Free actions of finite groups on rational homology 3-spheres.

D. Cooper and D.D. Long *

April 26, 1996

1 Introduction.

The purpose of this note is to prove the following:

Theorem 1.1 *Let G be a finite group. Then there is a rational homology S^3 on which G acts freely.*

That any finite group acts freely on some closed 3-manifold is easy to arrange: There are many examples of closed 3-manifolds whose fundamental groups surject a free group of rank two (for example, by taking a connected sum of $S^1 \times S^2$'s) and by passing to a covering space, one can obtain a manifold whose group surjects a free group of any given rank. This gives a surjection onto any finite group and hence a free action on the associated covering space. We also note that results of Milnor [2] easily imply that one cannot replace rational coefficients by integral coefficients and hope for a similar result.

The strategy for proving Theorem 1.1 is this: We begin with a free action of G on some 3-manifold M. This makes $H_1(M)$ into a representation module for the group G. (Here, as throughout, homology groups will be with rational coefficients.) Our first task is to gain some control over the representations which occur. To this end we recall that every finite group acts on its rational group algebra $\mathbb{Q}[G]$ by left multiplication to give the so-called *left regular representation*. We denote this representation by L_G. Then the control we seek is accomplished in Lemma 2.3, where, denoting the trivial representation by $< 1 >$ (that is to say, the one dimensional vector space with the trivial G-action) we show that one can find a possibly different 3-manifold and a free G-action, so that the G-module $H_1(M) \oplus < 1 >$ becomes a large number of copies of L_G.

We then show that one can systematically remove summands of this controlled type by Dehn surgery, a process which eventually yields a rational homology sphere with free G action. We conclude with a sketch that this rational homology sphere can be chosen to be hyperbolic.

2 The construction.

Suppose that M is a 3-manifold with a free G-action. Suppose that $\gamma_1, \ldots, \gamma_k$ is a set of disjoint smooth simple closed curves in M which are freely permuted by G. Equivariantly deleting open regular neighbourhoods of these curves, we form the manifold $X = M \setminus G \cdot N(\gamma_1 \sqcup \ldots \sqcup \gamma_k)$. We note that, by construction, X has a free G action.

We have a G map $i_* : H_1(\partial X) \to H_1(X)$ which is induced by inclusion and by duality we have a splitting into two submodules of the equal dimensions: $ker(i_*) \oplus Im(i_*) \cong H_1(\partial X)$. This is still an isomorphism of $\mathbb{Q}[G]$-modules, but is not natural since it arose by splitting a short exact sequence.

*Both authors supported in part by the NSF

Lemma 2.1 *As G modules we have: $Im(i_*) \cong (L_G)^k$.*

Proof. Since ∂X consists of tori which are freely permuted by G, there is an isomorphism $H_1(\partial X) \cong (L_G)^k \oplus (L_G)^k$. Duality implies that $dim(Im(i_*)) = dim((L_G)^k) = k \cdot |G|$. The intersection pairing on ∂X is G-equivariant. It induces a bilinear G-invariant pairing:

$$<,>: ker(i_*) \times Im(i_*) \to \mathbb{Q}$$

We note that this is well defined, since although the splitting which gives the direct sum decomposition is not natural, the ambiguity in a choice of element in $H_1(\partial X)$ representing an element of $Im(i_*)$ is an element of $ker(i_*)$. The intersection pairing vanishes on the subspace $ker(i_*)$ thus the ambiguity is erased by $<,>$.

The intersection pairing on a surface, . is non-degenerate and this implies that $<,>$ is nondegenerate. This gives an isomorphism of G-modules

$$Im(i_*) \cong Hom(ker(i_*), \mathbb{Q}) \cong ker(i_*).$$

The first isomorphism $\theta : Im(i_*) \cong Hom(ker(i_*), \mathbb{Q})$ is given by $[\theta(i_* x)](y) = x.y$ where x, y are in $H_1(\partial X)$ and . is the intersection pairing. Then θ is $\mathbb{Q}$-linear. Also, if $g \in G$ then

$$[\theta(g i_* x)](y) = (gx).y = x.(g^{-1} y)$$

since the intersection pairing is G-invariant. Thus $\theta(g i_* x) = \theta(i_* x) \circ g^{-1}$. The action of $g \in G$ on $\phi \in Hom(ker(i_*), \mathbb{Q})$ is $\phi \mapsto \phi \circ g^{-1}$. Hence θ is a $\mathbb{Q}[G]$-module map. The second isomorphism $\psi : Hom(ker(i_*), \mathbb{Q}) \cong ker(i_*)$ is defined the same way, but in place of $<,>$ we use a G-invariant positive-definite inner-product on $ker(i_*)$.

