The Area of Convex Projective Surfaces and Fock-Goncharov Coordinates

Ilesanmi Adeboye and Daryl Cooper

Abstract. The area of a convex projective surface of genus $g \geq 2$ is at least $(g-1)\pi^2/2 + \|\tau\|^2/8$ where $\tau = (\log t_i)$ is the vector of triangle invariants of Bonahon-Dreyer and t_i are the Fock-Goncharov triangle coordinates.

Mathematics Subject Classification (2010). Primary 57M50.

Keywords. Convex projective surfaces, Shape parameter, Volume.

A *convex projective surface* is $F = \Omega/\Gamma$ where $\Omega \subset \mathbb{R}P^2$ is the interior of a compact convex set disjoint from some projective line, and $\Gamma \subset \mathrm{PGL}(3, \mathbb{R})$ is a discrete subgroup that preserves Ω and acts freely on it. An example is a hyperbolic surface. Let $\mathbb{R}P(S)$ be the space of marked convex real-projective structures on a closed orientable surface S. Goldman and Choi [8] showed that this can by identified with the Hitchin component of $\mathrm{Hom}(\pi_1 S, \mathrm{PGL}(3, \mathbb{R}))/\mathrm{PGL}(3, \mathbb{R})$. Bonahon and Dreyer [7] showed:

Theorem 0.1. *Suppose S is a closed orientable surface of genus $g \geq 2$. There are real analytic maps, the* triangle invariant $\tau : \mathbb{R}P(S) \to \mathbb{R}^{4g-4}$, *and the* shear invariant $\sigma : \mathbb{R}P(S) \to \mathbb{R}^{12g-10}$, *so that $\beta = (\tau, \sigma) : \mathbb{R}P(S) \to \mathbb{R}^{16g-14}$ is a real-analytic parameterization. The image of β is the an open cone defined by 2 linear inequalities and 2 linear equalities. The image of τ is open in a codimension-1 subspace.*

The Hilbert metric d_Ω on a bounded convex set $\Omega \subset \mathbb{R}^2$ is a Finsler metric normalized so that when Ω is the interior of the unit disc the Hilbert metric is the hyperbolic metric (curvature -1). A Finsler metric determines a measure called *p-area* or just *area* that is the largest measure in the same measure class as Lebesgue so that the area of an infinitesimal parallelogram is at most the product of the side lengths, see Definition 1.1. In general this is *not Hausdorff measure* [4] but coincides with it for Riemannian metrics. For example the unit ball in the sup-norm is a square of side-length 2 which has p-area equal to the Euclidean area 4, and in the taxicab norm the unit

.

ball has p-area 2 but the Hausdorff measure of the unit ball of a norm is always π. When applied to the Hilbert metric we call p-area the *Hilbert area* or just *area*. In the positive quadrant the Hilbert area form is $dxdy/(4xy)$, see Lemma 2.4.

An ideal triangle in Ω has a *shape parameter* $t > 0$. This is one of the coordinates introduced by Fock and Goncharov [13]. The *triangle invariant* in the Bonahon-Dreyer theorem for this triangle is $\tau = \log t$ which is a certain *signed distance*, see Figures (2) and (4). Our main result is:

Theorem 0.2. *Suppose $\mathcal{T}$ is an ideal triangulation of a surface F. The map $\alpha_{\mathcal{T}} : \mathbb{R}P(S) \to \mathbb{R}$ defined by*

$$\alpha_{\mathcal{T}}(F) = (g-1)\pi^2/2 + \|\tau\|^2/8$$

satisfies area$(F) \geq \alpha(F)$. *Here $\tau = (\tau_1, \cdots, \tau_{4g-4})$ are the components of τ and $\|\tau\|^2 = \sum \tau_i^2$.*

This follows from:

Proposition 0.3. *If T is an ideal triangle with shape parameter t in a properly convex domain Ω then*

$$area_{\Omega}(T) \geq (\pi^2 + (\log t)^2)/8$$

with equality iff Ω is the interior of a triangle.

The area of a compact hyperbolic surface F is $2\pi|\chi(F)|$ and in particular is bounded below. A corollary of the above gives a *lower bound* on the area of a convex projective surface by using an ideal triangulation of the surface.

Theorem 0.4. *If F is a compact, properly convex projective surface then*

$$\text{area}(F) \geq (\pi/2)^2 \cdot |\chi(F)|.$$

It follows that if Q is a compact 2-orbifold with $\chi^{orb}(Q) < 0$ then every convex projective structure on Q has area at least $(\pi/2)^2 \cdot |\chi^{orb}(Q)|$. In particular this area is bounded below by $\pi^2/168$. The first author has given lower bounds on the volumes of *hyperbolic orbifolds* [1]; and with *Guofang Wei* see also [2] and for *complex hyperbolic orbifolds* [3].

