ORDINARY FAMILIES OF KLINGEN EISENSTEIN SERIES ON SYMPLECTIC GROUPS

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ABSTRACT. We construct $(n+1)$-variable Hida families of Klingen Eisenstein series on $\mathrm{Sp}(2n+2)$ for n-variable Hida families on $\mathrm{Sp}(2n)$, and relate their images under the Siegel operator to p-adic L-functions. We also carry out some preliminary calculation on the non-degenerate Fourier coefficients of the constructed Klingen Eisenstein families.

CONTENTS

1. INTRODUCTION

In his proof of the converse to Herbrand's Theorem, K. Ribet constructed Selmer classes for odd powers of Teichmüller characters via congruences between Eisenstein series and cuspidal modular forms on $\mathrm{GL}(2)_{/\mathbb{Q}}$. Later, the strategy of showing lower bound for Selmer groups via congruences between suitable automorphic forms with reducible and irreducible Galois representations has been greatly developed, and successfully exploited in the proof of the Iwasawa main conjectures for totally real fields [MW84, Wil90] and CM fields [HT94, Hsi14], as well as for symmetric square of elliptic curves over $\mathbb{Q}$ [Urb01, Urb06], elliptic curves over $\mathbb{Q}$ and over a imaginary quadratic field [SU14, Wan13].

The semi-simple Galois representation attached to an Eisenstein series is decomposable. Its congruence with irreducible Galois representations gives rise to nice classes in the first Galois cohomology group, and is expected to be closely connected with the "constant terms" of the Eisenstein series. Meanwhile, the "constant terms" are known to be closely related to special values of L-functions. In this article, we consider Klingen Eisenstein series attached to cuspidal automorphic representations generated by holomorphic Siegel modular forms.

Let $G = \mathrm{Sp}(2n)_{/\mathbb{Q}}$ and $G' = \mathrm{Sp}(2n+2)_{/\mathbb{Q}}$. The Klingen parabolic subgroup $P_{G'} \subset G'$ consists of elements of the form

$$\begin{array}{cc} \begin{matrix} n & 1 & n & 1 \end{matrix} & \\ \begin{pmatrix} a & 0 & b & * \\ * & x & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & x^{-1} \end{pmatrix} & \begin{matrix} n \\ 1 \\ n \\ 1 \end{matrix} \end{array}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2n)_{/\mathbb{Q}},\ x \in \mathrm{GL}(1)_{/\mathbb{Q}}.$$

Its Levi subgroup is isomorphic to $G \times \mathrm{GL}(1)_{/\mathbb{Q}}$. Let $\pi \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, and $\xi : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be a Dirichlet character. Given a section $\Phi(s,\xi)$ inside the π-isotypic part of the space $I_{P_{G'}}(s,\xi)$, which is isomorphic to $\mathrm{Ind}_{P_{G'}(\mathbb{A})}^{G'(\mathbb{A})} \pi \boxtimes \xi|\cdot|_{\mathbb{A}}^s$, (see §2.1 for the precise definition of $I_{P_{G'}}(s,\xi)$), one defines the Klingen Eisenstein series as

$$E^{\mathrm{Kl}}\left(g', \Phi(s,\xi)\right) = \sum_{\gamma \in P_{G'}(\mathbb{Q})\backslash G'(\mathbb{Q})} \Phi(s,\xi)(\gamma g').$$

Assume that the archimedean component of π is isomorphic to the holomorphic discrete series of weight $\underline{t} = (t_1, \dots, t_n)$, and suppose that $s = s_0$ is a critical point to the left of the center for the standard L-function $L(s, \pi \times \xi)$. Then $E^{\mathrm{Kl}}\left(\Phi(s,\xi)\right)(g')\big|_{s=s_0}$ is algebraic after suitable normalization [Shi00]. Let p be a prime number, and $\rho_\pi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(2n+1, \overline{\mathbb{Q}}_p)$ (resp. $\rho_\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \overline{\mathbb{Q}}_p^\times$) be the p-adic Galois representation associated to π (resp. ξ) constructed in [Art13, CHLN11, Shi11, CH13] (resp. by class field theory). The semi-simple Galois representation corresponding to $E^{\mathrm{Kl}}\left(\Phi(s,\xi)\right)(g')\big|_{s=s_0}$ is

