open subset $W \subset Z$ of particularly nice characters, and then showing that W is dense, smooth, and path connected. The results in this section are used to establish these properties of W.

5.1 Smoothness

The first two statements will be used in showing W is smooth.

Lemma 5.1 [8, 4.4] *The commutator map $\mathcal{C} : SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \to SL(2, \mathbb{C})$ given by $\mathcal{C}(A, B) = [A, B]$ is a submersion unless A and B commute.*

We remark that the commutator map is not open everywhere. Indeed, the pre-image of the identity under the commutator map has complex dimension 4 but other points have preimages of dimension 3. (Some 3–dimensional fibers are described explicitly in the proof of 5.8 below.) However, a holomorphic map ϕ between complex manifolds is open if and only if $\dim_{\mathbb{C}}(\phi^{-1}(z))$ is a constant function of z in the target [5, p. 145]. Alternatively one may show by direct computation that the commutator of two matrices which are small deformations of diagonal matrices is either parabolic or has fixed points in $\hat{\mathbb{C}}$ close to $\pm z$ for some $z \neq 0$. Such elements do not give a neighborhood of the identity.

Corollary 5.2 *The map $\chi : SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \to \mathbb{C}$ given by*

$$\chi(A, B) = \mathrm{trace}([A, B])$$

is a submersion unless $[A, B]$ is central.

Proof The trace map from $SL(2, \mathbb{C})$ to $\mathbb{C}$ is a submersion except at $\pm I$. Since the composition of submersions is a submersion, Lemma 5.1 implies the corollary. □

5.2 Genericity

The next two statements are used in showing that W is dense in Z. The first lemma should be contrasted with the genus 1 case, as discussed after Lemma 5.1.

Proposition 5.3 (punctured high genus) *Let S be a once punctured surface of genus $g \geq 2$. Then the restriction map*

$$f : R(\pi_1 S) \longrightarrow R(\pi_1 \partial S)$$

is open.

Proof Consider $\rho \in R(\pi_1 S)$. Suppose first that S contains a punctured torus T such that the restriction $\rho|\pi_1 T$ is nonabelian. Let β be the boundary of T, and let α denote ∂S. There is another subsurface T' of genus $g-1$ with boundary γ so that (connecting loops to basepoints correctly), we have $\alpha = \beta \cdot \gamma$. Notice that $\pi_1 S = \pi_1 T * \pi_1 T'$, so the restrictions of ρ to $\pi_1 T$ and $\pi_1 T'$ can be varied independently. Precisely, if R_T is the algebraic subset of $R(\pi_1 S)$ which agrees with ρ on $\pi_1 T'$, then R_T can be naturally identified with $R(\pi_1 T) \cong (SL(2, \mathbb{C}))^2$. The map $f|R_T$ then factors $f|R_T = L_\gamma \circ \mathcal{C}$, where $\mathcal{C}$ is the submersion from Lemma 5.1, and L_γ is right multiplication by $\rho(\gamma)$. In particular, $f|R_T$ is a submersion, and so f is also a submersion, and therefore open.

It remains to prove the result when the restriction of ρ to every punctured torus is abelian. This implies the image of ρ is abelian. First we consider the case that $tr(\rho(\alpha)) \neq \pm 2$ for some α, so the image of ρ is conjugate to a group of diagonal matrices. We can choose $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ so that $\partial S = \prod_{i=1}^{g}[\alpha_i, \beta_i]$ and none of $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ is mapped by ρ to $\pm I$. This is easy to ensure because the representation is abelian. The result now follows from two calculations. First we show that if $A = diag(p, 1/p)$ and $B = diag(q, 1/q)$ are diagonal matrices with $p, q \neq \pm 1$ there are nearby matrices whose commutator is

$$[A', B'] = \begin{pmatrix} 1 & u \\ v & 1+uv \end{pmatrix} \qquad u, v \text{ sufficiently small}$$

in fact, we can take:

$$A' = \begin{pmatrix} p & pu \\ 0 & \frac{1}{p} \end{pmatrix} \quad B' = \begin{pmatrix} q & \frac{u(1-p^2+p^2q^2-p^2uv)}{(p^2-1)q} \\ \frac{p^2qv}{1-p^2-p^2uv} & \frac{1}{q} - \frac{p^2uv(1-p^2+p^2q^2-p^2uv)}{(p^2-1)q(-1+p^2+p^2uv)} \end{pmatrix}.$$

The computation below shows that every matrix close to the identity is a product of two of these commutators close to the identity (x, y, z are small)

$$C = \begin{pmatrix} 1 & \sqrt{x} \\ \frac{z-\sqrt{x}}{1+x} & \frac{1+z\sqrt{x}}{1+x} \end{pmatrix} \qquad D = \begin{pmatrix} 1 & \frac{y-\sqrt{x}}{1+x} \\ \sqrt{x} & \frac{1+y\sqrt{x}}{1+x} \end{pmatrix} \qquad CD = \begin{pmatrix} 1+x & y \\ z & \frac{1+yz}{1+x} \end{pmatrix}.$$

Since we can obtain any matrix sufficiently close to I in this way, the map ρ is open in this case.