With the above normalization the Hilbert metric on the interior of the unit disc equals the hyperbolic metric. The above lower bound is $\pi/8 \approx 38\%$ of the hyperbolic area. In the remainder of this paper we normalize the Hilbert metric so that it is *twice* that above. This means the areas calculated in the rest of the paper should be divided by 4 to give the results announced.

This result extends to complete convex projective structures 2-orbifolds and to surfaces with cusps. Theorem 0.2 suggests various questions. For example there is a graph Γ with a vertex for each ideal triangulation $\mathcal{T}$ of F and edges that correspond to edge flips. Given a strictly convex structure on F what are the properties of the function defined on the vertices of Γ by $\mathcal{T} \mapsto \alpha_{\mathcal{T}}(F)$? Is the maximum attained ? Is there a uniform upper bound on the difference between the maximum and the area of F ?

It follows from the Margulis lemma for convex projective orbifolds [10], [11] that there is a lower bound on the Hilbert volume of a strictly convex projective n-orbifold. What are explicit lower bounds ?

The proposition is proved by using the following observation. An ideal triangle T in a domain Ω meets $\partial\Omega$ at its three vertices. There is a triangle Δ that contains Ω and is tangent to Ω at these three points, see Figure 2. The Hilbert metric given by Δ is smaller than that given by Ω. Thus the area of T in the Hilbert metric on Ω is bigger than its area using the Hilbert metric from Δ. We explicitly calculate the latter. The theorem follows by using ideal triangulations.

After the first version of this paper was written the authors learned of [9]. In that paper the authors show that the (Hausdorff) area of an ideal triangle is bounded below, and also that it is not bounded above. Their lower bound is different because they use a different definition of *area*. However, as Marquis remarks in [15], although there are different notions of area for Finsler metrics, because of the Benzécri compactness theorem [6], there is a universal bound on the ratios of different area forms for reasonable choices. The point of this paper is the relation to Fock-Goncharov coordinates.

1. Length and Area in Hilbert geometry

The **cross-ratio** of four distinct points $y_1, y_2, y_3, y_4 \in \mathbb{R}$ is

$$cr(y_1, y_2, y_3, y_4) = \frac{(y_1 - y_3)}{(y_1 - y_2)} \frac{(y_2 - y_4)}{(y_3 - y_4)}$$

Using the embedding $\mathbb{R} \subset \mathbb{R}P^1$ given by $y/x \mapsto [x : y]$ cross-ratio extends to a continuous map $cr : X \longrightarrow \mathbb{R}P^1$ where $X \subset \left(\mathbb{R}P^1\right)^4$ is the subset of quadruples of points at least 3 of which are distinct:

$$\begin{aligned} &cr([x_1 : y_1], [x_2 : y_2], [x_3 : y_3], [x_4 : y_4]) \\ &\qquad = [(x_2y_1 - x_1y_2)(x_4y_3 - x_3y_4) : (x_3y_1 - x_1y_3)(x_4y_2 - x_2y_4)]. \end{aligned}$$

Suppose $\Omega \subset \mathbb{R}^n \subset \mathbb{R}P^n$ is an open convex set that contains no affine line. Given $b, c \in \Omega$ there is a projective line ℓ in $\mathbb{R}P^n$ that contains them. This line meets $\partial\overline{\Omega} \subset \mathbb{R}P^n$ in two distinct points a, d. Label these points so a, b, c, d are in linear order along $\ell \cap \mathbb{R}^n$. Since Ω contains no affine line $a \neq d$. The *Hilbert metric on* Ω is

$$d_\Omega(b, c) = |\log cr(a, b, c, d)|$$

Some authors use (1/2) of this so that when Ω is the unit ball then d_Ω has curvature -1. This is a Finsler metric and

$$d_\Omega(x, x + dx) = \left(\frac{1}{|x - a|} + \frac{1}{|x - b|}\right) dx$$

In particular if $\Omega = (0, \infty) \subset \mathbb{R}^1$ then $a = 0, b = \infty$ and $d_\Omega(x, x+dx) = dx/x$.

The literature contains many distinct definitions of area for Finsler metrics [4],[5]. These depend on how *area* is defined for a normed vector space

$(V, \|\cdot\|)$. Area is a Borel measure on V that is preserved by translation (ie a Haar measure) so it is some multiple of Lebesgue measure. We will adopt the following definition which is particularly well suited for our purposes.

Definition 1.1. Suppose $(V, \|\cdot\|)$ is a normed 2-dimensional real vector space. Choose an inner product on V and let λ be the resulting Lebesgue measure. Set $K = \sup \lambda(\{\alpha a + \beta b \ : \ 0 \leq \alpha, \beta \leq 1 \})$ where the supremum is over all $a, b \in V$ with $\|a\|, \|b\| \leq 1$. Then define the *parallelogram measure* or *p-area* $\omega_{\|\cdot\|}$ on V by $\omega_{\|\cdot\|} = K^{-1} \cdot \lambda$.