$$\rho_\xi(s_0) \oplus \rho_\pi \oplus \rho_{\xi^{-1}}(-s_0). \tag{1.0.1}$$

Given an algebraic irreducible cuspidal automorphic representation Π of $G'(\mathbb{A})$ whose Hecke eigenvalues are congruent to those of the Klingen Eisenstein series modulo certain power of p, with a suitably chosen lattice for ρ_Π, we have

$$\rho_\Pi \equiv \begin{pmatrix} \rho_\xi(s_0) & * & * \\ 0 & \rho_\pi & \star \\ 0 & 0 & \rho_{\xi^{-1}}(-s_0) \end{pmatrix}$$

modulo that power of p. If ρ_Π is irreducible (for example if Π is stable), then $\star$ gives rise to nontrivial Selmer classes for $\rho_\pi \otimes \rho_\xi(s_0)$ with $\overline{\mathbb{Q}}_p/\mathcal{O}_{\overline{\mathbb{Q}}_p}$ coefficients. This consideration motivates the study of the Klingen Eisenstein congruence ideal, which measures the congruences between the Klingen Eisentein series and cuspidal automorphic forms. Such study for the groups $\mathrm{Sp}(4)$, $\mathrm{U}(2,2)$, $\mathrm{U}(2,1)$, $\mathrm{U}(3,1)$ has played a crucial role in [Urb06, SU14, Hsi14, Wan13].

In order to show desired properties for the Klingen Eisenstein congruence ideals, it is crucial to study the p-adic properties of Klingen Eisenstein series, especially to prove: (1) the relation between its "constant terms" and L-values, (2) its p-adic primitivity after appropriate normalization, (3) its p-adic interpolatability into p-adic families. We address (1) and (3) here, and carry out some preliminary computation on the non-degenerate Fourier coefficients which expects to help address (2).

Assume that $p \geq 3$. Fix a positive integer $N \geq 3$ prime to p and a sufficiently large finite extension F of $\mathbb{Q}_p$. For $m = n, n+1$, denote by T_m the standard maxinal torus of $\mathrm{GL}(m)$ and we identify T_{n+1} with $T_n \times \mathbb{G}_m$. A $\overline{\mathbb{Q}}_p^\times$-valued character of $T_m(\mathbb{Z}_p)$ is called arithmetic if it is a product of an algebraic character and a finite order character. For an arithmetic character $(\underline{\tau}, \kappa)$ of $T_{n+1}(\mathbb{Z}_p) = T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$, we denote by $(\underline{t}, k) = (t_1, \ldots, t_n)$ (resp. $(\underline{\epsilon}, \chi)$) its algebraic (resp. finite order) part. We call $(\underline{\tau}, \kappa)$ admissible if it is arithmetic and $t_1 \geq t_2 \geq \cdots \geq t_n \geq k \geq n+1$.

Define the m-variable Iwasawa algebra Λ_m as $\mathcal{O}_F[\![T_m(1+p\mathbb{Z}_p)]\!]$. By Hida theory [Hid02], there is an $\mathcal{O}_F[\![T_{n+1}(\mathbb{Z}_p)]\!]$-module $\mathcal{M}^0_{G',\mathrm{ord}}$ finite free over Λ_{n+1} (resp. $\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]$-module $\mathcal{M}^0_{G,\mathrm{ord}}$ finite free over Λ_n) consisting of cuspidal Siegel modular forms of genus $n+1$ (resp. genus n) and tame level principal level N. By the Hida theory for non-cuspidal Siegel modular forms [LR18], there is also an $\mathcal{O}_F[\![T_{n+1}(\mathbb{Z}_p)]\!]$-module $\mathcal{M}^1_{G',\mathrm{ord}}$ finite free over Λ_{n+1} consisting of ordinary families of Siegel modular forms of genus $n+1$ and tame principal level N, which vanish along the strata with cusp labels of rank > 1 in a toroidal compactification. In addition, there is the following short exact sequence