The next case is when $\rho(\alpha) = \pm I$ for every α. The proof is the same, except that for the first calculation we use

$$A' = \pm \begin{pmatrix} 1+a & u+ua \\ 0 & \frac{1}{1+a} \end{pmatrix}$$

$$B' = \pm \begin{pmatrix} 1 & \frac{u(1-(1+a)^2uv)}{a(2+a)} \\ -\frac{(1+a)^2v}{uv+2a(1+uv)+a^2(1+uv)} & 1 + \frac{(1+a)^2uv(-1+(1+a)^2uv)}{a(2+a)(uv+2a(1+uv)+a^2(1+uv))} \end{pmatrix}$$

We choose $|u|, |v| << |a| << 1$.

The last case is when some element is sent to a nontrivial parabolic. In this case the representation can be conjugated to be upper triangular. We can change generating set so that every generator is sent to a nontrivial parabolic. Suppose

$$A = \pm \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \quad B = \pm \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

are parabolic matrices with $p, q \neq 0$. We can change A slightly to

$$A' = \pm \begin{pmatrix} \sqrt{1+u+(v/p)} & q \\ \frac{-u/p}{\sqrt{1+u+(v/p)}} & \frac{-qu+p\sqrt{1+u+(v/p)}}{p+pu+v} \end{pmatrix} \qquad u, v \text{ are small.}$$

so that the commutator is

$$M_p(u,v) := [A', B] = \begin{pmatrix} 1+u & v \\ -\frac{u^2}{p+pu+v} & \frac{p+v-uv}{p+pu+v} \end{pmatrix}.$$

In this commutator we regard u and v as varying and p as fixed. Finally we show every matrix close to the identity is the product of two of these matrices close to the identity, $C = M_p(.,.)$ and $D = M_q(.,.)$, provided $p+q \neq 0$. We may always arrange $p+q \neq 0$ by choice of generating set.

$$\begin{aligned} C &= M_p((a-w+bw^2/q)/(1+w), b+bw) \qquad a, b, w \text{ small.} \\ D &= M_q(w, 0) \\ CD &= \begin{pmatrix} 1+a & b \\ \frac{-a^2q-(b+p+q)w^2+aw(2q-bw)}{(1+a)pq+b(pw^2+q(1+w)^2))} & \frac{pq-b^2w^2+bq(1-a+2w)}{(1+a)pq+b(pw^2+q(1+w)^2)} \end{pmatrix} \end{aligned}$$

It is easy to check that if c is small and $p+q \neq 0$, there is $w = O(\sqrt{|c|}+|a|)$ small so that

$$CD = \begin{pmatrix} 1+a & b \\ c & (1+bc)/(1+a) \end{pmatrix} \qquad a, b, c \text{ small.}$$

□

Lemma 5.4 (Extension Lemma) *Suppose that Σ is a closed surface of genus $g \geq 3$ and $S \subset \Sigma$ is the complement of a once-punctured incompressible subsurface of genus at least 2. If $\rho : \pi_1\Sigma \longrightarrow SL(2,\mathbb{C})$ is given then any sufficiently small deformation of $\rho|\pi_1 S$ can be extended to a small deformation over $\pi_1\Sigma$.*

Proof This follows from 5.3. □

5.3 Paths of representations

The remaining statements in this section will be used to show that W is path-connected.

Definition 5.5 A map $p : X \longrightarrow Y$ has *path-lifting with fixed endpoints* if for every continuous map $\gamma : [0,1] \longrightarrow Y$ and $x_0, x_1 \in X$ with $p(x_i) = \gamma(i)$ there is a continuous lift $\tilde{\gamma} : [0,1] \longrightarrow X$ with $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(i) = x_i$ for $i = 0, 1$.

Proposition 5.6 *If $p : X \longrightarrow Y$ is a surjective submersion of smooth manifolds and every fiber of p is path-connected then p has path-lifting with fixed endpoints.*

Proof Since p is a submersion there is a local product structure near each point in X so that p is given by coordinate projection $U \times V \longrightarrow V$. Since p is also surjective, we may lift paths locally. This means that given a path $\gamma : [0,1] \longrightarrow Y$ there is a finite open cover of $[0,1]$ by intervals $I_1, \dots I_k$ so that for each $n \in \{1, \dots, k\}$:

(1) $0 = \inf I_1 < \inf I_2 < \cdots \inf I_k$ and $\sup I_1 < \cdots \sup I_{k-1} < \sup I_k = 1$;
(2) For each $m \in \{1, \dots, k\}$, $I_m \cap I_n \neq \emptyset$ if and only if $|m-n| \leq 1$; and
(3) $\gamma|I_n$ lifts to $\tilde{\gamma}_n$ with image in a local product neighborhood.

Moreover, we may choose the first and last lifts so that $\tilde{\gamma}_1(0) = x_0$ and $\tilde{\gamma}_k(1) = x_1$, for the specified $x_0 \in p^{-1}(\gamma(0))$ and $x_1 \in p^{-1}(\gamma(1))$.

It suffices to change each $\tilde{\gamma}_n$ near the right-hand end of I_n so that it agrees with γ_{n+1} on $I_n \cap I_{n+1}$ without changing it at the left-hand end.

