Representation Theory and the A-polynomial of a knot.

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1 The motivation and basic idea.

A *knot* is a smooth simple closed curve, K, in the three-sphere, S^3. The *knot exterior* is the compact three-manifold $X = S^3 - \eta(K)$ where $\eta(K)$ is an open tubular neighborhood. The group $\pi_1(X)$ is called the *knot group* associated to the knot. We recall that the boundary of X is a torus T and there are two simple closed curves on X called a *longitude* and *meridian* which intersect transversely in a single point. These two curves generate $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$ which is usually referred to as the *peripheral subgroup*.

In the sixties, Waldhausen showed that the data consisting of the knot group plus the peripheral subgroup is a complete knot invariant. The difficulty is that although a powerful invariant it is extremely difficult to work with this data directly. One usually makes a trade-off and at the expense of some loss of information constructs a usable invariant. The Alexander polynomial is an example of this.

A classical way to study groups is to look at their linear representations and this suggests that one studies representations of knot groups into linear groups. Waldhausen's results suggest that one should also try and keep track of some of the peripheral data.

What is actually studied in this context is the totality of linear representations into $SL(2,\mathbb{C})$. (One can work with more general Lie groups but a good deal less is known. The isomorphism $PSL(2,\mathbb{C}) \cong Isom^+(\mathbb{H}^3)$ makes this choice occupy a special place in the current theory.) As we indicate below, given a finitely generated group, G, the set of representations of G into $SL(2,\mathbb{C})$ is an affine algebraic set which we will refer to as the *representation variety* even though it is typically a finite union of (irreducible) algebraic varieties in the sense of algebraic geometry.

Varieties are also difficult to work with directly, but when G is a knot group the peripheral subgroup of the knot provides a way to produce a more manageable algebraic set: One uses the longitude and meridian to project this representation variety into a smaller affine space which records the *restriction* of the representations of G to the peripheral subgroup.

One now obtains an invariant of the knot: an affine variety in an affine space with canonical coordinates determined by the longitude and meridian. This variety determines an ideal which is generated by a single integral polynomial in two variables, and this is the A-polynomial.

Work of Thurston on hyperbolic structures, and of Culler and Shalen imply that the A-polynomial is closely connected to many geometric and topological properties of the knot. In fact it turns out that the *Newton Polygon* of the A-polynomial contains a great deal of subtle information about the knot exterior.

*Both authors supported in part by the National Science Foundations

2 Definition of the A-polynomial.

In 1977, W. Thurston showed that many three-manifolds admit a so-called *hyperbolic structure* [25]. We do not give the technical definition since we shall not need it. However what will be important in the sequel is the consequence that the fundamental group of many manifolds have irreducible representations into the Lie group $Isom^+(\mathbb{H}^3) \cong PSL(2,\mathbb{C})$. It is shown in [9] that a representation corresponding to a complete hyperbolic structure can always be lifted to $SL(2,\mathbb{C})$, and it is $SL(2,\mathbb{C})$ with which we shall work.

Now observe that given any representation ρ of a group, we can obtain another representation ρ_A by stipulating that $\rho_A(g) = A \cdot \rho(g) \cdot A^{-1}$, where A is any matrix in $SL(2,\mathbb{C})$. One says that ρ and ρ_A are *conjugate*. In general these representations will be different, but the difference is not regarded as a substantial one, and one often works with representations up to conjugacy. (Or by using the closely related theory of characters.)

It makes sense then to ask how many different representations a 3-manifold group has up to conjugacy. In the case that the manifold has a torus boundary component, Thurston shows that there are many by proving that usually representations can be "deformed":

Theorem 2.1 *[25] Let X be a manifold with a single torus boundary component T^2. Suppose that $\rho : \pi_1(X) \to SL(2,\mathbb{C})$ is an irreducible representation with the property that $\rho(\pi_1(T^2))$ is not the trivial subgroup.*

Then the complex dimension of the space of small deformations of ρ up to conjugacy is at least one.

A word of explanation about the conclusion is in order as this will be the starting point of what follows. Suppose that $\pi_1(X)$ has a finite presentation $< g_1, ..., g_k \quad | \quad R_1, ..., R_l >$. Then we wish to assign matrices $G_i \in SL(2,\mathbb{C})$ to each g_i so that when we substitute these matrices into the relations $R_j(g_1, ..., g_k)$, we see the equation

$$R_j(G_1, ..., G_k) = I \qquad (*)$$

The representation ρ is then given by $\rho(g_i) = G_i$. Conversely, any representation describes such a collection of matrices G_i.

To construct such matrices, we regard each matrix G_i as a 2×2 matrix of indeterminates, $G_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ and formally make the substitution into all the relations: Each condition of the form $(*)$ yields 4 equations and hence we obtain a system of $4l$ polynomial equations which must be satisfied in order to have a representation. We also require each matrix has determinant 1, so that we obtain a further k equations of the form $a_i \cdot d_i - b_i \cdot c_i = 1$. Any complex solution to these equations gives rise to a representation of the group $\pi_1(X)$ and conversely, any representation gives rise to a solution to these equations. This shows that the set $Hom(\pi_1(X), SL(2,\mathbb{C}))$ has the structure of an *affine algebraic set* and that the tools from algebraic geometry can be brought to bear. The presence of the representation ρ shows that the set is not empty.

Thurston's theorem is proved by showing that the solutions near to the solution corresponding to ρ have complex dimension 4. The Lie group $SL(2,\mathbb{C})$ has complex dimension three, so that when we identify conjugate representations we obtain a one complex parameter family of solutions which turns out to be equipped with the structure of an affine algebraic curve.

One way of seeing this last statement is as follows. The boundary torus has fundamental group $\mathbb{Z} \oplus \mathbb{Z}$ and one can easily show the generic representation of this group into $SL(2,\mathbb{C})$ can be conjugated to be upper triangular and to have the shape:

$$\rho(\mu) = \begin{pmatrix} M & 1 \\ 0 & 1/M \end{pmatrix} \quad \rho(\lambda) = \begin{pmatrix} L & q \\ 0 & 1/L \end{pmatrix} \qquad (1)$$

No further conjugacy of ρ is possible once we have decreed it is of this form on the boundary torus, so that an informal description of the affine algebraic set of "representations up to conjugacy" is given by taking all the polynomial equations described above together with two more equations which specify that the lower-left entries of the two elements μ and λ are zero.