$$0 \longrightarrow \mathcal{M}^0_{G,\mathrm{ord}} \longrightarrow \mathcal{M}^1_{G',\mathrm{ord}} \xrightarrow{\mathscr{P}_{\mathrm{deg}}} \bigoplus_{\substack{L \in \mathfrak{C}_{\mathbb{V}'}/\Gamma_{G'}(N) \\ \mathrm{rk} L = 1}} \mathcal{M}^0_{L,\mathrm{ord}} \otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]} \mathcal{O}_F[\![T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times]\!] \longrightarrow 0. \tag{1.0.2}$$

The quotient map $\mathscr{P}_{\mathrm{deg}}$ is called the Siegel operator. It generalizes the operator of taking the constant term of modular forms on $\mathrm{GL}(2)$. The set $\mathfrak{C}_{\mathbb{V}'}/\Gamma_{G'}(N)$ parameterizes strata of the minimal compactification, and each $\mathcal{M}^0_{L,\mathrm{ord}}$ is isomorphic to $\mathcal{M}^0_{G,\mathrm{ord}}$. (See (1.0.7) for the definition of the congruence subgroups of $G(\mathbb{Z})$ and $G'(\mathbb{Z})$ we will use in this article.) The image of $\mathscr{P}_{\mathrm{deg}}$ of a family in $\mathcal{M}^1_{G',\mathrm{ord}}$ measures its congruences with cuspidal Hida families. For each $L \in \mathfrak{C}_{\mathbb{V}'}/\Gamma_{G'}(N)$, we denote by $\mathscr{P}_{\mathrm{deg},L}$ the projection of the Siegel operator $\mathscr{P}_{\mathrm{deg}}$ to the component indexed by L.

Denote by $\mathbb{T}^0_{G,\mathrm{ord}}$ the $\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]$-algebra consisting of unramified Hecke operators away from Np and $\mathbb{U}_p$-operators acting on $\mathcal{M}^0_{G,\mathrm{ord}}$. The natural map $\mathrm{Spec}\left(\mathbb{T}^0_{G,\mathrm{ord}}\right) \to \mathrm{Spec}\left(\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]\right)$ is called the weight projection map. A point (x, κ) is called admissible if $(\underline{\tau}_x, \kappa)$ is admissible with $\underline{\tau}_x$ being the projection of x to the weight space. Let $\mathscr{C}$ be a geometrically irreducible component of $\mathrm{Spec}\left(\mathbb{T}^0_{G,\mathrm{ord}} \otimes_{\mathcal{O}_F} F\right)$ with function field $F_{\mathscr{C}}$. Denote by $\mathbb{I}_{\mathscr{C}}$ the integral closure of Λ_n in $F_{\mathscr{C}}$.

Theorem 1.0.1 (Theorem 2.6.2, Theorem 3.6.1)**.** *Assume that $p, N \geq 3$, and η is a Dirichlet character with conductor dividing N. There exists an ordinary family*

$$\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}} \in F_{\mathscr{C}} \otimes_{\Lambda_n} \mathcal{M}^0_{G,\mathrm{ord}} \otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]} \mathcal{M}^1_{G',\mathrm{ord}}$$

which, at an admissible point $(x, \kappa) \in \mathscr{C}(\overline{\mathbb{Q}}_p) \times \mathrm{Spec}\left(\mathcal{O}_F[\![\mathbb{Z}_p^\times]\!]\right)$, specializes (up to an explicit constant) to the automorphic form

$$\sum_{\varphi \in \mathfrak{S}_x} \frac{\varphi \otimes E^{\mathrm{Kl}}\left(\cdot, \Phi_{f_{\underline{\tau}_x,\kappa}(s),\varphi}\right)\Big|_{s=n+1-k}}{\langle \varphi, \overline{\varphi} \rangle}$$