Fix n and suppose that $\tilde{\gamma}_n(s) \neq \tilde{\gamma}_{n+1}(s)$ for all $s \in I_n \cap I_{n+1}$. Choose $s_0 \in I_n \cap I_{n+1}$ and a smooth embedded path δ in $p^{-1}(\gamma(s_0))$ connecting $\tilde{\gamma}_n(s_0)$ to $\tilde{\gamma}_{n+1}(s_0)$. There is a local product structure near each point on δ, so by compactness there exists $0 = t_0 < t_1 < \cdots < t_k = 1$ so that δ maps $[t_{i-1}, t_i]$ into a local product structure $U_i \times V_i$. We can use the product structure $U_1 \times V_1$ to modify $\tilde{\gamma}_n$ near s_0 to produce a new lift of $\tilde{\gamma}_n$ that takes s_0 to $\delta(t_1)$. Repeating we get the required lift. □

In proving Theorem 5.8 we will make use of the following well known fact:

Lemma 5.7 *The set of points in $\mathbb{C}^n$ where finitely many polynomials are all nonzero is path connected.*

Proof By taking the product of the polynomials we may assume there is a single polynomial. Let $U \subset \mathbb{C}^n$ be the set where the given polynomial p is not zero. Given two distinct points $x, y \in U$ there is an affine line $L \cong \mathbb{C}$ containing them. The restriction $p|L$ is a polynomial in one variable which is nonzero at x and y therefore it has finitely many zeroes. There is a path in L from x to y that avoids these zeroes. □

Theorem 5.8 (commutator path lifting) *The restriction of the commutator map $\mathcal{C}$ to $\mathcal{C}^{-1}(\{M \in SL(2,\mathbb{C}) : trace(M) \neq \pm 2\})$ has path lifting with fixed endpoints.*

Proof By 5.1 $\mathcal{C}$ is a submersion on the given domain, so by 5.6 it suffices to show that for $M \in SL(2,\mathbb{C})$ with $trace(M) \neq \pm 2$ that $\mathcal{C}^{-1}(M)$ is path connected. The number of path components is not changed by conjugating M. Thus we may assume M is in Jordan normal form. For such an M we describe the set of all pairs $(A, B) \in SL(2,\mathbb{C})^2$ so that $[A, B] = M$.

We have

$$M = \begin{pmatrix} m & 0 \\ 0 & 1/m \end{pmatrix} \qquad m \notin \{0, \pm 1\}.$$

Fix a square root $m^{1/2}$ of m. Computation shows that $\mathcal{C}^{-1}(M)$ contains

$$(A_0, B_0) = \left(\begin{pmatrix} m^{1/2} & m-1 \\ 0 & m^{-1/2} \end{pmatrix}, \begin{pmatrix} m^{-1/2} & 0 \\ 1 & m^{1/2} \end{pmatrix} \right).$$

We will show that $\mathcal{C}^{-1}(M)$ is covered by two path-connected sets S_1 and S_2 so that $(A_0, B_0) \in S_1 \cap S_2$. Namely, let

$$S_1 = \left\{ (A, B) \,\middle|\, [A, B] = M, \text{and } B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c \neq 0 \right\},$$

and let

$$S_2 = \left\{ (A, B) \,\middle|\, [A, B] = M, \text{and } B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } a \neq 0 \right\},$$

We will reduce the proof of this to the following claim, and then prove the claim. □

Claim 5.8.1 *The intersection $S_1 \cap S_2$ contains paths connecting (A_0, B_0) to $(\epsilon_A A_0, \epsilon_B B_0)$ for any ϵ_A, ϵ_B in $\{\pm I\}$.*

For fixed $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the equation $[A, B] = M$ implies $AB = MBA$, which is linear in A. Basic linear algebra shows that the solution set to this equation is dimension either 0 or

2, with dimension 2 if and only if $d = am$(equivalently $\operatorname{tr}(B) = \operatorname{tr}(MB)$). In case $c \neq 0$ the general solution is:

$$A = \begin{pmatrix} cmt & bms + am(m-1)t \\ cs & ct \end{pmatrix} \qquad B = \begin{pmatrix} a & b \\ c & am \end{pmatrix}, \tag{6}$$

where s and t vary arbitrarily in $\mathbb{C}$. Two points p_1 and p_2 in S_1 thus can be described by two quintuples $(a, b, c, s, t) \in \mathbb{C}^5$ subject to the conditions $c \neq 0$, $\det A = c^2mt^2 - bcms^2 - acm(m-1)st = 1$, and $\det B = a^2m - bc = 1$. The set of points T_1 in $\mathbb{C}^5$ where the polynomials $\{c, \det A, \det B\}$ are all nonzero is path-connected, by Lemma 5.7. The set T_1 embeds into $GL(2,\mathbb{C})^2$ via Eq. (6), and the path connectedness of T_1 gives a path in $GL(2,\mathbb{C})^2$.

To obtain a path in $SL(2,\mathbb{C})^2$ we multiply the above matrices by the reciprocal of a square root of their determinants. A continuous choice of square root can be made along the path. At the end of the path our choices result in matrices which are the required matrices up to multiplication by -1. Rescaling matrices does not change their commutator, and so gives a path in $\mathcal{C}^{-1}(M)$. It follows that S_1 has at most 4 path components, and we can connect any point in S_1 by a path in $\mathcal{C}^{-1}(M)$ to one of the four points $(\epsilon_A A_0, \epsilon_B B_0)$ for ϵ_A, ϵ_B in $\{\pm I\}$. Claim 5.8.1 then implies that S_1 is connected.