It is easy to check the definition is independent of the choice of inner product. Having done this we dispense with the inner product, and refer to the norm of a vector as its *length.* The definition is equivalent to declaring that the maximum area of a parallelogram with sides of unit length is 1. If the norm is given by an inner product then parallelogram measure coincides with the usual area. This definition generalizes to n dimensions using parallelopipeds spanned by n vectors of norm one.

A *rectangle* in $(V, \|\cdot\|)$ is defined to be any parallelogram with side lengths x and y and area xy. Such parallelograms always exist. This enables the standard construction of Lebesque measure in the plane, starting from an inner product, to be extended to an arbitrary norm on the plane.

Parallelogram measure is an *increasing* function of the metric in the sense that if $\|\cdot\|$ and $\|\cdot\|'$ are two norms on V with $\|\cdot\| \leq \|\cdot\|'$ then $\omega_{\|\cdot\|} \leq \omega_{\|\cdot\|'}$. In particular if $\alpha > 0$ then $\omega_{\alpha\|\cdot\|} = \alpha^2 \omega_{\|\cdot\|}$.

A *Finsler surface* is a pair (S, ds) where S is a smooth surface and ds_x is a norm on T_xS for each $x \in S$. The *p-area form* on S is the p-area form for ds_x on T_xS. For a properly convex projective surface S the resulting area form ω_S is called the *Hilbert area form*, and the *Hilbert area of* S is $\int_s \omega_S$. If $\Omega' \subset \Omega$ are properly convex then $d_\Omega \leq d_{\Omega'}$ on Ω' and $\mu_\Omega \leq \mu_{\Omega'}$.

2. Hex geometry

A reference for this section is [12]. Let $u_0, u_1, u_2 \in \mathbb{R}^2$ be unit vectors with respect to the standard inner product such that $u_0 + u_1 + u_2 = 0$. We will use $u_0 = (1, 0)$ and $u_1 = (-1/2, \sqrt{3}/2)$ then $u_2 = -u_0 - u_1$. The convex hull of the vectors $\{\pm u_0, \pm u_1, \pm u_2\}$ is a regular Hexagon H.

Definition 2.1. The *Hex plane* $(\mathcal{H}, d_{\mathcal{H}})$ is the metric space obtained from the norm on $\mathbb{R}^2$ with unit ball H.

For the Hex plane p-area is different to Busemann volume, used for example in [15], or Holmes-Thompson used in [17]. On a normed plane all these measures are multiples of Haar measure, and so they are multiples of each other. We will describe some properties of p-area for the Hex plane which suggest this is the *right* definition to use.

Lemma 2.2. *Let $\omega_{\mathcal{H}}$ denote the area form on the Hex plane and λ be Lebesgue measure with respect to the standard inner product. Then $\omega_{\mathcal{H}} = (2/\sqrt{3})\lambda$*

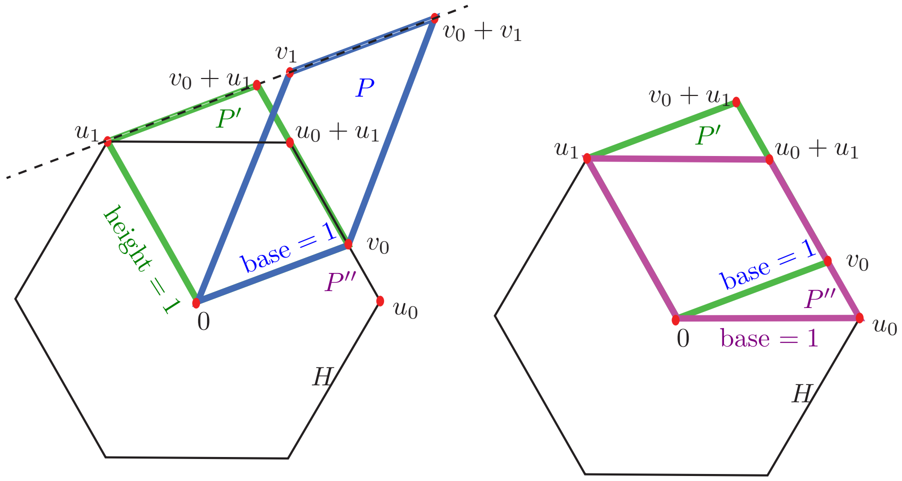


Figure 1. In Hex geometry area = base × height

Proof. Let a, b be unit vectors in the Hex norm. Then they lie on the boundary of the regular unit Hexagon H center at the origin and determine a parallelogram $P(a, b)$ with vertices $\{0, a, b, a + b\}$. Suppose a lies on the edge e of H. Then the area of $P(a, b)$ is maximized by taking b to be a vertex of H that that is not an endpoint of $\pm e$. The Euclidean area of $P(a, b)$ is then $\sqrt{3}/2$. □

Our choice of normalization of area has the following consequences in the Hex plane. A Hex circle of radius r is a Euclidean regular Hexagon so that the (Euclidean=Hex) distance from the center to a vertex is r. The circumference of this circle is $6r$ and the p-area is $3r^2$.