if x is classical and the weight projection is étale at x and $\kappa(-1) = \eta(-1)$. Here $\mathfrak{S}_x$ consists of an orthogonal basis of algebraic ordinary cuspidal holomorphic Siegel modular forms on $G(\mathbb{A})$ of *weight $\underline{t}_x$, nebentypus $\underline{\epsilon}_x$ at p and tame level $\widehat{\Gamma}_{G,1}(N)$ on which the Hecke algebra acts through the eigenvalues parameterized by x. The precise formulas for the section $\Phi_{f_{\mathfrak{I}_x,\kappa}(s),\varphi} \in I_{P_{G'}}(s,\eta\chi)$ are given as (2.2.2) and §2.3.*

Moreover, the image of the ordinary Klingen family $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}$ under the Siegel operator is divisible by the (n+1)-variable p-adic L-function associated to $\mathscr{C}$. More precisely, for the family

$$(\mathrm{id}\times \mathscr{P}_{\deg,L})\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right) \in \mathcal{M}^0_{G,\mathrm{ord}} \otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]} \mathcal{M}^0_{L,\mathrm{ord}} \otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]} \mathbb{I}_{\mathscr{C}}[\![\mathbb{Z}_p^\times]\!] \otimes_{\mathbb{I}_{\mathscr{C}}} F_{\mathscr{C}},$$

let

$$\varepsilon^{\gamma_{N,1},\gamma_{N,2}}_{q\text{-}\mathrm{exp}}\left(\beta_1,\beta_2,\mathscr{P}_{\deg,L}\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)\right) \in \mathbb{I}_{\mathscr{C}}[\![\mathbb{Z}_p^\times]\!] \otimes_{\mathbb{I}_{\mathscr{C}}} F_{\mathscr{C}} \tag{1.0.3}$$

be the coefficient indexed by $(\beta_1,\beta_2) \in N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*,\oplus 2}_{>0}$ of its p-adic q-expansion at the cusp associated to $\gamma_{N,1},\gamma_{N,2} \in G(\mathbb{Z}/N)$. Let $\gamma'_L \in G'(\mathbb{Z}/N)$ be associated with the stratus L. Then (1.0.3) vanishes unless $\gamma'_L \in P_{G'}(\mathbb{Z}_v) w_{PG'} U_{P_{G'}}(\mathbb{Z}_v)$ for all $v \mid N$. If nonvanishing, then it satisfies the interpolation properties of the p-adic L-function for $\mathscr{C}$, i.e. at (x,κ) as above, it takes the value

$$\begin{aligned}
&\varepsilon^{\gamma_{N,1},\gamma_{N,2}}_{q\text{-}\mathrm{exp}}\left(\beta_1,\beta_2,\mathscr{P}_{\deg,L}\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)\right)(x,\kappa)\\
=&p^{2n^2}(p-1)^n \operatorname{vol}\big(\widehat{\Gamma}_{G,1}(pN)\big)(\eta\kappa)_N\left(p_{\mathrm{GL}(1)}(\gamma'_L)\right)\\
&\times \frac{\sqrt{-1}^{\frac{n^2-n}{2}} 2^{-\sum t_i + \frac{n^2-n}{2}}}{\dim\left(\mathrm{GL}(n),\underline{t}_x\right)} \sum_{\varphi\in\mathfrak{s}_x} \frac{\varepsilon^{\gamma_{N,1}}_{q\text{-}\mathrm{exp}}(\beta_1,\varphi)\,\varepsilon^{p_G(\gamma'_L),\gamma_{N,2}}_{q\text{-}\mathrm{exp}}(\beta_2,e_G\mathcal{W}(\varphi))}{\langle\varphi,\overline{\varphi}\rangle}\\
&\times E^+_p(k-n,\pi\times\eta^{-1}\chi^{-1})\,E^+_\infty(k-n,\pi\times\eta^{-1}\chi^{-1})\,L^{Np\infty}(k-n,\pi\times\eta^{-1}\chi^{-1}).
\end{aligned}$$