The proof that S_2 is connected is similar. The general solution to $AB = MBA$ can now be described by

$$A = \begin{pmatrix} cs + bmt & a(m-1)s \\ a(m-1)t & \frac{c}{m}s - bt \end{pmatrix} \qquad B = \begin{pmatrix} a & b \\ c & am \end{pmatrix}, \tag{7}$$

where s and t vary arbitrarily in $\mathbb{C}$. Two points in S_2 can thus be described by quintuples $(a, b, c, s, t) \in \mathbb{C}^5$ subject to the conditions $a \neq 0$, $\det A = 1$, and $\det B = 1$. We argue as before that Claim 5.8.1 implies S_2 is path connected.

Proof (Claim 5.8.1) The path

$$(A_\theta, B_\theta) = \left(\begin{pmatrix} m^{1/2} & (m-1) \\ 0 & m^{-1/2} \end{pmatrix}, \begin{pmatrix} e^{i\theta}m^{-1/2} & e^{i\theta} - e^{-i\theta} \\ e^{i\theta} & e^{i\theta}m^{1/2} \end{pmatrix} \right), \theta \in [0, \pi]$$

connects (A_0, B_0) to $(A_0, -B_0)$. To connect $(A_0, -B_0)$ to $(-A_0, -B_0)$ use

$$(A_\theta, B_\theta) = \left(\begin{pmatrix} e^{i\theta}m^{1/2} & e^{i\theta}(m-1) \\ \frac{e^{i\theta} - e^{-i\theta}}{m-1} & e^{i\theta}m^{-1/2} \end{pmatrix}, \begin{pmatrix} -m^{-1/2} & 0 \\ -1 & -m^{1/2} \end{pmatrix} \right), \theta \in [0, \pi].$$

Finally, the path

$$(A_\theta, B_\theta) = \left(\begin{pmatrix} e^{i\theta}m^{1/2} & e^{i\theta}(m-1) \\ \frac{e^{i\theta} - e^{-i\theta}}{m-1} & e^{i\theta}m^{-1/2} \end{pmatrix}, \begin{pmatrix} m^{-1/2} & 0 \\ 1 & m^{1/2} \end{pmatrix} \right), \theta \in [0, \pi]$$

connects (A_0, B_0) to $(-A_0, B_0)$. One can verify by computation or by examining (6) and (7) that these paths lie in $S_1 \cap S_2$. $\square$

Lemma 5.9 *Let ρ_0 and ρ_1 be representations of $\mathcal{F}_2 = \langle \alpha, \beta \rangle$ into the solvable group*

$$S = \left\{ \begin{pmatrix} z & w \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^*, w \in \mathbb{C} \right\}.$$

There is then a path ρ_t of reducible representations of $\mathcal{F}_2$ into S joining ρ_0 to ρ_1, and satisfying, for all $t \in (0, 1)$:

(1) *ρ_t has nonabelian image, and*
(2) *neither $\rho_t(\alpha)$ or $\rho_t(\beta)$ has trace ± 2.*

Proof For each $i \in \{0, 1\}$, define λ_i, μ_i, d_i and c_i by

$$\rho_i(\alpha) = \begin{pmatrix} \lambda_i & d_i \\ 0 & \lambda_i^{-1} \end{pmatrix}, \rho_i(\beta) = \begin{pmatrix} \mu_i & e_i \\ 0 & \mu_i^{-1} \end{pmatrix}$$

First we choose paths λ_t from λ_0 to λ_1 in $\mathbb{C}^*$ and μ_t from μ_0 to μ_1 in $\mathbb{C}$ so that λ_t and μ_t do not intersect $\{-1, 0, 1\}$ at any point in their interiors. Now choose a path d_t from d_0 to d_1 so that $d_t \neq 0$ for $0 \in (0, 1)$. We need to choose a path e_t from e_0 to e_1 so that the commutator of

$$\rho_t(\alpha) = \begin{pmatrix} \lambda_t & d_t \\ 0 & \lambda_t^{-1} \end{pmatrix}$$

with

$$\rho_t(\beta) = \begin{pmatrix} \mu_t & e_t \\ 0 & \mu_t^{-1} \end{pmatrix}$$

is nontrivial for all $t \in (0, 1)$. A quick computation shows that the commutator is nontrivial if and only if

$$e_t \mu_t (\lambda_t^2 - 1) - d_t \lambda_t (\mu_t^2 - 1) \neq 0;$$

in other words, for $t \in (0, 1)$ we need

$$e_t \neq g(t) = \frac{d_t \lambda_t (\mu_t^2 - 1)}{\mu_t (\lambda_t^2 - 1)}.$$

Now $g(t)$ is some path in $\mathbb{C}$, and it is easy to see that a path e_t can be found from e_0 to e_1 so that e_t and $g(t)$ are distinct for all $t \in (0, 1)$. □

6 Irreducibility

The next theorem is the chief technical result we need.