Recall that a module is *simple* if it has no proper submodules, and *semi-simple* if it is a direct sum of simple modules. Maschke's theorem ([1], p455) states that $k[G]$ is semi-simple if the characteristic of k does not divide the order of G. In our situation $k = \mathbb{Q}$ has characteristic zero, so the theorem applies.Since $\mathbb{Q}[G]$ is semi-simple, every $\mathbb{Q}[G]$-module, M, is semi-simple ([1], p446). The number of times a simple module appears (up to isomorphism) in a decomposition of M into simple submodules is independent of the decomposition ([1], p440). Now

$$(L_G)^k \oplus (L_G)^k \cong H_1(\partial X) \cong ker(i_*) \oplus Im(i_*) \cong ker(i_*) \oplus ker(i_*).$$

If we consider the decompositions of both sides into simple submodules and compare the number of times each simple module appears, we deduce that $ker(i_*) \cong Im(i_*) \cong (L_G)^k$. □

Corollary 2.2 *If ,in addition, the map i_* is surjective, then $H_1(X) \cong (L_G)^k$.*

Lemma 2.3 *In the above notation, suppose that the map i_* is surjective.*
Then $H_1(DX) \cong (L_G)^k \oplus (L_G)^k - < 1 >$, where DX denotes the double of X.

Proof. Choose basepoints p_L and p_R in the left and right copies of X inside DX and form the graph Γ by connecting the basepoints by one arc for each copy of a boundary torus of X which lies inside DX. The graph Γ admits an obvious G action and there is a retraction mapping $r : DX \to \Gamma$. This retraction is not G-equivariant, but the map induced on homology is. The exact sequence of the pair $(\Gamma, \{p_L, p_R\})$, together with the observation that our construction gives that $H_1(\Gamma, \{p_L, p_R\}) \cong (L_G)^k$ yields the short exact sequence

$$0 \to H_1(\Gamma) \to H_1(\Gamma, \{p_L, p_R\}) \cong (L_G)^k \to \mathbb{Q} \cong < 1 > \to 0$$

so that as G-modules we have $H_1(\Gamma) \cong (L_G)^k - < 1 >$.

Since $H_1(X)$ is carried by the boundary, $H_1(X) \rightarrow H_1(DX)$ is injective and we have a short exact sequence of $\mathbb{Q}[G]$ modules:

$$0 \rightarrow H_1(X) \cong (L_G)^k \rightarrow H_1(DX) \rightarrow H_1(\Gamma) \cong (L_G)^k - <1> \rightarrow 0$$

which implies the result. $\square$

We now seek to improve the module provided by Lemma 2.3. To this end we define a submodule V of a $\mathbb{Q}[G]$-module A to be **canonical in** A if it has the property that if V' is any submodule of A with $V' \cong V$, then $V' = V$. For our purposes it is sufficient to note:

Lemma 2.4 *The submodules L_G, $<1>$ and $L_G - <1>$ are all canonical submodules of L_G.*

Proof. Though this is standard, (see [3]) we include a proof for convenience. That L_G is canonical in itself is transparent. It is clear that the element $\sum_{g \in G} \lambda_g g$ of $\mathbb{Q}[G]$ is invariant under the action of G if and only if all the $\lambda_g \in \mathbb{Q}$ are equal. Thus there is a unique 1-dimensional subspace, $<1>$, on which G acts trivially. Hence $<1>$ is canonical. It follows that $L_G - <1>$ is canonical in L_G since it is the sum of all the simple submodules in L_G which are not isomorphic to $<1>$ and it is recovered as the orthogonal complement of any G-invariant inner product on L_G. $\square$

Proposition 2.5 *Suppose that M admits a free G-action so that*

$$H_1(M) \cong V \oplus W$$

where V is a canonical submodule of L_G. Then by doing Dehn surgeries on M, we may find another manifold with free G-action so that

$$H_1(M') \cong W$$

Proof. Firstly we note that the module L_G is cyclic, that is to say, there is a vector $v \in L_G$ so that the smallest $\mathbb{Q}[G]$-module containing v is all of L_G. This implies that any submodule V of L_G is also cyclic. This is because the G-action admits a G-invariant positive definite form. Form the orthogonal decomposition $L_G \cong V \oplus V^{\perp}$ then the orthogonal projection into V of any cyclic vector for L_G is a cyclic vector for V.

Choose a cyclic vector for the module V and represent it by an embedded simple closed curve $\gamma \subset M$. By general position we may assume that the G-translates of γ are all disjoint, so that as above, we may remove a small equivariant neighbourhood of γ to form a manifold $X = M \setminus G \cdot N(\gamma)$ with free G-action and $|G|$ torus boundary components.

Let $i : \partial X \longrightarrow X$ and $j : X \rightarrow M$ be the inclusion maps. Denoting the projection onto W by $\pi_W : H_1(M) \rightarrow W$, we obtain a G map $p = \pi_W \circ j_* : H_1(X) \rightarrow W$ which is clearly surjective.

Clearly $j_*(Im(i_*)) = <G \cdot \gamma> = V$ since γ generates the submodule V of $H_1(M)$.