Here $E^+_p(s,\pi\times\eta^{-1}\chi^{-1})$ (resp. $E^+_\infty(s,\pi\times\eta^{-1}\chi^{-1})$) is the modified Euler factor at p (resp. at infinity) for our p-adic interpolation, and it aligns with the conjecture of Coates–Perrin-Riou [Coa91] (see Remarks 3.5.8, 3.5.3 for their explicit formulas). e_G denotes the ordinary projector on G, and the operator $\mathcal{W}$ on holomorphic Siegel modular forms is defined as (3.5.19). See also (3.5.10) for the definition of $(p_G,p_{\mathrm{GL}(1)}) : P_{G'}w_{P_{G'}}U_{P_{G'}} \to G\times\mathrm{GL}(1)$.

Remark 1.0.2. Compared to [Liu16, Theorem 1.0.1], we do not assume the nontriviality of η^2 (denoted as ϕ^2 *loc. cit*). Thanks to the assumption $N \neq 1$ and our choice of sections at $v \mid N$, the degenerate Fourier coefficients at the infinity cusp of our Siegel Eisenstein series on G' vanish by Proposition 2.4.1, so the issue of the pole of p-adic zeta function does not show up. In fact the condition $\phi^2 \neq 0$ is also unnecessary *loc. cit* as long as $N \neq 1$. In Proposition 4.4.1 *loc. cit*, for $\phi_{\boldsymbol{\beta}}\chi = \mathrm{triv}$, the pole of the p-adic zeta function is canceled by $\prod_{v\mid N} L_v(k-n,\phi_{\boldsymbol{\beta}}^{-1}\chi^{-1})$.

The above theorem together with the short exact (1.0.2) reduces proving the divisibility of the Klingen Eisenstein ideals by the p-adic L-functions to showing the primitivity of $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}$. One possible approach, which has been successfully carreid out in the cases $\mathrm{Sp}(4)$ and $\mathrm{U}(2,2)$ [Urb06, SU14], is to show that there does not exist height one prime ideal of $\mathbb{I}_{\mathscr{C}}[\![\mathbb{Z}_p^\times]\!]$ that divides simultaneously the p-adic L-function and all the non-degenerate Fourier coefficients of $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}$. (Strictly speaking, we need a further normalization so that everything becomes integral and we can discuss divisibility and primitivity. Such a normalization is related with the congruence of $\mathscr{C}$ with other cuspidal Hida families on G and can be done by assuming that $\mathbb{T}^0_{G,\mathrm{ord}}$ is Gorenstein in a similar way as in [EHLS16], but we do not pursue it here.) Therefore, an accurate computation of the non-degenerate Fourier coefficients of $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}$ is of great interest.

Theorem 1.0.3. *Given $\mathfrak{a} \in \mathrm{GL}(n+1,\mathbb{A}_f)$ and $\beta' \in \mathrm{Sym}(n+1,\mathbb{Z})^*_{>0}$ such that ${}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v \in \mathrm{Sym}(n+1,\mathbb{Z}_v)^*$ and $2\,{}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v \in \mathrm{GL}(n+1,\mathbb{Z}_v)$ for all $v \nmid Np\infty$, let $\varepsilon^{m(\mathfrak{a})}_{q\text{-}\mathrm{exp},\beta'} : \mathcal{M}^1_{G',\mathrm{ord}} \to \mathcal{O}_F[\![T_{n+1}(\mathbb{Z}_p)]\!]$ be* *the map of taking the β'-th coefficient in the p-adic q-expansion at the cusp associated to $m(\mathfrak{a}) = \left(\begin{smallmatrix}\mathfrak{a} & 0\\ 0 & {}^{t}\mathfrak{a}^{-1}\end{smallmatrix}\right)$. Then for (x,κ) as in Theorem 1.0.1 with $\underline{t}_x$ being a scalar weight and a sufficiently large (with respect to η,χ) integer r, the evaluation of*