The *positive quadrant* is $Q = \{(x, y) \ : \ x, y > 0\}$. A *triangle* in $\mathbb{R}P^2$ is a compact convex subset, Δ, bounded by 3 segments of projective lines. There is a projective transformation taking the interior of Δ to $\{[x : y : 1] \ : \ x, y > 0\}$ which may be identified with Q. Thus the Hilbert metric on the interior of Δ is isometric to (Q, d_Q).

Lemma 2.3 (Proposition 7 in [12]**).** *The Hilbert metric on Q is isometric to the Hex plane $(\mathcal{H}, d_{\mathcal{H}})$.*

Proof. There is an isometry $A : (\mathbb{R}^2, \|\cdot\|_{Hex}) \to (Q, d_Q)$ given by

$$A(u, v) = \left(e^{u + \frac{1}{\sqrt{3}}v}, e^{\frac{2}{\sqrt{3}}v}\right) = (x, y).$$

This may be checked as follows. The map A conjugates the action of $\mathbb{R}^2$ on itself by translations to the action of the positive diagonal group on Q. Thus it suffices to check A is infinitesimally an isometry at the origin. □

If $\Omega \subset \mathbb{R}P^2$ is a properly convex domain the *Hilbert area form* ω_Ω is the 2-form given by the Hilbert metric.

Lemma 2.4. *The Hilbert area form on the positive quadrant $Q = \{(x,y) : x,y > 0\}$ is*

$$\omega_Q = \frac{dxdy}{xy}.$$

Proof. The isometry A in Lemma 2.3 infinitesimally multiplies Lebesgue measure λ on $\mathbb{R}^2$ by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^{u+\frac{1}{\sqrt{3}}v} & \frac{1}{\sqrt{3}} e^{u+\frac{1}{\sqrt{3}}v} \\ 0 & \frac{2}{\sqrt{3}} e^{\frac{2}{\sqrt{3}}v} \end{vmatrix} = \frac{2}{\sqrt{3}} \left(e^{u+\frac{1}{\sqrt{3}}v}\right)\left(e^{\frac{2}{\sqrt{3}}v}\right) = \frac{2}{\sqrt{3}} xy.$$

Since $\omega_{\mathcal{H}} = (2/\sqrt{3})\lambda$ it follows that

$$\omega_Q = \frac{\omega_{\mathcal{H}}}{\frac{2}{\sqrt{3}}xy} = \frac{\lambda}{xy} = \frac{dxdy}{xy}.$$

□

Lemma 2.5. *The p-area of every parallelogram in the Hex plane is base × height.*

Proof. Affine maps of the plane multiply Euclidean area (and hence p-area) by the determinant of the linear part. Hence it suffices to prove the result in the special case of a parallelogram P with vertices $0, v_0, v_1, v_0 + v_1$ for which the length of the base is $\|v_0\|_{Hex} = 1$ and the height is also one. Moreover a rotation through an angle of $\pi/3$ is an isometry of Hex. Thus we may assume v_0 is on the side of H between u_0 and $u_0 + u_1$.

Refer to Figure 1. We transform P to P' to P'' and show that each of these parallelograms has the same p-area. A shear parallel to the base of P preserves base, height and p-area. Shear P parallel to v_0 sending v_1 to u_1 to give a parallelogram P' with vertices $0, v_0, u_1, v_0 + u_1$. Now shear P' parallel to the u_1-direction to get a parallelogram P'' with vertices $0, u_0, u_1, u_0 + u_1$. This shear preserves base and height because of the special properties of H. The area of P'' is 1. □

3. Ideal triangles in Hex geometry

Suppose $\Omega \subset \mathbb{R}P^2$ is an open properly convex set. If T is a triangle with vertices in $\partial\Omega$ then $T \cap \Omega$ is called an *ideal triangle* in Ω. It is *proper* if $T \cap \partial\Omega$ consists of only the vertices of T. We will only be concerned with proper ideal triangles, and will henceforth omit the term *proper*.