Theorem 6.1 *Suppose Σ is a closed orientable surface of genus $g \geq 4$ and C is a simple closed curve in Σ which bounds a punctured torus in Σ. Let Z denote the set of characters of representations $\rho : \pi_1 \Sigma \to SL(2, \mathbb{C})$ for which $trace(\rho(C)) = 2$. Then Z is an irreducible affine variety.*

Proof Clearly Z is an affine algebraic subset of $X = X(\pi_1 \Sigma)$. We will construct a path-connected, dense, open subset, W, of the smooth part of Z. Theorem 8.4 then implies that Z is irreducible.

We choose a simple closed curve C' disjoint from C so that $C \cup C'$ separates Σ into three connected components whose closures are F_1, F_2, F_3 as shown in the diagram as shown in Fig. 2. They are labelled so that $F_1 \cap F_2 = C$ and $F_2 \cap F_3 = C'$ and F_1 is disjoint from F_3. The surfaces F_1 and F_3 are genus 1 and $k = genus(F_2) = genus(\Sigma) - 2 \geq 2$.

We choose standard generators for $\pi_1 \Sigma$ given by loops that can be freely homotoped to be disjoint from C and C'. We will not be careful with basepoints; the diligent reader may fill in the details. We choose loops $\alpha_1, \beta_1 \subset F_1$ and $\alpha_2, \beta_2, \ldots, \alpha_{k+1}, \beta_{k+1} \subset F_2$

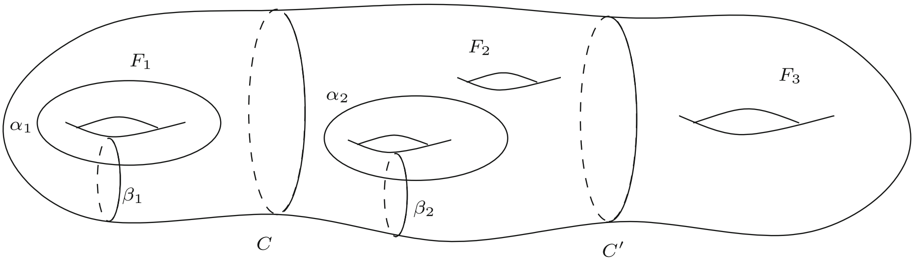


Fig. 2 The surface Σ, cut into pieces

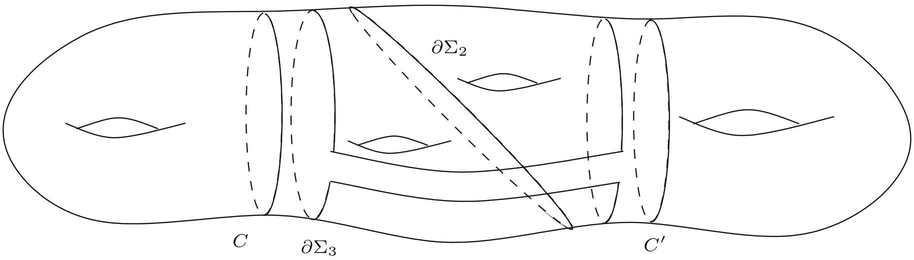


Fig. 3 Subsurfaces used in Claim 6.1.1

and $\alpha_{k+2}, \beta_{k+2} \subset F_3$ which gives a generating set for $\pi_1\Sigma$. This is done so that $C = [\alpha_1, \beta_1], \alpha_2, \beta_2, \ldots, \alpha_{k+1}, \beta_{k+1} \subset F_2$ is a basis for the free group $\pi_1 F_2$. We also arrange that $C' = [\alpha_{k+2}, \beta_{k+2}]$.

Define W to be the subset of $X(\pi_1\Sigma)$ consisting of all characters x satisfying the following conditions:

(W-1) $x(C) = 2$
(W-2) $x(\beta_1) \neq \pm 2$
(W-3) $x([\alpha_2, \beta_2]) \neq \pm 2$
(W-4) $x([C, \alpha_2]) \neq 2$
(W-5) $x(C') \neq \pm 2$

Condition (W-1) is equivalent to the statement $W \subset Z$. Conditions (W-2) and [(W-3) with Lemma 5.1 imply certain transversality results. Condition (W-4) implies $\rho C \neq \pm I$.

It is clear that W is an open subset of Z in both the classical and Zariski topologies. We will show that W is a path connected, dense subset of the smooth part of Z. This will prove the theorem. □

Claim 6.1.1 *W is dense in Z.*

Proof of Claim. Suppose ρ is a representation whose character x is in Z. Condition (W-1) and Lemma 3.1 imply the restriction of ρ to the free group generated by α_1, β_1 is reducible. Thus we may assume $\rho|\langle\alpha_1, \beta_1\rangle$ is upper triangular. We can change $\rho|\langle\alpha_1, \beta_1\rangle$ a small amount, keeping it upper triangular, so that condition (W-2) holds and ρC is a nontrivial parabolic with fixed point at ∞. We now use the Extension Lemma 5.4 to extend this change of ρ to a small change over the rest of the surface.

Now we make further small changes to ensure conditions (W-3) to (W-5) hold. There is a genus 2 surface $\Sigma_2 \subset \Sigma$ containing $\alpha_1, \beta_1, \alpha_2$, and β_2 (bounded by the diagonally oriented curve in Fig. 3).