$$\left(\mathrm{id}\times\varepsilon^{m(\mathfrak{a})}_{q\text{-exp},\beta'}\right)\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)\in\mathcal{M}^{0}_{G,\mathrm{ord}}\otimes_{\Lambda_n}\mathbb{I}_{\mathscr{C}}[\![\mathbb{Z}_p^\times]\!]\otimes_{\mathbb{I}_{\mathscr{C}}}F_{\mathscr{C}}$$

equals

$$(*)\sum_{\varphi\in\mathfrak{S}_x}\frac{\varphi\cdot\left\langle\left(c_{k,\eta\chi}\prod_{v|Np}U_{Q_G,v}\right)^{r}E^{\mathrm{Si}}\left(\cdot,f_{\widetilde{G},\underline{\mathfrak{T}}_x,\kappa}(k-n-1)\right)\theta_{2\beta'}\left(\cdot,\phi^{r}_{2\beta',\mathfrak{a},\underline{\mathfrak{T}}_x,\kappa}\right),\overline{\varphi}\right\rangle}{\langle\varphi,\overline{\varphi}\rangle},$$

where $\widetilde{G}$ denotes the metaplectic group $\widetilde{\mathrm{Sp}}(2n)$. See Theorem 4.8.1 for the detailed formulas of the scalar $(*)$, the section $f_{\widetilde{G},\underline{\mathfrak{T}}_x,\kappa}$ and the Schwartz function $\phi^{r}_{2\beta',\mathfrak{a},\underline{\mathfrak{T}}_x,\kappa}$ on $M_{n,n+1}(\mathbb{Q}_v)$. The operator $U_{Q_G,v}$ is defined as $\displaystyle\int_{\mathrm{Sym}(n,\mathbb{Z}_v)}$ right translation by $\left(\begin{smallmatrix}\mathbf{1}_n & \sigma\\ 0 & \mathbf{1}_n\end{smallmatrix}\right)\left(\begin{smallmatrix}q_v\mathbf{1}_n & 0\\ 0 & q_v^{-1}\mathbf{1}_n\end{smallmatrix}\right)d\sigma$, *and* $c_{k,\eta\chi} = \eta_p\chi_p(p)^n\,p^{n(k-2n-2)}\prod_{v|N}\eta_v\chi_v(q_v)^{-n}|q_v|_v^{nk}$.

By Siegel–Weil formula, the Siegel Eisenstein series on $\widetilde{G}$ can be obtained as a theta lift of the trivial representation on the orthogonal group $\mathrm{O}(2k-n-1)$. At the end of §4, we discuss expressing the product of the Siegel Eisenstein series and the theta series in the above theorem in terms of theta lift from $\mathrm{O}(2k)$. This point of view is more convenient for dealing with general vector weight at the archimedean place and the place 2 when n is even.

Compared to the construction of Klingen Eisenstein families on unitary groups in [Wan15], we have a more refined study on the properties of the constructed families. Firstly, we do not assume the condition that the nebentypus at p are sufficiently ramified for the evaluations of the "constant terms" and the non-degenerate Fourier coefficients. (The evaluations for trivial nebentypus are of more interest and even crucial particularly if the size of the group is not very small.) Secondly, the archimedean weights are assumed to be scalar in [Wan15] and we include all vector weights. Our results rely on a good understanding of Maass–Shimura differential operators and the computation of archimedean zeta integrals in [Liu19]. Finally, we include a discussion on expressing the local sections in Theorem 1.0.3 in terms of theta lifts from $\mathrm{O}(2k)$, which can be useful for potential applications as explained in §4.9.