We shall denote the affine algebraic set constructed in this way by $R(\pi_1(X), SL(2,\mathbb{C}))$; Thurston's theorem shows that this set has a component (in the sense of algebraic geometry) of complex dimension at least one.

This idea was then taken up by Culler and Shalen in [9] and [10]. Somewhat amazingly, they showed that this information has implications for the geometry of the manifold. To state their results we need some definitions.

We will consider embeddings of connected orientable surfaces $i : S \to X$ which are *proper*, that is to say $i : \partial S \to \partial X$. We say such an embedding is *incompressible* if the induced map $i : \pi_1(S) \to \pi_1(X)$ is injective. A closed surface in the interior of X is called *boundary parallel* if it can be homotoped into the boundary of X. Usually one makes another stipulation that the surfaces are *boundary incompressible* but we will not define this formally. One can roughly think of this condition as requiring that there are no redundant boundary components. An incompressible, boundary incompressible surface which is not boundary parallel is called *essential*.

Essential surfaces are a powerful tool for understanding a manifold. In the context of knot complements, it was classically known that there is always at least one essential surface, a *Seifert surface.* This is an essential surface in the knot exterior with boundary a longitude of the knot. However it was a longstanding open question whether there was always a separating essential surface. This was resolved by:

Theorem 2.2 *(Culler-Shalen [10]) Suppose that X is an irreducible 3-manifold with a single torus boundary component. Then either $M \cong D^2 \times S^1$ or X contains a separating essential surface with nonempty boundary.*

We outline the major steps which lead to this result; a slightly more detailed (geometric) description of the ingredients in contained in §7.

Some classical 3-manifold topology shows that the crucial case is when X is hyperbolic; Thurston's theorem guarantees an affine algebraic curve of representations $\mathcal{C}$ of the fundamental group. Any affine curve has points at infinity (the so-called *ideal points*) and one fixes one of these and examines what happens to representations which are very close to this ideal point. One applies Bass-Serre theory [22] and deduces that the group $\pi_1(X)$ acts on a certain simplicial tree. This gives the algebraic information that $\pi_1(X)$ has a decomposition as a *graph of groups.*

Now there is a topological idea which goes back to Stallings [24] and Epstein [11] that a graph of groups decomposition of the fundamental group of a three-manifold, X, can be changed into a (possibly different) decomposition which is realized topologically in X. For example, if $\pi_1(X) = A*B$ is a free product decomposition, then there is a sphere embedded in X which separates X into two complementary components whose fundamental groups are A and B [24]. More generally if $\pi_1(X) \cong A *_C B$ is the free product of A and B amalgamated along a subgroup C then there is an essential surface F properly embedded in X with $\pi_1(F)$ injecting into the subgroup C of $\pi_1(X)$.

In summary, the argument so far has shown that to each ideal point one can associate a splitting of the group and hence an incompressible surface. However, these surfaces are possibly closed or possibly the Seifert surface. The theorem is completed by showing that for at least one ideal point one obtains a surface which is neither of these types; we now outline this step as it will be of interest to us in the sequel.

As representations move around on the curve $\mathcal{C}$, we can examine the behaviour of the pair of "eigenvalues" (M, L) which arise by restricting the representation to the boundary torus. In fact, if we fix coprime integers, a and b we can ask about the eigenvalue of the element $\mu^a\lambda^b \in \pi_1(T)$,

(namely $M^a \cdot L^b$) and with some care one can make sense out of the idea that for each (a,b), there is a holomorphic map $f_{a,b} : \mathcal{C} \to \mathbb{C}$ which assigns to a representation the eigenvalue at the representation of the element $\mu^a \lambda^b$.

Fixing some ideal point of $\mathcal{C}$, two kinds of behaviour are now possible for the functions $g_{a,b} = f_{a,b} + 1/f_{a,b}$ near this ideal point. One possibility is that for every choice of coprime (a,b), the functions $g_{a,b}$ remain bounded. One then shows that the surface associated to the splitting can be chosen to be closed. Otherwise, there is a single choice $\pm(a,b)$ for which $g_{a,b}$ remains bounded and for every other pair, the function $g_{a,b}$ has a pole at the ideal point. A surface $(S, \partial S)$ associated to this ideal point has boundary and has all its boundary curves parallel to the curve $\mu^a \lambda^b$. One says that the pair (a,b) is a *boundary slope*. Usually these are encoded as $a/b \in \mathbb{Q} \cup \{1/0\}$.

The argument of [10] concludes as follows. One can show that in this setting, the functions $g_{a,b}$ are always nonconstant, so no $g_{a,b}$ can be bounded at every ideal point. It follows that there are at least two distinct values of (a,b) which are boundary slopes and now an easy homology argument shows that one of them must correspond to a separating surface.

Some years later, these ideas were used to define a knot invariant, the *A-polynomial* [3]. Suppose that M is a knot complement and project the affine algebraic set $\xi : R(\pi_1(X), SL(2,\mathbb{C})) \to \mathbb{C}^2$, a map defined by $p(\rho) = (f_\mu(\rho), f_\lambda(\rho))$. It turns out that although the image of ξ need not be an affine algebraic set, it is very close to being one and if we take the closure (either in the classical or Zariski topology) we obtain a new affine algebraic set, the *eigenvalue variety* $\mathcal{E} = \overline{\xi(R)} \subset \mathbb{C}^2$. One can show (see below 3.8) that all the components of $\mathcal{E}$ have complex dimension zero or one. Discard the zero dimensional components.