The fundamental group of Σ_2 is freely generated by $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$, so we can deform the representation on this free subgroup holding $\rho|\langle\alpha_1, \beta_1\rangle$ fixed, but changing $\rho|\langle\alpha_2, \beta_2\rangle$ by an arbitrarily small amount, and ensuring that conditions (W-3) and (W-4) hold. Achieving (W-4) is possible since we have already ensured $\rho C \neq \pm I$. The Extension Lemma 5.4 applied to this deformation tells us we can extend this deformation to all of $\pi_1\Sigma$. Since $\rho|\langle\alpha_1, \beta_1\rangle$ is fixed during this deformation, conditions (W-1) and (W-2) are preserved.

Next we perform a small deformation to ensure condition (W-5) holds. There is another embedded genus 2 surface with one boundary component $\Sigma_3 \subset \Sigma$ whose fundamental group is freely generated by $\{\alpha_1, \beta_1, \alpha_{k+2}, \beta_{k+2}\}$. (See Fig. 3.) We can make an arbitrarily small deformation of $\rho|\pi_1\Sigma_3$ holding $\rho|\pi_1 F_1$ fixed, and so that trace$(\rho C') \neq \pm 2$. Applying the Extension Lemma 5.4, this deformation again extends to all of $\pi_1\Sigma$. Since $\rho|\pi_1 F_1$ is fixed, conditions (W-1) and (W-2) are undisturbed; since conditions (W-3) and (W-4) are open, they will still hold for sufficiently small deformations which ensure (W-5). □

Claim 6.1.2 *W is path connected.*

Proof of Claim Choose representations ρ_0, ρ_1 with characters x_0 and x_1.By Condition (W-1) and Lemma 3.1, we may assume ρ_0 and ρ_1 restrict to upper triangular representations of $\langle\alpha_1, \beta_1\rangle$. We will construct a path ρ_t of representations in W with these endpoints. We construct the path ρ_t by successively extending the definition of ρ_t over $\pi_1 F_1$ then $\pi_1 F_2$ and finally $\pi_1 F_3$.

First we define $\rho_t|\pi_1 F_1$ using Lemma 5.9 so that $\rho_t|\pi_1 F_1$ is reducible but nonabelian for every t.

Next we need to extend ρ_t over $\pi_1 F_2$. This must be done in such a way that conditions (W-3) , (W-4) and (W-5) hold on the interior of the path. We have a path of representations defined on

$$\pi_1(\Sigma) \times \{0, 1\} \cup \pi_1 F_1 \times (0, 1)$$

and wish to extend over $\pi_1(F_1 \cup F_2) \times [0, 1]$. That this can be done follows by noticing that C, α_2, β_2 is part of a basis of the free group $\pi_1 F_2$, and $C = F_1 \cap F_2$. For each element γ of the basis of $\pi_1 F_2$ we can choose any path in $SL(2, \mathbb{C})$ from $\rho_0(\gamma)$ to $\rho_1(\gamma)$ to get a representation. We first make sure to choose $\rho_t(\alpha_2)$ so that condition (W-4) holds for $t \in (0, 1)$. Geometrically, we do this by making sure that $\rho_t(\alpha_2)$ always moves the fixed point, ∞, of the parabolic $\rho_t([\alpha_1, \beta_1])$. Algebraically, this amounts to choosing a path

$$\rho_t(\alpha_2) = \begin{pmatrix} a_{11,t} & a_{12,t} \\ a_{21,t} & a_{22,t} \end{pmatrix}$$

from $\rho_0(\alpha_2)$ to $\rho_1(\alpha_2)$ so that $a_{21,t} \neq 0$ for $t \in (0, 1)$. Having done so, we can then choose a path $\rho_t(\beta_2)$ from $\rho_0(\beta_2)$ to $\rho_1(\beta_2)$ so that $\rho_t([\alpha_2, \beta_2]) \neq \pm 2$ when $0 < t < 1$. This ensures condition (W-3) holds on the interior of the path. We can extend the representation over the rest of $\pi_1 F_2$ so condition (W-5) holds. This is easy to do because we are free to deform α_3, β_3 in any way.

Condition (W-5) and Theorem 5.8 allow us to extend ρ_t over $\pi_1 F_3$ compatible with $\rho_t(C')$. We have defined ρ_t on all of $\pi_1\Sigma$ and the character of ρ_t satisfies condition (W-1) , (W-3) and (W-3) on the interior of the path. This proves Claim 6.1.2. □

It only remains to show that W is contained in the smooth part of Z. By Theorem 4.7 the smooth part $X_s(\pi_1\Sigma)$ of $X(\pi_1\Sigma)$ contains the set of characters of irreducible representations. Condition (W-3) implies $W \subset X_s(\pi_1\Sigma)$. We show that W is a codimension-1 smooth submanifold of $X_s(\pi_1\Sigma)$ by showing the map $P : X(\pi_1\Sigma) \to \mathbb{C}$ given by $P(x) = x(C)$ is a submersion along W.

Fix $x_0 \in W$ and$\rho_0 \in R(\pi_1\Sigma)$ so that $[\rho_0] = x_0$. Let G denote the subgroup of $\pi_1\Sigma$ generated by $\{\alpha_3, \beta_3 \cdots, \alpha_{k+2}, \beta_{k+2}\}$. Let $R_G \subset R(\pi_1\Sigma)$ denote those representations σ such that $\sigma|G = \rho_0|G$.

