Mutations of Links in Genus 2 Handlebodies

by

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Abstract A short proof is given to show that a link in the 3-sphere and any link related to it by genus 2 mutation have the same Alexander polynomial. This verifies a deduction from the solution to the Melvin-Morton conjecture. The proof here extends to show that the link signatures are likewise the same and that these results extend to links in a homology 3-sphere.

1. Introduction A remarkable correlation between the Jones polynomials of a classical knot and its Alexander polynomial was established by D.Bar-Natan and S.Garoufalidis in [1], where they gave a proof of a conjecture of P.M.Melvin and H.R.Morton. The background to the conjecture was established in [4] and heuristic evidence in its favour was given by L.Rozansky [7]. The conjecture asserted that, with a specific formula, it was possible to calculate each coefficient in the Alexander polynomial of a knot from knowledge of all the coefficients in all the coloured Jones polynomials of the knot (with zero framing). These coloured Jones polynomials can be regarded as the generalisations of the Jones invariant to framed knots equipped with irreducible representations of SU(2) or, interpretted by means of the Kauffman bracket skein, as the invariant of the zero-framed knot decorated by a Chebyshev polynomial in the skein generator of the solid torus. The proof given in [1] is an intricate and skillful deployment of Vassiliev invariants analysed by means of arguments with chord diagrams and weight systems. Indeed, it was in the world of these ideas that the conjecture was originally conceived.

A consequence of the solution to the Melvin-Morton conjecture is, of course, that if two knots have all the same coloured Jones polynomials, then they have the same Alexander polynomials. To verify this it is necessary to find a pair of knots that satisfy this hypothesis.

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Two knots related by a classical mutation (in the sense of J.H.Conway) are the first example and it is a standard result that they have the same Alexander polynomial. A second example considers a pair of knots related by what will here be called *genus* 2 *mutation*: Suppose K is a knot in a genus 2 handlebody H that is contained in some arbitrary (complicated) way in S^3. Let ρ be the involution of H depicted abstractly as a π-rotation in Figure 1.

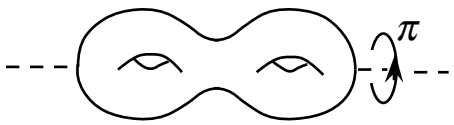

Figure 1

Then K and ρK have all coloured Jones invariants the same. An easy proof is simply that the Kauffman bracket skein of H has a base consisting of sets of disjoint closed unoriented curves in a twice punctured disc in H and each such set of curves is invariant under ρ. This idea has been employed by A.Kawauchi in [3]. It thus follows from the above mentioned results that K and ρK have the same Alexander polynomial. It is impressive that this simple result, readily expressible in the language of classical, pre-Jones knot theory, should have emerged from the technicalities of Vassiliev invariants. It may be the first such new result to arrive in this way. The first purpose of this note is to give an independent proof, in terms of classical knot theory, of this result thus verifying the above mentioned chain of reasoning. However the method here used depends only on elementary homology theory and so the result is equally valid for links in a homology 3-sphere. Furthermore, as the proof produces Seifert matrices for K and ρK, it is simple to deduce that these knots have the same classical signature [6] and the same ω-signatures [9]. Thus, these latter results are inspired by, but not directly deducible from, the solution of the Melvin-Morton conjecture (which used Vassiliev theory). It is interesting to note that it has been shown in [2] and [8] that the signature of a knot is not a Vassiliev invariant.

It is fairly easy to see that if two *knots* differ by a classical mutation then they differ by a sequence of at most two of the genus 2 mutations described above. In particular, in

the famous example of the Conway knot and the Kinoshita-Terasaka knot, the two knots are related by a single genus 2 mutation. This implies that a satellite of one of these knots is related by genus 2 mutation to the satellite, with the same pattern, of the other. Computer calculations of Morton and H.B.Short (see [5]) have show that a certain cabling of one of these knots has its HOMFLY polynomial different from that of the same cabling of the other knot. That implies that a genus 2 mutation is not just a sequence of classical mutations, for the latter preserve HOMFLY polynomials. In addition, it makes it hard to see how skein theory could be used to show that K and ρK always have the same Conway polynomial. They do have the same Conway polynomial as that is but a normalisation of the Alexander polynomial. However, Conway skein theory is very similar to HOMFLY skein theory.

2. Preliminary results There follow here four elementary results that will be used later.

Lemma 1. *Suppose that H is a handlebody of genus g contained in S^3 and let X be the closure of $S^3 - X$. The inclusion maps of ∂H into H and X induce an isomorphism $H_1(\partial H) \to H_1(H) \oplus H_1(X)$.*

Proof. The Mayer-Vietoris exact homology sequence applied to S^3 expressed as $H \cup X$ provides this isomorphism at once. □

This result implies that each of $H_1(H)$ and $H_1(X)$ is a copy of $\mathbb{Z}^g$. Furthermore, letting $i : \partial H \to X$ be the inclusion map, it follows that the kernel of $i_\star : H_1(\partial H) \to H_1(X)$ is a direct summand of $H_1(\partial H)$ and that this kernel maps isomorphically onto $H_1(H)$.

Lemma 2. *Suppose that a collection of disjoint, oriented, simple, closed curves, contained in the boundary of an orientable 3-manifold M, represents the zero element of $H_1(M)$. Then those curves form the boundary of some oriented surface properly embedded in M.*

This is a standard result that can, for example, be proved by means of transversality.

Now let H be a genus 2 handlebody and ρ be the homeomorphism of H to itself given by rotation through π about the axis shown in Figure 1. In that diagram H is shown in a standard position in $\mathbb{R}^3$. If H is embedded in some other way then it is the conjugate, by the embedding, of this rotation that is to be called ρ.

Lemma 3. *If C is any simple closed curve in ∂H, then C is isotopic to a curve $\widetilde{C}$ that is fixed, as a set, by ρ. If C separates ∂H then ρ preserves the orientation of $\widetilde{C}$, otherwise ρ reverses the orientation of $\widetilde{C}$.*

Proof. It is a classical result that $\rho|\partial H$ represents the only non-trivial element in the centre of the mapping class group of ∂H. In fact, any orientation preserving automorphism ϕ of ∂H is isotopic to some $\widetilde{\phi}$ that commutes with ρ. If C separates ∂H there is some such ϕ for which ϕC is equal to a "standard curve" C_1 that separates and is preserved, together with its orientation, by ρ. Then $\widetilde{\phi}^{-1}C_1$ is isotopic to C and is preserved by ρ. Similar remarks apply when C does not separate. □

Suppose that, with entries from some commutative ring, M is an $r \times r$ matrix and N is an $s \times s$ matrix and that $M_{r,r} = N_{1,1}$. Let $M \oplus_1 N$ be the $(r+s-1) \times (r+s-1)$ matrix that has M for its first $r \times r$ terms, N for its final $s \times s$ terms and zeros elsewhere (see Figure 2). Let $\widetilde{M}$ and $\widetilde{N}$ be obtained by deleting the last row and column from M and the first row and column from N.

$$M \oplus_1 N = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

Figure 2

Lemma 4. *With the above notation,*

$$\det(M \oplus_1 N) = \det \widetilde{M} \det N + \det M \det \widetilde{N} - M_{r,r} \det \widetilde{M} \det \widetilde{N}.$$

Proof. This follows at once by expanding the determinant by the r^{th} row. □

Note that it follows at once from this result that $\det(M \oplus_1 N) = \det(M \oplus_1 N^\tau)$.

3. Seifert matrices It will now be shown that there are closely related Seifert matrices