Now we appeal to the crucial fact (see [15] Proposition 1.13) is that an irreducible algebraic hypersurface in $\mathbb{C}^2$ is the zero set of a single polynomial of two variables (in the language of algebraic geometry, the associated radical ideal is principal). Thus for each curve component $\mathcal{C}_i$ of $\mathcal{E}$ we may associate a two variable polynomial $c_i(M,L)$. We define

$$A_K(M,L) = \frac{1}{L-1} \prod_i c_i(M,L)$$

where the product is taken over the polynomials associated to each curve component of $\mathcal{E}$. The point of dividing out by the factor $L-1$ is that $H_1(X) \cong \mathbb{Z}$ for any knot so that we always have a component corresponding to abelian representations; these give rise to a factor $L-1$ since λ lies in the commutator subgroup of the knot group. This normalization arranges:

Corollary 2.3 *Let $K \subset S^3$ be the unknot. Then $A_K(M,L) = 1$.*

There is some ambiguity in A since one can regard it as the generator of an ideal, but it turns out that we may scale so that $A_K(M,L)$ is an integer polynomial and it is then defined up to multiplication by $\pm L^\alpha M^\beta$. There are two alternative normalizations. The first is to choose α, β so that the A-polynomial is unchanged by simultaneously replacing both L, M by their reciprocals, this is possible by 3.5. This results in a Laurent polynomial symmetric about the origin. The second is to choose α, β so that $A_K(L,M)$ is a polynomial of minimal total degree. With either of these conventions, the A-polynomial is defined up to sign.

Another matter of convention is whether or not one counts components of the variety corresponding to the A-polynomial with *multiplicity.* This corresponds to deciding on whether or not the definition of the A-polynomial should allow repeated factors. As these issues at the present time do not seem to be of much importance, we will ignore them here.

3 Properties of A_K.

It would appear that the A-polynomial, being a description of a polynomial ideal associated to the eigenvalues of representations encodes information about a knot which is somewhat obscure. However it turns out that a good deal of geometric information is captured. The most primitive example of this comes from a comparison with the Alexander polynomial; the precise result needs to be stated a little carefully (For which we reference [3] and [7]), but one can summarize roughly by saying that it is known that the A-polynomial contains at least as much information as the Alexander polynomial in the sense that if the Alexander polynomial is nontrivial then so is the A-polynomial. Moreover, conjecturally, one can actually recover the factors occurring in the Alexander polynomial from A, but although this is proven in many special cases, a proof that this works in general is lacking.

We now address some of the most basic issues attaching to any invariant. For example, one immediate question which arises is the question of how powerful an invariant A is. In this direction we note that it follows immediately from the definition:

Theorem 3.1 *Suppose that $K \subset S^3$ is any hyperbolic knot. Then A_K is nontrivial.*

This already covers a very large class of knots. In general, it seems probable that the A polynomial is nontrivial for any nontrivial knot, but this remains open because the following is unresolved:

Question 3.2 *Suppose that $K \subset S^3$ is a nontrivial knot. Does $\pi_1(S^3 - K)$ admit an irreducible representation into $SL(2, \mathbb{C})$?*

It is known that the A-polynomial is not a complete invariant of knots: it is shown in [5], that there are different knots with the same A-polynomial.

A related issue is how the invariant changes as one applies various symmetries to knot. It is easily seen that reversal of string orientation is not detected by A, but reflection potentially is:

Theorem 3.3 *If K and K' are mirror images, then $A_K(M, L) = A_{K'}(M, 1/L)$ up to powers of M and L.*

Example 3.4 One trefoil has polynomial $1 + M^6 L$, the mirror image has polynomial $L + M^6$. In particular, A is powerful enough to show that these are distinct knots.

Another basic issue with any invariant is the question of the set of values the invariant can take on. Some restrictions are easily seen:

Theorem 3.5 *Suppose that K is a knot in S^3. Then:*

- *$A_K(M, L)$ contains only even powers of M*
- *$A_K(M, L) = A_K(1/M, 1/L)$ up to powers of L and M*

These two properties can safely be left as an exercise in the definitions for the untiring reader. Slightly more subtle is:

Theorem 3.6 *Suppose that K is a knot in S^3 with the additional property that every closed incompressible surface embedded in $S^3 - N(K)$ is parallel to the boundary torus.*

Then $A(\pm 1, L) = n \cdot (L + 1)^{\alpha +} \cdot (L - 1)^{\alpha -} \cdot L^{\beta}$ for some nonzero integer n.

In general, specializing $M = \pm 1$ in A either corresponds to representations where the meridian has eigenvalue ± 1 or to points at infinity in the eigenvalue variety. However it turns out that this latter possibility is excluded by the hypothesis on closed surfaces. (See [8])

Once we know that the roots of $A(\pm 1, L)$ correspond to representations of the knot complement, it is not difficult to see that the only possibilities for such roots are $\{\pm 1, 0\}$. (In fact, it follows from [6] below that $n = \pm 1$.)

Question 3.7 *Do the constants $\alpha_\pm$ or β have any geometric significance?*

There is a much more interesting and less well-understood condition. It is shown in [3] (4.5) that one can associated to a representation of ρ of $\pi_1(X)$ into $SL(2,\mathbb{C})$ a *volume* and this assignment defines a continuous function $V : Hom(\pi_1(X), SL(2,\mathbb{C})) \longrightarrow \mathbb{R}$.

Briefly, the idea is that given a representation ρ, one chooses a nice ρ-equivariant map of the universal cover $\widetilde{M}$ of M into hyperbolic 3-space, $\mathbb{H}^3$. Use this map to pull-back the volume form on $\mathbb{H}^3$ to a $\pi_1(X)$-equivariant 3-form on $\widetilde{M}$. This descends to a form on M and integrating this over M gives the volume of the representation. In the case that the representation is the holonomy of a hyperbolic structure on M the volume defined this way equals the volume of M as a Riemannian manifold.

Fix a representation ρ and as usual let L, M be the eigenvalues of the longitude and meridian. Set

$$log(M) = \ell_\mu + i\theta_\mu \qquad log(L) = \ell_\lambda + i\theta_\lambda$$

Then define a 1-form ω on $(\mathbb{C}-0)^2$ by the formula $\omega = \ell_\mu d\theta_\lambda - \ell_\lambda d\theta_\mu$. This form is not exact since $d\omega = d\ell_\mu \wedge d\theta_\lambda - d\ell_\lambda \wedge d\theta_\mu$. However pulling back to a curve C in $Hom(\pi_1(X), SL(2,\mathbb{C}))$ gives the form $\xi^*\omega$ which is exact since it equals $\pm 2dV$. This formula is due to Hodgson, [16] see also [3]. One can use the fact that ω is not exact on $(\mathbb{C}-0)^2$ to obtain:

Corollary 3.8 *[3]* $\quad dim_{\mathbb{C}}(R) \leq 1.$

This leads to an obstruction to a polynomial arising as the A-polynomial of a knot, namely if γ is any loop on the curve $A = 0$ then the integral of ω around γ must be zero.