The article is organized as follows. In §2, we construct the Klingen Eisenstein family $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}$. The construction relies on Garrett's generalization of the doubling method and the p-adic interpolation of p-adic q-expansions of a collection of nice Siegel Eisenstein series on $\mathrm{Sp}(4n+2)_{/\mathbb{Q}}$. Extra care needs to be taken for selecting the sections at p in order to ensure the nonvanishing of the ordinary projection of the resulting Klingen Eisenstein series on $\mathrm{Sp}(2n+2)_{/\mathbb{Q}}$. In §3, we identify the coefficients of the p-adic q-expansions of $\mathscr{P}_{\deg}\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)$ with p-adic L-functions by computing their evaluations at admissible points. §§3.2-3.4 reduce the problem to local computations, and 3.5 computes the local integrals place by place. §4 is about computing the non-degenerate Fourier coefficients of specializations of $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}$ at admissible points. The strategy is to compute the partial Fourier expansions of the Siegel Eisenstein series on $\mathrm{Sp}(4n+2)_{/\mathbb{Q}}$. After the standard unfolding in §4.1, involved computation is done in §§4.4-4.7 to work out the precise formulas for local sections which are crucial for further study of the p-adic properties of the non-degenerate Fourier coefficients.

Notation. We fix an odd prime p and a positive integer $N\geq 3$ coprime to p. We also fix an embedding $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$ and an isomorphism between $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$.

Fix the standard additive character $\mathbf{e}_{\mathbb{A}} = \bigotimes_v \mathbf{e}_v : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ with local component $\mathbf{e}_v$ defined as $\mathbf{e}_v(x) = \begin{cases} e^{-2\pi i\{x\}_v}, & v \neq \infty \\ e^{2\pi i x}, & v = \infty \end{cases}$ where $\{x\}_v$ is the fractional part of x. All the gamma factors appearing in this article are with respect to this fixed additive character, and we omit $\mathbf{e}_v$ from the notation for gamma factors.

For a positive integer m, define the algebraic group $\mathrm{Sp}(2m)$ over $\mathbb{Z}$ as

$$\mathrm{Sp}(2m) = \left\{ g \in \mathrm{GL}(2n) : {}^{\mathrm{t}}g \begin{pmatrix} 0 & \mathbf{1}_m \\ -\mathbf{1}_m & 0 \end{pmatrix} g = \begin{pmatrix} 0 & \mathbf{1}_m \\ -\mathbf{1}_m & 0 \end{pmatrix} \right\}.$$

The standard Siegel parabolic subgroup $Q \subset \mathrm{Sp}(2m)$ consists of elements whose lower left $m \times m$ blocks are 0. The modulus character δ_Q is given as $\delta_Q \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = |\det a|_{\mathbb{A}}^{\frac{m+1}{2}}$. The unipotent radical $U_Q \subset Q$ is identified with $\mathrm{Sym}(m)$, the space of symmetric $m \times m$ matrices, via

$$\varsigma \longmapsto u(\varsigma) = \begin{pmatrix} \mathbf{1}_n & \varsigma \\ 0 & \mathbf{1}_n \end{pmatrix}, \qquad \varsigma \in \mathrm{Sym}(m),$$

and the Levi subgroup $M_Q \subset Q$ is identified with $\mathrm{GL}(n)$ via

$$a \longmapsto m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^{\mathrm{t}}a^{-1} \end{pmatrix}, \qquad a \in \mathrm{GL}(m). \tag{1.0.4}$$

For each finite place v of $\mathbb{Q}$, the group $\mathrm{Sp}(2m, \mathbb{Z}_v)$ is a maximal open compact subgroup of $\mathrm{Sp}(2m, \mathbb{Q}_v)$. For the archimedean place, the maximal compact subgroup

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a + \sqrt{-1}b \in \mathrm{U}(m, \mathbb{R}) \right\} \tag{1.0.5}$$

is isomorphic to the rank m definite unitary group over $\mathbb{R}$.