If $\Delta = \{[x_0 v_0 + x_1 v_1 + x_2 v_2] \ : \ x_i > 0 \ \}$ is the interior of triangle in $\mathbb{R}P^2$ there is an isometry $\phi : (\Delta, d_\Delta) \longrightarrow (\mathcal{H}, d_{\mathcal{H}})$ given by

$$\phi[x_0 v_0 + x_1 v_1 + x_2 v_2] = (\log x_0) u_0 + (\log x_1) u_1 + (\log x_2) u_2$$

Suppose T is an ideal triangle in Δ. We refer to $\phi(T)$ as an *ideal triangle* in the Hex plane. Then $T \subset \Delta$ together with an ordering of the vertices of Δ determines a *shape parameter* $t = t(T, \Delta) \in \mathbb{R}$ defined as follows, see Figure 2. If the vertices of T are $[w_0 = v_0 + a v_1], [w_1 = v_1 + b v_2], [w_2 = v_2 + c v_0]$

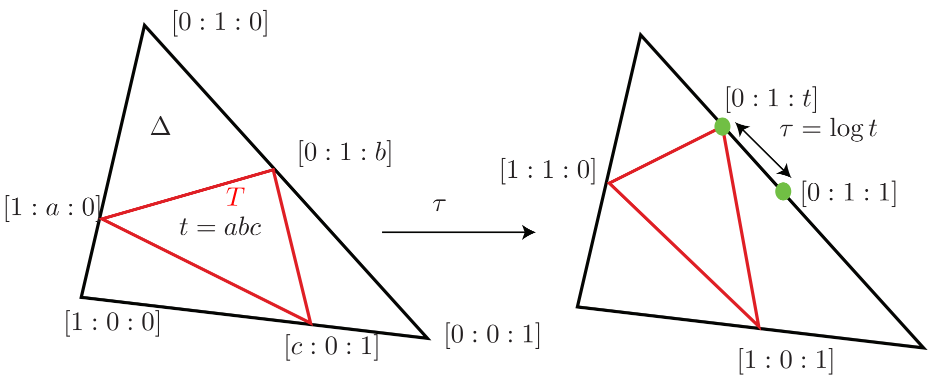


FIGURE 2. The shape parameter $t > 0$ and triangle invariant $\tau = \log t$

with $a, b, c > 0$ then $t = abc$. This depends only on the cyclic ordering of the vertices. Changing this ordering replaces t by $1/t$. Observe that when $a = b = 1$ then $|\log t|$ is the Hilbert distance in (v_1, v_2) between $[w_2]$ and the midpoint $[v_1 + v_2]$.

The group $PGL(\Omega)$ is the subgroup of $PGL(3, \mathbb{R})$ which preserves Ω. Using the basis v_0, v_1, v_2 of $\mathbb{R}^3$ the identity component $PGL_0(\Delta)$ consists of positive diagonal matrices. This group acts transitively on the interior of Δ. There is a unique element $\tau \in PGL_0(\Delta)$ which takes two of the vertices of T to $[u_1 + u_2], [u_1 + u_3]$. The remaining vertex is taken to $[u_2 + tu_3]$, see Figure (2). The *regular* ideal triangle is given by $t = 1$. It has maximal isometry group: dihedral of order 6.

Proposition 3.1. *Isometry classes of ideal Hex triangle are in 1-1 correspondence with shape parameters $t \in [1, \infty)$.*

Lemma 3.2. *The Hilbert area of an ideal triangle in the Hex plane with shape parameter t is*

$$B(t) = \int_1^\infty \frac{1}{s} \log\left(\frac{(st+1)(s+t)}{t(s-1)^2}\right)\, ds$$

Proof. By Lemma 2.3, the ideal triangle is isometric to an ideal triangle T in Q. By means of a projective transformation preserving Q we can arrange that one side of T is given by $x + y = 1$ and the other two sides are then the parallel rays in Q given by $y - 1 = tx$ and $y = t(x - 1)$. Refer to Figure 3. It is easy to check that t is the shape parameter.

For $s \geq 1$ define $\alpha(s), \beta(s)$ to be the points of intersection of the line $x + y = s$ with the sides $y - 1 = tx$ and $y = tx + 1$ of T. Let $\gamma(s)$ be the line segment with these endpoints. For $s \geq 1$ these lines foliate T. For $s > 1$ define

$$\ell(s) = d_Q(\alpha(s), \beta(s))$$

to be the Hilbert length of $\gamma(s)$. The distance between $\gamma(s)$ and $\gamma(s + ds)$ is ds/s. This is easily seen by projecting onto the x-axis along the direction

$x+y=0$. It follows from Lemma 2.5 that the Hilbert area of the infinitesimal parallelogram shown is $\ell(s)ds/s$, thus

$$area_\Delta(T) = \int_1^\infty \ell(s)\frac{ds}{s}$$

Next we compute

$$\ell(s) = \log\left(\frac{(s+t)(st+1)}{t(s-1)^2}\right)$$

This may be done as follows. Projection of one line onto another preserves Hilbert distance. Let π be vertical projection of the line segment $x+y=s$ in Q onto the x-axis. The image of this segment is $[0,s]$. Then

$$\pi(\alpha(s)) = \frac{s-1}{t+1}, \qquad \pi(\beta(s)) = \frac{s+t}{t+1}$$

Then $\ell(s) = d_Q(\alpha(s),\beta(s)) = d_{[0,s]}(\alpha(s),\beta(s)) = |\log CR(0,\alpha(s),\beta(s),s)|$. □

Next we calculate this integral. This implies Proposition 0.3.

