THE HEISENBERG GROUP ACTS ON A STRICTLY CONVEX DOMAIN.

DARYL COOPER

Every linear group acts by isometries on some properly convex domain in real projective space. This follows from the fact that action of $\mathrm{SL}(n,\mathbb{R})$ on the space of quadratic form in n variables preserves the projectivization, $\mathrm{Pos}(n)$, of the properly convex cone consisting of positive definite forms. If Γ is the holonomy of a properly convex orbifold of finite volume then every virtually nilpotent group is virtually abelian, moreover every unipotent subgroup is conjugate into $\mathrm{PO}(n,1)$. A reference for all this is [1]. This paper gives the first example of a unipotent group that is not virtually abelian and preserves a strictly convex domain. It answers a question asked by Misha Kapovich.

The *Heisenberg group* is the subgroup $H \subset \mathrm{SL}(3,\mathbb{R})$ of unipotent upper-triangular matrices. Define $\theta : H \to \mathrm{SL}(10,\mathbb{R})$ and $G = \theta(H)$ where

$$\theta\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2a & 2c & a & a^2/2 & a^3/6 & b & 2a^2+b^2/2 & b^3/6+2ac & (a^4+b^4)/24+c^2\\ 0 & 1 & b & 0 & 0 & 0 & 0 & 2a & ab+c & bc\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a & c\\ 0 & 0 & 0 & 1 & a & a^2/2 & 0 & 0 & 0 & a^3/6\\ 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & a^2/2\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 & b^3/6\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

It is clear that θ is injective and easy to check that it is a homomorphism.

Theorem 0.1. *There is a strictly convex domain $\Omega \subset \mathbb{RP}^9$ that is preserved by G. This is an effective action of the Heisenberg group on Ω by parabolic isometries that are unipotent.*

Proof. The group G acts affinely on the affine patch $[x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7 : x_8 : x_9 : 1]$ that we identify with $\mathbb{R}^9$. Let $p \in \mathbb{R}^9$ be the origin. Then $G \cdot p$ is

$$((a^4+b^4)/24+c^2, bc, c, a^3/6, a^2/2, a, b^3/6, b^2/2, b)$$

This orbit is an algebraic embedding $\mathbb{R}^3 \hookrightarrow \mathbb{R}^9$ which limits on the single point

$$q = [1:0:0:0:0:0:0:0:0:0] \in \mathbb{RP}^9$$

in the hyperplane at infinity, P_∞. This follows from the fact that $(a^4+b^4)/24+c^2$ dominates all the other entries whenever at least one of $|a|, |b|, |c|$ is large.

Let $S \subset \mathbb{R}^9$ be this orbit. Choose 10 random points on $S \subset \mathbb{RP}^9$ and compute the determinant, d, of the corresponding 10 vectors in $\mathbb{R}^{10}$. Then $d \neq 0$ therefore the interior $\Omega^+ \subset \mathbb{R}^9$ of the convex hull of S has dimension 9.

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Moreover the closure Ω' of Ω^+ in $\mathbb{RP}^9$ is disjoint from the closure of the affine hyperplane $x_1 = -1$, hence Ω^+ is properly convex. Since $\Omega' \cap P_\infty = q$ and G preserves q and P_∞ and G is unipotent, it follows from (5.8) in [1] that G preserves some strictly convex domain $\Omega \subset \Omega'$. □

Corollary 0.2. *There is a strictly convex real projective manifold Ω/Γ of dimension 9 with nilpotent fundamental group $\Gamma \cong \langle \alpha, \beta : [\alpha, [\alpha, \beta]], [\beta, [\alpha, \beta]] \rangle$ that is not virtually abelian. Moreover Γ is unipotent.*

Proof. If Γ is a lattice in G then Ω/Γ is a strictly convex manifold with unipotent holonomy and Γ is nilpotent but not virtually abelian. □

The genesis of this example is as follows. The image of H in $\mathrm{SL}(6, \mathbb{R})$ under the irreducible representation $\mathrm{SL}(3, \mathbb{R}) \to \mathrm{SL}(6, \mathbb{R})$ is

$$\begin{pmatrix} 1 & 2a & a^2 & 2c & 2ac & c^2 \\ 0 & 1 & a & b & ab+c & bc \\ 0 & 0 & 1 & 0 & 2b & b^2 \\ 0 & 0 & 0 & 1 & a & c \\ 0 & 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and preserves $Q := \mathrm{Pos}(3) \subset \mathbb{RP}^5$. The boundary of the closure of Q consists of semi-definite forms and contains flats, so Q is *not strictly convex*. Let $A, B, C \in \mathrm{SL}(6, \mathbb{R})$ be the elements corresponding to one of a, b, c being 1 and the others 0. Each of A, B, C has a parabolic fixed point in ∂Q corresponding to a rank 1 quadratic form. Every point in Q converges to this parabolic fixed point under iteration by the given group element. The fixed point for A and B are distinct and lie in a flat in ∂Q.

The idea is to increase the dimension of the representation and use the extra dimensions to add parabolic blocks of size 5 onto A (row 1 and rows 7-10) and onto B (row 1 and rows 11-14) that commute and the parabolic fixed point of each block is the rank-1 form that is a fixed point of C. This gives a 14-dimensional representation of H:

$$\begin{pmatrix} 1 & 2a & a^2 & 2c & 2ac & c^2 & a & a^2/2 & a^3/6 & a^4/24 & b & b^2/2 & b^3/6 & b^4/24 \\ 0 & 1 & a & b & ab+c & bc & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2b & b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a^2/2 & a^3/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a^2/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 & b^3/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The top-left 6×6 block is the image of H in $\mathrm{SL}(6, \mathbb{R})$. The entries in A^n and B^n grow like n^2. This is beaten by the growth of some entries in the added blocks of size 5 which grow like n^4. This gives rise to a representation of H of dimension $6 + 4 + 4 = 14$. The orbit of

$$[0 : 0 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 1]$$

is

$$[(a^4+b^4)/24+c^2 : bc : b^2 : c : b : 1 : a^3/6 : a^2/2 : a : 1 : b^3/6 : b^2/2 : b : 1]$$

so there is a codimension-4 projective hyperplane that is preserved, and which is defined by

$$x_6 = x_{10} = x_{14} \qquad x_5 = x_{13} \qquad x_3 = 2x_{12}$$

The restriction to this hyperplane gives θ.

References

[1] D. Cooper, D. D. Long, and S. Tillmann. On convex projective manifolds and cusps. *Adv. Math.*, 277:181–251, 2015.

Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

E-mail address: cooper@math.ucsb.edu