A specific example of this is the following: One finds that the polynomial of the figure eight knot is:

$$A(L,M) = -2 + M^4 + M^{-4} - M^2 - M^{-2} - L - L^{-1}.$$

Changing this slightly gives a different polynomial:

$$f(L,M) = -2 + M^6 + M^{-6} - M^2 - M^{-2} - L - L^{-1}.$$

One can use the volume form to show that this is not the A-polynomial of any knot. (See [5] for some details.) Nonetheless it does satisfy every other condition that we know of to be a knot polynomial.

Question 3.9 Which affine curves $\mathcal{C}$ in $(\mathbb{C}-0)^2$ satisfy the condition that ω is exact on $\mathcal{C}$?

A final basic problem is the issue of whether the invariant can be calculated. It turns out (see [3] or [5] for details) that in principle, one can compute A_K for any knot K. Nonetheless it involves rather a lot of calculation and all but the simplest examples require the use of a computer. In this sense, the invariant appears to have a different character to other knot polynomials (e.g. Alexander and Jones), in that it seems unlikely that there are iterative crossing change formulae.

4 The Newton polygon and boundary slopes

The most important and deepest properties of the A-polynomial come from the fact that it encodes information about the boundary slopes of the knot, via its so-called Newton polygon.

If $p(x,y)$ is a polynomial in two variables the *Newton polygon* of p is the convex hull of the finite set of points in the plane:

$$Newt(p) = \{\ (i,j)\ :\ \textit{the coefficient of } x^i y^j \textit{ in } p(x,y) \textit{ is not zero}\ \}.$$

The sides of the Newton polygon describes the geometry of the curve $\mathcal{C}$ defined by $p = 0$ when at least one of the coordinates is near zero or infinity. To see this, suppose that (X, Y) is a point on C and that at least one of the variables, for example X, has large modulus. The polynomial p is a linear combination of monomials of the form $x^a y^b$ and the logarithm of the modulus of this monomial at (X, Y) is $\phi(a, b) = a \log |X| + b \log |Y|$.

Since $p(X, Y)$ vanishes there cannot be a single monomial which is far larger in modulus than all the other monomials. One thinks of ϕ as a linear map defined on $\mathbb{R}^2$ and in particular on the Newton polygon. The level sets of ϕ are straight lines with slope $-\log |X|/\log |Y|$. By the previous discussion there is a side of $Newt(p)$ which is nearly parallel to these lines. Similar considerations hold if X is very close to zero. From this one sees that to each topological end of the curve $\mathcal{C}$ one may assign an edge e of $Newt(p)$ consisting of those terms of p of approximately largest modulus for points on $\mathcal{C}$ near the given end.

This gives rise to a connection between sides of the Newton polygon and boundary slopes which we now sketch. Suppose that ρ_n is a sequence of representations which are "blowing up" on the boundary torus in the sense that at least one of the longitude or meridian eigenvalue is converging to 0 or ∞. (See §7) Then as we argued above, there is an edge, e, of $Newt(A(L, M))$ containing the terms of largest magnitude. If $L^a M^b$ and $L^c M^d$ both lie on e then $L^a M^b \approx L^c M^d$ and so $M^{b-d} L^{a-c} \approx 1$. Thus $\rho_n(\mu^{b-d} \lambda^{a-c})$ has bounded eigenvalues and we are the situation of Theorem [9] so that $\mu^{b-d} \lambda^{a-c}$ is a boundary slope. The slope of this curve on the boundary torus is $\frac{b-d}{a-c}$ which equals the slope of e. This proves one of the most important properties of A:

Theorem 4.1 *[3] Slopes of edges of the Newton polygon A_K are boundary slopes of the knot K.*

As remarked above, although in general quite difficult, in principle the A-polynomial can always be computed so that this theorem gives a way of computing boundary slopes of any knot.

The question which this immediately raises and which remains open is:

Question 4.2 *Does A_K detect **all** the boundary slopes of K except possibly $0/1$ for fibered knots?*

There seems to be some evidence that this is true. A proof would give a new proof of a theorem of Hatcher:

Theorem 4.3 *[13] A manifold with a single torus boundary component has only finitely many boundary slopes.*

We briefly explain the reason for the exclusion of the slope $0/1$ in Question 4.2. A knot may have the property that it can be written as a punctured surface bundle over a circle; such knots are called *fibered* knots and the surface in question (which is always unique up to isotopy) is called the fiber. It always has slope $0/1$ and is the only incompressible nonseparating surface with this slope. It is known (see [3]) that this machinery applied to *irreducible* representations can never detect fibers and it follows that if the knot K is fibered and has no separating surface of slope $0/1$, the A-polynomial will not detect $0/1$. This is not an uncommon situation. However, if one adopts the convention that the abelian representations of the knot complement are also included, the above machinery does detect fiber surfaces.

A beautiful example of how this apparent disadvantage can actually be exploited is given by the A-polynomial of the knot 8_{20} shown in Figure 1. This knot is known to be fibered, however one sees that the Newton polygon has detected a slope of $0/1$; this means that there is also a *separating* surface in this knot complement with slope $0/1$. This is the first knot in the table with this property.

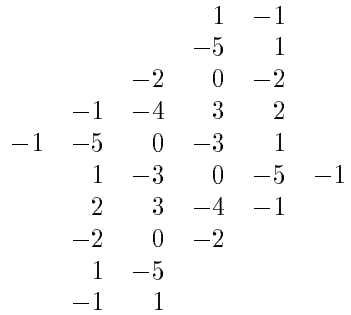

Figure 1.

The convex hull of the polygon in the figure shows that 8_{20} has slopes $\{0, 8/3, -10\}$. It is possible to show that these are only slopes of the knot. The presence of the nonintegral slope $8/3$ is also interesting; it is the first knot in the table with such a slope.