Lemma 3.3.

$$B(t) = \frac{\pi^2 + (\log t)^2}{2}$$

Proof.

$$\frac{dB}{dt} = \frac{d}{dt}\left(\int_1^\infty \frac{1}{s}\log\left(\frac{(st+1)(s+t)}{t(s-1)^2}\right)\, ds\right)$$

Taking the derivative inside the integral gives

$$\begin{aligned}\int_1^\infty \frac{1}{s}\frac{d}{dt}\left(\log\left(\frac{(st+1)(s+t)}{t(s-1)^2}\right)\right)\, ds &= \int_1^\infty \frac{1}{s}\left(\frac{s}{1+st} + \frac{1}{s+t} - \frac{1}{t}\right)\, ds \\ &= \int_1^\infty \left(\frac{1}{1+st} - \frac{1}{t(s+t)}\right)\, ds \\ &= \left[\frac{1}{t}\left(\log(1+st) - \log(s+t)\right)\right]_{s=1}^{s=\infty} \\ &= \left[t^{-1}\log\left(\frac{1+st}{s+t}\right)\right]_{s=1}^{s=\infty} \\ &= t^{-1}\log t\end{aligned}$$

Thus

$$\frac{dB}{dt} = t^{-1}\log t$$

Integrating gives $B(t) = (1/2)(\log t)^2 + C$. The next lemma shows $C = \pi^2/2$. □

Lemma 3.4.

$$B(1) = \frac{\pi^2}{2}$$

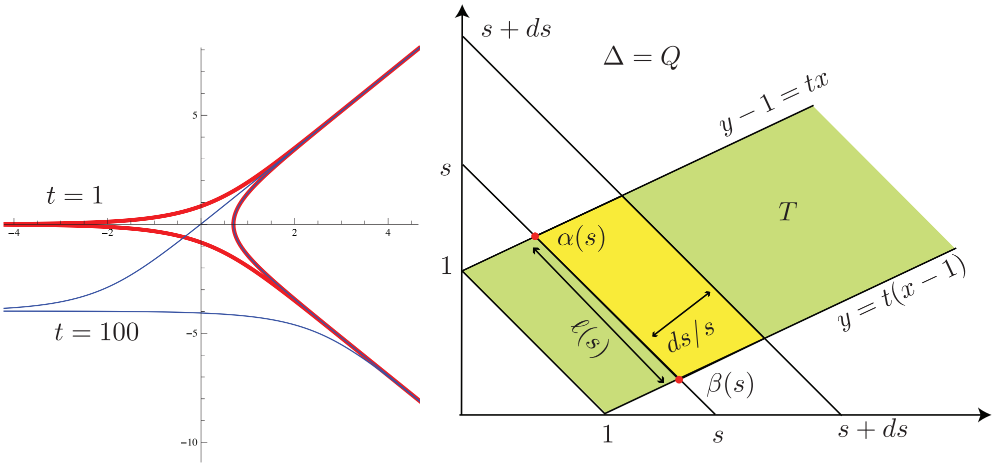


FIGURE 3. Ideal triangles in the Hex plane and the positive quadrant.

Proof.

$$B(1) = \int_1^\infty \frac{1}{s}\log\left(\frac{(s+1)^2}{(s-1)^2}\right)\ ds = \int_1^\infty \frac{2}{s}\log\left(\frac{s+1}{s-1}\right)\ ds$$

Set $w = (s+1)/(s-1)$ then

$$s = \frac{w+1}{w-1} = 1 + \frac{2}{w-1} \quad \text{and} \quad \frac{ds}{dw} = -2(w-1)^{-2}$$

hence

$$B(1) = \int_1^\infty \frac{2}{s}\log\left(\frac{s+1}{s-1}\right)\ ds = \int_\infty^1 2\left(\frac{w-1}{w+1}\right)\log w\ (-2)(w-1)^{-2}dw$$
$$= \int_1^\infty \frac{4\log w\ dw}{w^2-1}.$$

Set $w = e^x$ this becomes

$$B(1) = \int_0^\infty \frac{4x\ e^x dx}{e^{2x}-1} = \int_0^\infty \frac{4x\ dx}{e^x - e^{-x}}.$$