We claim that in fact $ker(p) = Im(i_*)$. To this end, note that

$$ker(\pi_W) = V = <G \cdot \gamma> = j_* Im(i_*),$$

so that $Im(i_*) \subset ker(p)$. Moreover, if $\xi \in H_1(X)$ lies in $ker(p)$, this implies that $j_*(\xi)$ lies in $ker(\pi_W) = V$, and so by the observation of the above paragraph, $j_*(\xi) = j_*(\tau)$ for some $\tau \in Im(i_*)$. This implies that $\xi - \tau \in ker(j_*) \subset Im(i_*)$. It follows that $ker(p) = Im(i_*)$ and we have a short exact sequence:

$$0 \rightarrow Im(i_*) \rightarrow H_1(X) \rightarrow W \rightarrow 0$$

from which it follows that $H_1(X) \cong Im(i_*) \oplus W$, whence by Lemma 2.1 that $H_1(X) \cong L_G \oplus W$.

We wish to do an equivariant surgery on a boundary torus of X which kills all of the L_G part of $H_1(X)$.

Now $Im(i_*) \cong L_G \cong V \oplus A$ and the submodules of $Im(i_*)$ corresponding to V and A are unique because V is canonical in L_G. Thus there is a unique internal direct-sum decomposition $Im(i_*) = V' \oplus A'$ with $V' \cong V$. Doing equivariant surgery along the meridian $\mu \subset \partial X$ recovers the manifold M, and

$$H_1(M) \cong H_1(X)/i_* < G.\mu > \cong (Im(i_*)/ < G.(i_*\mu) >) \oplus W \cong V \oplus W.$$

Therefore $< G.(i_*\mu) >= A'$. Fix an element $\lambda \subset H_1(\partial X)$ represented by a simple closed curve so that $j_*(i_*\lambda) = \gamma$. Denote the projection onto V by $\pi_V : H_1(M) \longrightarrow V$. Thus $\pi_V j_* < G.(i_*\lambda) >= V$. Write $i_*\mu = (0, m)$ and $i_*\lambda = (\ell_1, \ell_2)$ as elements of $V' \oplus A' = Im(i_*) < H_1(X)$. Since $\pi_V j_*(i_*\lambda)$ is a cyclic vector for V it follows that ℓ_1 is a cyclic vector for V'. Also m is a cyclic vector for A'.

Fix some integer q and consider the $\mathbb{Q}[G]$-submodule, B, of $H_1(X)$ generated by the vector $i_*(\mu + q^{-1} \cdot \lambda) = (q^{-1}\ell_1, m + q^{-1}\ell_2) \in V' \oplus A'$. Let $\pi_{V'}, \pi_{A'}$ be the projection of $V' \oplus A'$ onto the factors. Then $\pi_{V'}B = V'$. Moreover the set of cyclic vectors for a given representation is open, so that for sufficiently large q, the vector $m + q^{-1}\ell_2$ continues to be cyclic for the submodule A'. Thus $\pi_{A'}B = A'$. It follows that the submodule $B = V' \oplus A'$ because it surjects to both factors, and the factors are canonical. Thus equivariant Dehn filling along $q \cdot \mu + \lambda$ produces a manifold with $H_1(M') \cong H_1(X)/B \cong W$ as required. □

The proof of Theorem 1.1 follows. Our introductory remarks constructed free G actions on some closed 3-manifold for any finite group G, we then perform the modifications to achieve the situation of Lemma 2.3 and multiple applications of Proposition 2.5 prove the result.

We conclude with a sketch that the homology sphere can also be chosen to be hyperbolic:

Theorem 2.6 *Let G be a finite group. Then there is an hyperbolic rational homology S^3 on which G acts freely.*

Proof. We have shown that there is a rational homology sphere M on which G acts freely. Consider the manifold M/G. By standard results, this manfold contains a simple closed curve K, so that $(M/G) - K$ is a complete hyperbolic 3-manifold with a single cusp. With a little more care one can arrange that the loop K lies in the kernel of the map $\pi_1(M/G) \to G$ defining the covering of M over M/G, so that K lifts to M with $|G|$ preimages.

By standard results (see [4]), all but finitely many surgeries on K yield a hyperbolic manifold, so that all but finitely many equivariant surgeries on $p^{-1}K \subset M$ give hyperbolic 3-manifolds. By Corollary 2.2, the action of G on $M - p^{-1}K$ gives an isomorphism of G-modules $L_G \cong H_1(M - p^{-1}K)$ and the meridian is a cyclic vector for L_G; whence all sufficiently close vectors on one of the boundary tori are also cyclic vectors; equivariant surgery along such a slope yields a hyperbolic manifold as required. □

References

[1] S. Lang. *Algebra.* pub. Addison-Wesley (1972).

[2] J. Milnor. *Groups which act on S^n without fixed points.* Amer. J. of Math. vol. 79 (1957) pp. 623 - 630.

[3] J. P. Serre. *Linear representations of finite groups.* Graduate texts in Mathematics, No. 42. Springer-Verlag 1977.

[4] W.P. Thurston. *The geometry and topology of* 3*-manifolds.* Princton University Press.

Department of Mathematics,
University of California
Santa Barbara, CA 93106.