In fact there is even more geometric information which comes from the edges. The terms of the A-polynomial appearing along an edge e of its Newton polygon may be viewed as a polynomial in a single variable t called an *edge polynomial* $f_e(t)$. It turns out that the variable t may be identified with an eigenvalue of the loop β on the boundary torus which is the boundary slope of an incompressible surface in the knot complement. Thus $t = L^q M^p$ if the slope of β is p/q.

Theorem 4.4 *[3],[4],[6] The edge polynomial $f_e(t)$ is a product*

$$n.f_1(t).f_2(t)\cdots f_n(t)$$

where n is an integer and $f_i(t)$ is a cyclotomic polynomial.

If ω is a p'th root of unity which is a zero of $f(t)$ then p divides the number of boundary components of every component of an incompressible surface associated to the action on a tree arising from a degeneration corresponding to the edge e.

In summary, with the interpretation of the previous paragraph, if one goes to infinity in the eigenvalue variety, then the limiting value of the eigenvalue of the boundary slope is a root of unity. Moreover, the degree of this root of unity has consequences for the surfaces which arise from the degeneration. One interesting aspect of this theorem is the diverse methods of proof which are available to prove it. It can be proved geometrically, or by using Algebraic K-theory or by using the volume form together with Theorem 4.6 below.

Definition 4.5 *A **corner** of a polynomial $p(x, y)$ in two variables is a term appearing in a corner of the Newton polygon of p.*

One might view corners as analogous to the first and last term in a polynomial in a single variable, in which case one interprets the theorem which follows as saying that the A-polynomial is monic if we assume that it has been normalized so that the greatest common divisor of its coefficients is 1.

Theorem 4.6 *[6] The coefficients of terms in the corners of the A-polynomial are ± 1.*

This is perhaps the deepest property currently known about A, for example it is stronger than the roots of unity theorem. It has various other consequences, here are two simple ones:

Corollary 4.7 *The constant n appearing in Theorem (4.4) is ± 1.*

Corollary 4.8 *The edge polynomials of a 2-bridge knot are all $\pm(t-1)^k(t+1)^l$.*

Proof. It is shown in [14] that an incompressible surface in a 2-bridge knot has one or two boundary components. The above theorems now give the result. □

Example.

Figure 2 below illustrates all these results. It shows the Newton polygon of the two-bridge knot 8_1 exhibiting the boundary slopes $\{12, 0, -4\}$, corner entries of ± 1 and edge polynomials $t-1$, $(t-1)^2$ and $(t-1)^3$.

		1	−2	1		
		−1	6	−5		
				6	−3	
			−4	5	10	
				3	0	3
				−10	−12	−5
−1	−4	−10	−20	−10	−4	−1
	−5	−12	−10			
	3	0	3			
		10	5	−4		
		−3	6			
			−5	6	−1	
			1	−2	1	

Figure 2.

5 Dehn surgery and Property P.

Given a knot complement $X = S^3 - N(K)$ and a boundary slope a/b one can obtain a closed manifold as follows. Fix a closed curve $\alpha \subset \partial X = T^2$ representing the slope a/b. Then a regular neighbourhood of α in ∂X is homeomorphic to $S^1 \times I$. Attach a $D^2 \times I$ to X along ∂X by identifying $\partial D^2 \times I = S^1 \times I$ with this regular neighbourhood of α. This yields a manifold with a single boundary component which is a 2-sphere. We cap this off with a 3-ball to obtain a closed manifold we denote by $X(a/b)$ or $X(\alpha)$ if we have not chosen meridian-longitude coordinates. We say that $X(a/b)$ is obtained from X by *Dehn filling* along the slope a/b. (Or *Dehn surgery* depending whether your preference is for dentists or doctors.)

Example 5.1 *If $K \subset S^3$, then $X(1/0) = S^3$*

An easy exercise in the Van Kampen theorem shows that $\pi_1(X(\alpha))$ is obtained from $\pi_1(X)$ by adding the single relation which says that the element of fundamental group represented by α is trivial, that is to say, $\pi_1(X(\alpha))$ has presentation $\pi(X(\alpha)) \cong < \pi_1 X \mid \alpha = 1 >$.

One of the major thrusts of knot theory in recent years has been directed towards an understanding of which manifolds can be obtained as a result of Dehn surgery along some knot and the extent to which the geometry of the manifold must persist in the Dehn filled manifold. The usual flavour of the theorems proved in recent years has been that although a complicated manifold may become simpler (the above example shows that a hyperbolic knot complement may admit a surgery which is S^3) this either happens finitely often or only for a very controlled set of surgeries. A very striking example is:

Theorem 5.2 *[8] Suppose that $X = S^3 - N(K)$ is a hyperbolic knot complement. Then there are at most three surgeries on X for which X_α has cyclic fundamental group.*

Examples exist which show that the number three cannot be improved.

The Poincaré conjecture asserts that a closed, simply-connected three-manifold is homeomorphic to S^3. One motivation for this theorem comes from the question of whether one can do surgery on a knot and obtain a counterexample to the Poincaré conjecture. A knot for which is it impossible to do surgery and obtain a counterexample to the Poincaré conjecture is said to have *Property P*. Many classes of knots are known which have Property P, but it remains open in general.

The sharpest methods available at the moment split knots into two types:

- Knots which contain an essential closed surface in the exterior.
- Knots which contain no such surface.

The A-polynomial contains information about this problem at least in the case of the latter type of knot:

Theorem 5.3 *[3] Suppose that there is no essential closed surface in the exterior of a knot K. Then K has property P if the degree of the A-polynomial in M is more than twice the degree in L.*

All known A-polynomials satisfy this condition. It is easy to show that the M degree is always at least the L degree.

In fact in our current state of ignorance, the restriction to knots which do not contain an essential, closed surface may be completely unnecessary. As hinted above, the reason for imposing such a condition arises from ideal points lying at infinity in $R(\pi_1(X), SL(2,\mathbb{C}))$ which project to finite points in $\mathcal{E}$. A sequence ρ_n in $R(\pi_1(X), SL(2,\mathbb{C}))$ *blows up* if there is α in G such that $trace(\rho_n\alpha) \to \infty$. A *hole* is a point in $(\mathbb{C}-0)^2$ which is the limit of the projection, by ξ, of a sequence ρ_n that blows up. If there is an essential torus T in X then one may slide the representation along T by conjugating the representation on one side of T by an element that commutes with $\pi_1 T$. A similar construction can be done if there is an essential annulus. These give rise to holes. However the situation is somewhat special, in particular if X is hyperbolic then there are no such annuli or tori.