Let $C'' = [\alpha_1, \beta_1][\alpha_2, \beta_2]$. The map

$$\text{res}\colon\ \sigma \mapsto (\sigma(\alpha_1), \sigma(\beta_1), \sigma(\alpha_2), \sigma(\beta_2))$$

sends R_G homeomorphically to a subset L of $(SL(2,\mathbb{C}))^4$:

$$L = \left\{(A_1, B_1, A_2, B_2) \in (SL(2,\mathbb{C}))^4 \mid [A_1, B_1][A_2, B_2] = \rho_0(C'')\right\}.$$

Claim 6.1.3 *The restriction map $\phi\colon\ R_G \to R(\langle\alpha_1, \beta_1\rangle)$ is a submersion atρ_0.*

Proof Using condition (W-3) and Lemma 5.1 it follows that the map $\psi\colon\ (SL(2,\mathbb{C}))^4 \to (SL(2,\mathbb{C}))^3$ given by

$$\psi(A_1, B_1, A_2, B_2) = (A_1, B_1, [A_1, B_1][A_2, B_2])$$

is a submersion at ρ_0. Hence ϕ, which may be regarded as the restriction of ψ to $L = \psi^{-1}((SL(2,\mathbb{C}))^2 \times \rho_0\,(C''))$, is a submersion at ρ_0.

By condition (W-4) the commutator of the matrices $A_1 = \rho_0(\alpha_1)$ and $B_1 = \rho_0(\beta_1)$ is not central. By Corollary 5.2, the map $\chi : R(\langle\alpha_1, \beta_1\rangle) \longrightarrow \mathbb{C}$ given by $\chi(A_1, B_1) = \text{trace}([A_1, B_1])$ is a submersion at $\phi(\rho_0)$. Since ϕ is also a submersion, so is the composition $\chi \circ \phi$. This map factors through the restriction of P. Therefore P is a submersion at $x(\rho_0)$. This completes the proof that W is smooth. □

7 Avoiding real traces

In this section, we assume the genus of Σ is at least 4.

Lemma 7.1 *Suppose $\alpha \in \pi_1\Sigma$ then*

(1) *If $x(\alpha)$ is constant on Z then $x(\alpha) = 2$.*
(2) *If $x(\alpha)$ is not constant on Z then the subset of Z on which it is real has real codimension* 1.

Proof The trivial representation gives a point in Z and at this point $x(\alpha) = 2$. By 6.1 Z is irreducible, hence it is connected. Thus if $x(\alpha)$ is constant on Z then it equals 2. If $x(\alpha)$ is not constant then at every point in the smooth part of Z it is a non-constant polynomial. Therefore the subset of the smooth part of Z on which it is real has real codimension 1. The singular part of Z has complex codimension 1 and the result follows. □

We now can prove Theorem 1.2, which implies Theorem 1.1 as explained in the introduction. Recall that Σ is a closed orientable surface of genus g; the subsets Z, Y, and E of the character variety are described in the introduction.

Proof of Theorem 1.2 The fact that representations whose character lies in Z are non-injective follows from Corollary 3.2.

The statement about Y is Corollary 2.7.

Finally, we suppose that the genus of Σ is at least 4, and describe E. For $\gamma \in \pi_1\Sigma$, let $E_\gamma = \{x \in Z \mid x(\gamma) \in \mathbb{R} \setminus \{2\}\}$. Lemma 7.1 implies that E_γ is either empty or has real codimension 1, so $E = \bigcup_{\gamma\in\pi_1\Sigma} E_\gamma$ is a countable union of subsets of Z of real codimension at least one. The theorem is proved. □

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8 Appendix: Algebraic geometry in $\mathbb{C}^n$

General references for this section are chapter 1 of [15] and chapter 2 of [19]. A nonempty subset $V = V(S) \subset \mathbb{C}^n$ is an *(affine) algebraic set* if it is the set of common zeroes of a collection $S \subset \mathbb{C}[\mathbb{C}^n]$ of polynomial functions on $\mathbb{C}^n$. This set is *reducible* if $V = A \cup B$ with A and B nonempty algebraic sets and $A \neq V \neq B$. Otherwise V is *irreducible* and called an *(affine algebraic) variety*. A *regular map* between algebraic sets is the restriction of a rational map defined on a subset of affine space that contains the domain. A *regular isomorphism* is bijective regular map.

In this appendix we state two results we need which relate the smooth topology and the algebraic properties of algebraic sets. Although these follow easily from well-known results, we have not been able to locate these exact statements in the literature. In what follows we use the classical (Euclidean) topology.

Every algebraic set V has a *decomposition into varieties*: $V = V_1 \cup \cdots \cup V_k$ with each V_i a variety and $V_i \nsubseteq V_j$ whenever $i \neq j$. Moreover this decomposition is unique up to re-ordering.

The set of all polynomials which vanish on V is an ideal $I = I(V)$ in $\mathbb{C}[\mathbb{C}^n]$ and V is irreducible iff I is prime. More generally $I(V) = \prod I(V_i)$ where the product is over the decomposition of V into varieties. If V is irreducible and $f \in \mathbb{C}[\mathbb{C}^n]$ is a polynomial which is zero on an open set in V then $f \in I(V)$. Thus if W is algebraic and contains an open subset of the variety V then W contains V.

