Symplectic mapping class groups of rational 4-manifolds

Quantitative aspects in symplectic geometry, QG&T@OSU

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Questions on Torelli symplectomorphism groups

Definition (Symplectic rational 4 manifolds, reduced forms)

X is a **Rational 4 Manifold** *if it is diffeomorphic to* $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k \geq 0$ *or* $S^2 \times S^2$ *with the standard smooth structure.*

Choose basis $\{H, E_1, \cdots, E_n\}$ *of* $H_2(X, \mathbb{Z})$*. Any symplectic form* ω *on* X *is diffeomorphic to a* **reduced form** *with area* $(1|c_1, \cdots, c_n)$ *on the basis such that* $1 > c_1 \geq c_2 \geq \cdots \geq c_n > 0 \quad and \quad 1 \geq c_i + c_j + c_k$.

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Theorem (With Tian-Jun Li and Weiwei Wu)

A1: Yes. $Symp_h(X,\omega) \subset Diff_0(X)$, $\forall \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, $k \geq 0$, $\forall \omega$.

A2: Not always. For almost all ω, $\pi_0(Symp_h(X,\omega)) = \{1\}$; *But,*
if $\omega = (1|a, \frac{1-a}{2}, \frac{1-a}{2}, \frac{1-a}{2}, \frac{1-a}{2}, \cdots)$, $\frac{1}{3} < a < 1$, $\pi_0(Symp_h) = PB_m(S^2)$.

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Quantitative idea, 5-point blowup as an example

Without changing $\pi_0(Symp_h)$, ω can be deformed to be a **semi-toric** $\mathbb{R}P^2$**-packing form**, i.e. $c_k < \frac{1}{2}, \sum_k c_k < 2$.

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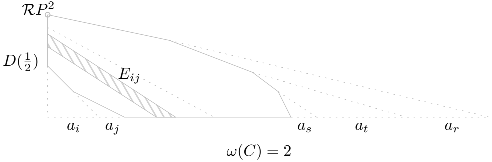

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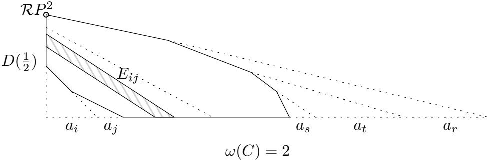

Thank You!