Question 5.4 *If X is hyperbolic does $\mathcal{E}$ have any holes?*

If the eigenvalue variety of a knot has a hole, there must be a closed incompressible surface in the complement of the knot. No holes have ever been seen but this may be due to the complication of the knots which could possibly have holes. It would be convenient if the answer to this question were negative, but there seems to be no reason to believe this.

Our remarks at the beginning of this section show that a representation of the knot exterior which kills $\lambda^p\mu^q$ induces a representation of the corresponding Dehn-filled manifold $X(p/q)$. Killing $\lambda^p\mu^q$ is almost the same as requiring that the eigenvalue, L^qM^p, of $\lambda^p\mu^q$ is ± 1. It is not quite the same thing because of parabolics. However one can show:

Proposition 5.5 *[8] also [2](2.1). Suppose that there is no essential closed surface in the exterior of a knot K. Suppose that L, M are complex numbers with $L^qM^p = \pm 1$ and that $A_K(L,M) = 0$ and that either $L \neq \pm 1$ or $M \neq \pm 1$. Then Dehn filling K along $\lambda^q\mu^p$ produces a closed manifold with non-cyclic fundamental group.*

Because of the problems they cause, we will call a boundary slope, α, a *pesky boundary slope*, or *pb*-slope if it satisfies two conditions:

- Dehn-filling along α yields a manifold with cyclic fundamental group.
- There is a degeneration of the knot exterior which produces an essential surface in the knot exterior with slope α and such that the limiting eigenvalue of α is ± 1.

We will call a knot *nice* if it has no pb-slopes. In his thesis [23], Shanahan gives a necessary condition based on the Newton polygon for a Dehn-filling on a nice knot to give a manifold with cyclic fundamental group. Shanahan defines, for each rational direction, a *width* of the Newton polygon in that direction as follows. Suppose that $\mathcal{P}$ is a polygon in the plane whose vertices have integer coordinates. Given $p/q \in \mathbb{Q} \cup \{1/0\}$ define the *p/q-width* of $\mathcal{P}$ to be one less than the number of lines of slope p/q which intersect $\mathcal{P}$ and which contains a point with integer coordinates. Thus the 1/1-width of a unit square is 2. The following is derived in [23] using the work of Culler and Shalen, [9], [8].

Theorem 5.6 *[23] Suppose K is a nice knot. If p/q is a cyclic filling, the p/q-width is minimal over all rational directions. Given a polygon, there are at most three such minimal-width directions.*

For nice knots, this shows that the *shape* of the Newton polygon determines the possibilities for cyclic surgeries. In view of this the following question seems natural. An affirmative answer would also enable the hypothesis of no essential closed surface to be removed from 5.3 and 5.5.

Question 5.7 *Is every knot nice?*

6 Geometric Structures and the A-polynomial.

In this section we discuss how for hyperbolic knot complements the A-polynomial gives information about the hyperbolic 3-manifolds obtained by Dehn-filling along the knot. There are in general many components of $R(\pi_1 X) \equiv Hom(\pi_1 X, SL_2\mathbb{C})$, but if the knot is hyperbolic there is a particular component, R_0, which contains the holonomy, ρ_0, of the complete hyperbolic structure. One corollary of Thurston's theorem 2.1 is that it there is a neighborhood, U, of ρ_0 in R_0 which corresponds to *incomplete* hyperbolic structures on X. This means that for each $\rho \neq \rho_0$ in U there is a hyperbolic metric on X which is metrically incomplete. Taking the metric completion yields a compactification $\overline{X}_\rho$ of X. This compactification is either the one-point compactification of X, which is therefore not a manifold. Or else it is a closed 3-manifold obtained by *Dehn filling.* Which of these possibilities arises depends on ρ.

The topology of the completion depends only on the restriction of ρ to the fundamental group of T. If α is an essential simple closed curve on T and if $\rho(\alpha) = \pm I$ then $\overline{X}_\rho$ is homeomorphic to $X(\alpha)$. Since the A-polynomial encodes which representations of $\pi_1 T$ one obtains by restriction of a representation of $\pi_1 X$, this means that the A-polynomial contains information about the hyperbolic 3-manifolds obtained by Dehn-filling along the knot. The *core curve* of a Dehn-filling is an unknotted simple closed curve in the solid torus glued on to the knot exterior which generates the fundamental group of the solid torus.

Proposition 6.1 *Given a hyperbolic knot exterior X then for all but finitely many homotopy classes of α the manifold $X(\alpha)$ is hyperbolic. The holonomy, ρ, of $X(\alpha)$ lifts to a representation ρ of $\pi_1 X$ satisfying $\rho(\alpha) = \pm I$. If $\alpha = \mu^p \lambda^q$ then there are integers r, s with $pr - qs = 1$ and the core curve is homotopic in $X(\alpha)$ to $\mu^s \lambda^r$. The holonomy of the core curve satisfies trace$(\rho\beta) = T + 1/T$ where $T = M^s L^r$ and L, M satisfy the two equations*

$$A(L, M) = 0 \qquad M^p L^q = \pm 1.$$

However, in general there are other components of $R(\pi_1 X)$ other that R_0 and so this is not the end of the story. It does raise the question of what subset of R_0 corresponds to incomplete hyperbolic structures on X, the so-called *hyperbolic region* of Dehn-surgery space. This is intimately connected to the Geometrization conjecture for closed 3-manifolds.

For a hyperbolic knot, the unique complete hyperbolic structure on the knot complement gives rise to a Euclidean similarity structure on the boundary torus T. Such a structure is determined by a complex number $x + iy$ with $y > 0$ called the *cusp constant.* In [5] it is shown that the cusp constant is a root of a polynomial derived from the A-polynomial.