The *Zariski tangent space* of V at p is $T_p^Z V = \cap \ker_{f \in I} d_p f \subset \mathbb{C}^n$ with complex dimension $d(p)$. It is easy to see that the subset of V with $d(p) \geq r$ is algebraic. If V is irreducible the *(topological) dimension*, $\dim V$, of V is twice the minimum of this function and in general $\dim(V_1 \cup \cdots \cup V_k) = \max \dim(V_i)$.

Definition 8.1 Let $V \subset \mathbb{C}^n$ be an algebraic set. The point $p \in V$ is a *smooth point* of V if there is a neighborhood U of p in V such that U is a smooth submanifold of $\mathbb{C}^n$ with dimension $\dim V$. Following Shafarevich (section 1.4) it is a *nonsingular point* if the real dimension of the complex vector space $T_p^Z V$ is $\dim V$ and otherwise is *singular*.

It is easy to check that a nonsingular point is a smooth point. The set $\Sigma(V) \subset V$ of singular points is an algebraic set of smaller dimension than V. The *nonsingular part* of V is $V^s = V \setminus \Sigma(V)$ and is a smooth manifold of dimension $\dim V$ with finitely many components, and is open in V. It follows that an algebraic set is the disjoint union of finitely many smooth connected submanifolds of even dimensions.

The next result says the notions of nonsingular and smooth points coincide. This is well known for varieties. However we will use it to prove certain algebraic subsets are varieties.

Lemma 8.2 *Let $V \subset \mathbb{C}^n$. Let $V = V_1 \cup \cdots \cup V_k$ be the decomposition into varieties. For $p \in V$ the following are equivalent*

(1) *p is a smooth point of V.*
(2) *p is nonsingular point of V.*

(3) *($\exists!\ i$ with $p \in V_i$), and p_i is nonsingular point of V_i, and* $\dim V_i = \dim V$.

Proof (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is clear. For (1) $\Rightarrow$ (3), without loss of generality, we may assume $p \in V_i$ for all i.Milnor shows [13, p. 13] that if p is a smooth point of a variety V then p is a nonsingular point of V. It only remains to show $k = 1$.

Let $U \subset V$ be a connected open smooth manifold of dimension $\dim V$ that contains p. Since $\Sigma(V)$ is codimension 2 in V it follows that $W = U \setminus \Sigma(V)$ is connected. Define $W_i = V_i \cap W$. If $\dim V_i < \dim V$ then $V_i \subset \Sigma(V)$ and $W_i = \phi$. Otherwise $W_i \neq \phi$ implies $\dim W_i = \dim V_i = \dim V$. Then $\Sigma(V_i) \subset V_i \cap \Sigma(V)$ and W_i is open in W. But $W = \cup W_i$ is connected so if more than one of these sets in not empty then for some $j \neq k$ then $W_j \cap W_k \neq \phi$. This implies $V_j \cap V_k$ is a nonempty algebraic subset of codimension-0 in both V_j and V_k. Irreducibility implies $V_j = V_k$ a contradiction. Thus $W = W_1$ and $\dim V_j < \dim V$ for all $j \geq 2$. Since $W \subset V_1$ is dense in U and V_1 is closed it follows that $U \subset V_1$. By Milnor p is a nonsingular point of V_1. But $V_i \cap V_1$ contains a neighborhood of p in V_i. Since V_i is irreducible this implies $V_i \subset V_1$ a contradiction unless $k = 1$. □

Proposition 8.3 *If $V \subset \mathbb{C}^n$ is a variety and $W \subset V$ is an algebraic set then $V \setminus W$ is connected in the classical topology.*

Proof By homogenization of polynomials we obtain a projective variety $X = \overline{V} \subset \mathbb{C}P^n$ such that $V = X \setminus \mathbb{C}P^{n-1}$. Then $Y = W \cup \mathbb{C}P^{n-1}$ is an algebraic set and $V \setminus W = X \setminus Y$. Corollary (4.16) on page 68of [15] states that if $X \subset \mathbb{C}P^n$ is a projective variety and $Y \subsetneq X$ is a (closed) algebraic subset then $X \setminus Y$ is connected in the classical topology, thus so is $V \setminus W$. □

Theorem 8.4 *Suppose $V \subset \mathbb{C}^n$ is an algebraic subset. Then V is a variety if and only if V contains a connected, dense, open, subset of smooth points.*

Proof First assume that V is a variety. Then $V^s = V \setminus \Sigma(V)$ is connected by 8.3. Now V^s is open in V, and since V is a variety, this implies it is dense in V.

For the converse, suppose $U \subset V$ is a connected, dense, open subset of smooth points. Then U is a smooth submanifold of $\mathbb{C}^n$. Let $V = V_1 \cup \cdots \cup V_k$ be the decomposition into varieties. Since $U \subset V$ is dense $U_i = U \cap V_i \neq \phi$. By 8.2 $U_i \subset V_i^s$. Since U is open in V it follows that U_i is open in V_i^s. But V_i^s is a manifold so U_i ismanifold. By invariance of domain U_i is also open in U. By 8.2 the U_i are pairwise disjoint. Since U is the disjoint union of the open nonempty sets U_i, and U is connected, it follows that $k = 1$ and V is a variety. □

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