The integrand is even so

$$B(1) = \int_{-\infty}^\infty \frac{2x\ dx}{e^x - e^{-x}}.$$

Define

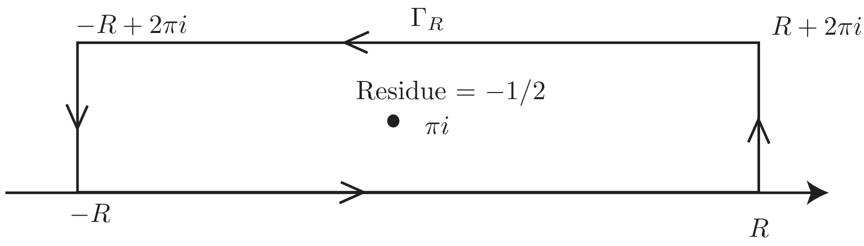
$$g(z) = \frac{i}{4\pi} z(z - 2\pi i),$$

then

$$g(z) - g(z + 2\pi i) = z.$$

Hence

$$B(1) = \int_{-\infty}^\infty \frac{2x\ dx}{e^x - e^{-x}} = 2\int_{-\infty}^\infty \frac{g(x) - g(x+2\pi i)}{e^x - e^{-x}}\ dx.$$



We claim this equals the limit of the contour integrals

$$B(1) = \lim_{R\to\infty} \oint_{\Gamma_R} \frac{2g(z)}{e^z - e^{-z}}\ dz.$$

Where Γ_R is the rectangle

$$([-R,R]\times\{0,2\pi i\}) \cup (\{\pm R\}\times[0,2\pi i])$$

oriented counterclockwise. Observe that on the vertical sides $(\{\pm R\}\times[0,2\pi i])$ the integrand goes to 0 as $R\to\infty$. The integral along the horizontal sides $([-R,R]\times\{0,2\pi i\})$ with the orientation specified is

$$2\int_{-R}^{R} \frac{g(z)-g(z+2\pi i)}{e^z-e^{-z}}\ dz.$$

This proves the claim. To evaluate the contour integral we observe the only singularity inside Γ_R of

$$\frac{2g(z)}{e^z-e^{-z}}$$

is a simple pole at $z = i\pi$. Now

$$2g(i\pi) = 2\ \frac{i(i\pi)(-i\pi)}{4\pi} = \frac{i\pi}{2}.$$

Also the denominator is $e^z - e^{-z} = 2z + \cdots$ has residue $-1/2$ at $z = i\pi$ because

$$\begin{aligned}
\text{residue} &= \lim_{z\to i\pi} \frac{z-i\pi}{e^z-e^{-z}} \\
&= \lim_{w\to 0} \frac{w}{e^{w+i\pi}-e^{w-i\pi}} \\
&= \lim_{w\to 0} \frac{w}{e^{-i\pi}(e^w-e^{-w})} \\
&= -\lim_{w\to 0} \frac{w}{(1+w+\cdots)-(1-w+\cdots)} \\
&= -\lim_{w\to 0} \frac{w}{2w+\cdots} \\
&= -\frac{1}{2}.
\end{aligned}$$

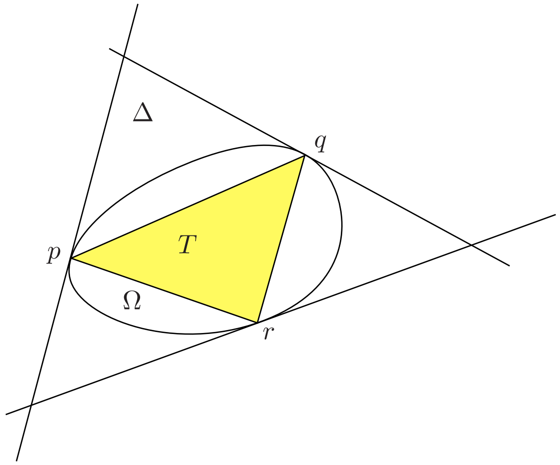


FIGURE 4. An ideal triangle T in Ω

Thus the residue of the integrand at $i\pi$ is $(-1/2)(i\pi/2) = -i\pi/4$. Cauchy's theorem gives the contour integral is

$$(2\pi i)(-i\pi/4) = \pi^2/2.$$

□

4. Ideal triangulations

Ideal triangulations of surfaces were introduced by Thurston for hyperbolic surfaces, see [14],[16]. The extension to properly convex surfaces is routine.

Definition 4.1. An *ideal triangulation* of a convex projective surface $F = \Omega/\Gamma$ is a decomposition of F into closed subsets with disjoint interiors called *ideal triangles* such that each component of the preimage of an ideal triangle in F is an ideal triangle in Ω.

Proposition 4.2. *If F is a closed convex projective surface then F has an ideal triangulation. The number of ideal triangles is* $2|\chi(F)|$.