7 Degenerations, Actions on Trees, and Incompressible Surfaces.

If G is a finitely generated group then a sequence of representations of G into $SL(2, \mathbb{C})$ which goes to infinity gives rise to an action of G on a tree. By classical topology, if M is a 3-manifold and $G = \pi_1 M$ acts on a tree without a common fixed-point then one obtains an incompressible surface in M. This theory was described in [9] in terms of the Bass-Serre theory applied to $SL(2, K)$ where K is a field equipped with a discrete rank-1 valuation. There is a less elegant and more geometric way to describe the idea, and this was done in [1], [4]. We will recapitulate the geometric approach, treading lightly over details which may be found in the above.

First one must confront the problem that conjugacy poses. We think of a representation of a finitely generated group, G, as an assignment of matrices in $SL(2, \mathbb{C})$ to each of some fixed finite set of generators of G. Given a representation one may conjugate it to obtain another representation. Doing this repeatedly sometimes gives a sequence of representations which goes to infinity in the sense that some matrix entries go to infinity. Conjugation corresponds to change of basepoint (and base-frame) and does not tell us anything about G. An obvious way to overcome this is to consider, in place of the representation, the *character* of the representation, the point being that the trace of a matrix is not changed by conjugacy. This approach leads to the consideration of the closely related *character variety* instead of the representation variety. We will not do this in this article.

The idea for understanding degenerations geometrically is roughly the following. Let G be a finitely generated group and (W, d_W) a metric space. Consider a sequence, ρ_n, of representations of G into the group, $Isom(W)$, of isometries of W. We will say that this sequence *blows up* if there is an element g of G such that the sequence $\rho_n(g)$ cannot be conjugated to lie in any compact subset of $Isom(W)$. Suppose that it is possible to linearly rescale the metric on W to obtain a sequence of metric spaces, W_n, which converge in some sense to a limiting metric space $\overline{W}$. We want to do this in such a way that the actions ρ_n on the rescaled spaces W_n have a subsequence that converges to an interesting action, $\overline{\rho}$, by isometries on $\overline{W}$. The particular case we have in mind is that $W = \mathbb{H}^3$ and $\overline{W}$ is some sort of tree.

Example 7.1 Let G be free non-abelian group on two generators α, β, and suppose that W is $\mathbb{R} \times S^1$ metrized as a right circular cylinder. The isometry group of W is $Isom(W) \cong Isom(\mathbb{R}) \times Isom(S^1)$ which has four connected components. The component containing the identity is isomorphic to the Lie group $\mathbb{R} \times S^1$. Suppose that we have a sequence, $\rho_n : G \longrightarrow Isom(W)$ which is blowing up. Then we can choose a subsequence (which for simplicity of notation we assume to be the original sequence) and $\lambda_n \to \infty$ with the following property. Rescale the metric on W by multiplying by λ_n^{-1} to obtain a metric space $W_n \equiv (W, \lambda_n^{-1} d_W)$. Then W_n converges as a metric space to $\overline{W} = \mathbb{R}$ with the usual metric. Furthermore we may arrange that the action of ρ_n on W_n converges to an isometric action of G on $\mathbb{R}$ such that some element of G is a non-zero translation. Rescaling the metric on W shrinks the circle factor until in the limit it has become single point.

A direct attempt to do this with hyperbolic space $\mathbb{H}^3$ in place of W leads to complications. Instead one pulls back the metric from $\mathbb{H}^3$ to a metric on G. This metric is rescaled, and under suitable hypotheses converges to a generalization of a tree, a so-called $\mathbb{R}$-tree.

Here are a few more details, see [1] for a treatment from this point of view. Consider the *complete graph* $K(G)$ with vertex set the elements of G and with an edge connecting every pair of vertices. Given a representation $\rho : G \longrightarrow Isom(\mathbb{H}^3)$ we choose a G-equivariant *straight map* $\phi : K(G) \longrightarrow \mathbb{H}^3$ as follows. First, given ρ, one chooses a point x in $\mathbb{H}^3$ called an *approximate center* for ρ. This is a point with the property that the greatest distance x is moved by any of a fixed finite generating set of G is within 1 of the infimum of of this quantity. Given x define $\phi(g) = (\rho g)(x)$ and then extend ϕ over the edges of $K(G)$ by mapping an edge e with endpoints g_1, g_2 to the unique geodesic segment in $\mathbb{H}^3$ with endpoints $\phi(g_1), \phi(g_2)$.

The map ϕ need not be injective, hence the pull-back via ϕ of the hyperbolic metric to $K(G)$ is a pseudo-metric. If one identifies points in $K(G)$ which are zero distance apart in this pseudo-metric then one obtains a metric space which is isometric to the image of ϕ. Given a sequence of representations of G one obtains a sequence of pseudo-metrics on $K(G)$. Now a pseudo-metric is just a symmetric function

$$K(G) \times K(G) \longrightarrow \mathbb{R}$$

satisfying the triangle inequality. One shows that this sequence of pseudo-metrics can be rescaled suitably so that a subsequence converges to a pseudo-metric on $K(G)$. The metric space obtained from this pseudo-metric on $K(G)$ obtained by identifying points which are zero distance apart is an $\mathbb{R}$-tree. The reason for this is that a triangle in $\mathbb{H}^3$ is *thin* meaning that every point on one side of the triangle is within a distance of at most $\Delta = ln(1+\sqrt{2})$ of the union of the other two sides. The rescaled hyperbolic metrics pulled back to $K(G)$ have the property that each side of a triangle in $K(G)$ is within a distance $\Delta\lambda_n^{-1}$ of the union of the other two sides. The limit as $\lambda_n \to \infty$ gives a metric such that each side of a "triangle" is within a distance 0 of the union of the two other sides. Thus every triangle in $K(G)$ has collapsed to an interval or a figure Y, and $K(G)$ has collapsed to an $\mathbb{R}$-tree T.

The next step is to find a G-invariant simplicial subtree T_0 contained in T. Given an isometry τ of metric space X, the *translation length*, $t(\tau)$, of τ is defined to be the infimum of the distance between x and τx as x varies over X. In particular if τ has a fixed point then the $t(\tau) = 0$. Given an action $\rho : G \longrightarrow Isom(W)$ the *translation length function* associated to ρ is

$$L_\rho : G \longrightarrow [0, \infty)$$

given by $L_\rho(g) = t(\rho(g))$. We have produced an action of G on the $\mathbb{R}$-tree T. The metric on T is only defined up to rescaling by multiplying by a constant. Suppose that for this action the translation length of every element of G is integral. One then shows by a combinatorial argument that T contains a G-invariant simplicial subtree.

To obtain the condition on integral translation length one shows that if C is an affine algebraic curve contained in $R(G)$ and that ρ_n is a sequence of points in C which blow up and which converge to some end ϵ of C then the translation length function for the action on the $\mathbb{R}$-tree, T, obtained above can be scaled to be integral valued. This is shown as follows. Suppose we have a sequence of representations $\rho_n : G \longrightarrow SL(2, \mathbb{C})$ which converge in the above sense to an action on an $\mathbb{R}$-tree $\sigma : G \longrightarrow Isom(T)$. Let $L_{\rho_n} : G \longrightarrow \mathbb{R}$ be the translation length function of ρ_n and L_σ that for σ. Then there are positive real numbers λ_n such that $\lim_{n\to\infty} \lambda_n^{-1} L_{\rho_n} = L_\sigma$.

Given an element A of $SL(2, \mathbb{C})$ the translation length of A thought of as an isometry of $\mathbb{H}^3$ is $2cosh^{-1}|trace(A)/2|$. Given an element g of G and a representation $\rho \in C$ the translation length of $\rho(g)$ is a real valued function on the curve C, denoted $t_g : C \longrightarrow \mathbb{R}$. Now $p(\rho) = trace(\rho g)$ is a Laurent polynomial function on the curve C. Also $t_g(\rho) = 2cosh^{-1}|t_g(\rho)| \approx 2ln|t_g(\rho)|$. The rate at which this polynomial goes to infinity as $\rho \to \epsilon$ determines the translation length of g on the limiting $\mathbb{R}$-tree. We illustrate this with an example:

Example 7.2 Consider the special case that the affine curve C is just the complex plane $\mathbb{C}^1$. Then $p(\rho)$ is a Laurent polynomial $z^k + a_{k-1}z^{k-1} + \cdots + a_l z^{k-l}$. This has the same magnitude as z^k when

z is large. Also $cosh^{-1}|z^k| \approx k\ ln|z|$ when z is large. Thus scaling the translation lengths of ρ_n by dividing by $ln|z_n|$ shows that L_σ may be scaled to be integral. One thus sees that $L_\sigma(g)$ is the degree, k, of the Laurent polynomial $p(\rho)$ on the curve C.

More generally, given an affine curve C and an end ϵ there is a function which assigns to each polynomial function an integer which generalizes the idea of the degree of a polynomial. This integer measures the rate at which the polynomial goes to infinity as you go out along C towards the end ϵ. This function is the *valuation* on C associated to the end ϵ. (This is the valuation used on the function field in [9]. See [1] for an geometric description.) The translation length $L_\sigma(g)$ of the element g on the $\mathbb{R}$-tree T is the valuation evaluated on the polynomial $p(\rho)$, and in therefore integer valued.

A simple example of this theory arises from looking at abelian representations of the knot group, G, into $SL(2,\mathbb{C})$. The abelianization of G is $\mathbb{Z}$ with the meridian abelianizing to 1 and the longitude to 0. Given a non-zero complex number, t, there is an abelian representation ρ_t which sends the meridian to the diagonal matrix with diagonal entries $t, 1/t$. As $t \to \infty$ the tree produced by re-scaling is the real line, and G acts by integer translations.

If a group G acts on a simplicial tree T then there is associated to this action a decomposition of G as a *graph of groups* constructed as follows: The space obtained from T by identifying points in the same orbit under G is a simplicial graph Γ. If G is finitely generated and T has no invariant subtree, then Γ is a finite graph. To each vertex and edge of Γ one assigns a group and also a monomorphism from the group assigned to an edge to the group assigned to the vertex at either end, see [22] also [1]. For each vertex, v, in Γ one chooses a vertex in $\tilde{v}$ in T which projects to v, then the group assigned to v is the *stabilizer* of $\tilde{v}$, in other words the elements of G which fix $\tilde{v}$. Similarly groups are assigned to edges.

As described earlier, an action of the fundamental group of a 3-manifold gives rise to an essential surface. The A-polynomial gives information about how these incompressible surfaces in the knot exterior intersect the boundary torus ∂X. The idea is that representations of $\pi_1 X$ which degenerate can be restricted to $\pi_1 \partial X$ to give limiting actions of $\pi_1 \partial X$ on a tree. Either the action of $\pi_1 \partial X$ on the tree has a common fixed point. In this case there is a *closed* surface F in X which is disjoint from ∂X associated to the degeneration. The other possibility is that the action of $\pi_1 \partial X$ on the tree T contains an invariant subtree which is a copy of $\mathbb{R}$. Then $\pi_1 \partial X$ acts by integer translations on $\mathbb{R}$. The kernel of this action is an infinite cyclic subgroup of $\pi_1 \partial X$ and a generator of this subgroup is represented by a simple closed curve, α, on ∂X. Then any surface F associated with this degeneration intersects ∂X in curves parallel to α. Such a curve is a *boundary slope* of the knot.

The A-polynomial tells us which boundary slopes α arise this way. Recall that the traces of an element is large if and only if the translation length of that element is large. One is thus looking for complex numbers L, M such that $A(L,M) = 0$ and also so that at least one of $L + 1/L, M + 1/M$ is near infinity. The elements of $\pi_1 \partial X$ act by translation on $\mathbb{R}$ and the translation distance is determined by how fast the trace of the element goes to infinity under the degeneration. The element α is some element whose translation length, hence trace, remains bounded as the representation goes to infinity. Now as elements of $\pi_1 \partial X$ we have $\alpha = \mu^p \lambda^q$ for some coprime integers p, q. The eigenvalues of α are thus $(M^p L^q)^{\pm 1}$. We thus seek L, M as above with $M^p L^q$ bounded away from 0 and ∞. Clearly this information is contained in the A-polynomial, in fact via its Newton polygon.

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