Suppose $F = \Omega/\Gamma$ is a compact, strictly convex projective surface with $\chi(F) < 0$. Then $\Omega \subset \mathbb{R}P^2$ has a unique tangent line at each point of $\partial\overline{\Omega}$ by [6]. If $T \subset \Omega$ is an ideal triangle with vertices $p, q, r \in \partial\Omega$. The tangent lines at p, q, r contain the sides of a triangle Δ which contains Ω. Following Fock and Goncharov [13] we define the *shape* $t = t(T, \Omega) \in [1, \infty)$ of T in Ω to be the shape $t(T, \Delta)$ of T in Δ previously defined.

For a properly convex set Ω we denote the Hilbert area form on Ω by ω_Ω. This pushes down to a 2-form ω_F on F. We denote the *Hilbert area* of a measurable subset $X \subset F$ by

$$area_F(X) = \int_X \omega_F$$

Since $\Omega \subset \Delta$ it follows that $d_\Delta \leq d_\Omega$ and $\omega_\Delta \leq \omega_\Omega$. It follows that if $T \subset F$ is an ideal triangle with shape parameter t then

$$area_F(T) \geq B(t) \geq \pi^2/2$$

Since F contains $2|\chi(F)|$ ideal triangles with disjoint interiors Theorem 0.2 follows from Propositions 4.2 and 0.3.

Acknowledgment

The authors thank the MRC at Snowbird 2011 and the MSRI during spring of 2015 for hospitality while parts of this paper were written. The authors acknowledge support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties" (the GEAR Network). Cooper was partially supported by NSF grants 1065939, 1207068 and 1045292 and thanks IAS and the Ellentuck Fund for partial support during completion of this paper.

References

[1] I. Adeboye. Lower bounds for the volume of hyperbolic n-orbifolds. *Pacific J. Math.*, 237(1):1–19, 2008.

[2] I. Adeboye and G. Wei. On volumes of hyperbolic orbifolds. *Algebr. Geom. Topol.*, 12(1):215–233, 2012.

[3] I. Adeboye and G. Wei. On volumes of complex hyperbolic orbifolds. *Michigan Math. J.*, 63(2):355–369, 2014.

[4] J. C. Álvarez Paiva and G. Berck. What is wrong with the Hausdorff measure in Finsler spaces. *Adv. Math.*, 204(2):647–663, 2006.

[5] J. C. Álvarez Paiva and A. C. Thompson. Volumes on normed and Finsler spaces. In *A sampler of Riemann-Finsler geometry*, volume 50 of *Math. Sci. Res. Inst. Publ.*, pages 1–48. Cambridge Univ. Press, Cambridge, 2004.

[6] J.-P. Benzécri. Sur les variétés localement affines et localement projectives. *Bull. Soc. Math. France*, 88:229–332, 1960.

[7] F. Bonahon and G. Dreyer. Parameterizing Hitchin components. *Duke Math. J.*, 163(15):2935–2975, 2014.

[8] S. Choi and W. M. Goldman. Convex real projective structures on closed surfaces are closed. *Proc. Amer. Math. Soc.*, 118(2):657–661, 1993.

[9] B. Colbois, C. Vernicos, and P. Verovic. L'aire des triangles idéaux en géométrie de Hilbert. *Enseign. Math. (2)*, 50(3-4):203–237, 2004.

[10] D. Cooper, D. D. Long, and S. Tillmann. On convex projective manifolds and cusps. *Adv. Math.*, 277:181–251, 2015.

[11] M. Crampon and L. Marquis. Un lemme de Kazhdan-Margulis-Zassenhaus pour les géométries de Hilbert. *Ann. Math. Blaise Pascal*, 20(2):363–376, 2013.

[12] P. de la Harpe. On Hilbert's metric for simplices. In *Geometric group theory, Vol. 1 (Sussex, 1991)*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 97–119. Cambridge Univ. Press, Cambridge, 1993.

[13] V. V. Fock and A. B. Goncharov. Moduli spaces of convex projective structures on surfaces. *Adv. Math.*, 208(1):249–273, 2007.

[14] M. Lackenby. Taut ideal triangulations of 3-manifolds. *Geom. Topol.*, 4:369–395 (electronic), 2000.

[15] L. Marquis. Around groups in Hilbert geometry. In *Handbook of Hilbert geometry*, volume 22 of *IRMA Lect. Math. Theor. Phys.*, pages 207–261. Eur. Math. Soc., Zürich, 2014.

[16] L. Mosher. Tiling the projective foliation space of a punctured surface. *Trans. Amer. Math. Soc.*, 306(1):1–70, 1988.

[17] C. Vernicos. Asymptotic volume in Hilbert geometries. *Indiana Univ. Math. J.*, 62(5):1431–1441, 2013.

Ilesanmi Adeboye
Department of Mathematics and Computer Science
Wesleyan University
265 Church Street
Middletown, CT 06459
USA
e-mail: `iadeboye@wesleyan.edu`

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106
USA
e-mail: `cooper@math.ucsb.edu`