For a finite place v, we fix the Haar measure on $\mathbb{Q}_v$ (resp. $\mathrm{Sp}(2m, \mathbb{Q}_v)$) with $\mathbb{Z}_v$ (resp. $\mathrm{Sp}(2m, \mathbb{Z}_v)$) having volume 1. We take the usual Lebesgue measure for $\mathbb{R}$, and for the group $\mathrm{Sp}(2m, \mathbb{R})$, we take the product measure where the one on the maximal compact subgroup (1.0.5) has total volume 1 and the one on the upper half space $\mathbb{H}_m = \{z = x + iy \in \mathrm{Sym}(m, \mathbb{C}) : y > 0\}$ is $\det(y)^{-m-1} \prod_{1 \leq i \leq j \leq m} dx_{ij}\, dy_{ij}$. The Haar measures on $\mathbb{A}$ and $\mathrm{Sp}(2m, \mathbb{A})$ are obtained by taking products of the local ones.

Let $\mathbb{V}$ (resp. $\mathbb{V}'$) be a vector space over $\mathbb{Q}$ with a fixed basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ (resp. $e'_1, \ldots, e'_{n+1}$, $f'_1, \ldots, f'_{n+1}$) equipped with the symplectic pairing given by $\begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & \mathbf{1}_{n+1} \\ -\mathbf{1}_{n+1} & 0 \end{pmatrix}$) with respect to the fixed basis. Let $\mathbb{V}'' \subset \mathbb{V}'$ be the $2n$-dimensional subspace spanned by $e'_1, \ldots, e'_n$, $f'_1, \ldots, f'_n$ with a symplectic pairing induced from that of $\mathbb{V}'$. Put $\mathbb{W} = \mathbb{V} \oplus \mathbb{V}''$ with basis $e_1, \ldots, e_n, e'_1, \ldots, e'_n, f_1, \ldots, f_n, f'_1, \ldots, f'_n$, and $\mathbb{W}' = \mathbb{V} \oplus \mathbb{V}'$ with basis $e_1, \ldots, e_n$, $e'_1, \ldots, e'_n$, $f_1, \ldots, f_{n+1}$, $f'_1, \ldots, f'_{n+1}$. The spaces $\mathbb{W}$, $\mathbb{W}'$ are endowed with symplectic pairings induced from those on $\mathbb{V}, \mathbb{V}'', \mathbb{V}'$.

The following four symplectic groups will be used.

$$\begin{aligned} G &= \mathrm{Sp}(\mathbb{V}) \cong \mathrm{Sp}(2n)_{/\mathbb{Q}}, & G' &= \mathrm{Sp}(\mathbb{V}') \cong \mathrm{Sp}(2n+2)_{/\mathbb{Q}}, \\ H &= \mathrm{Sp}(\mathbb{W}) \cong \mathrm{Sp}(4n)_{\mathbb{Q}}, & H' &= \mathrm{Sp}(\mathbb{W}') \cong \mathrm{Sp}(4n+2)_{/\mathbb{Q}}, \end{aligned}$$

where the isomorphisms are given by the above fixed basis. The Siegel modular forms we are interested in are on G and the families of Klingen Eisenstein series we are going to construct and study are on G'. The auxiliary groups H, H' appear in the doubling method on which the auxiliary Siegel Eisenstein series live. We fix the following embeddings

$$\begin{aligned} \iota_H : G \times G &\hookrightarrow H \\ \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} &\longmapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}, \end{aligned} \qquad \begin{aligned} \iota_{H'} : G \times G' &\hookrightarrow H' \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix} &\longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & \mathfrak{a} & 0 & \mathfrak{b} \\ c & 0 & d & 0 \\ 0 & \mathfrak{c} & 0 & \mathfrak{d} \end{pmatrix}. \end{aligned}$$

We will also view G as a subgroup of G' via the embedding

$$\begin{aligned} G &\hookrightarrow G' \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{1.0.6}$$

For $R = \mathbb{Z}, \mathbb{Z}_v$, we denote by $\mathrm{Sym}(m, R)^*$ the subset of $\mathrm{Sym}(m, R \otimes \mathbb{Q})$ consisting of β's such that $\mathrm{Tr}\beta\varsigma \in R$ for all $\varsigma \in \mathrm{Sym}(m, R)$. We fix the Haar measures on $\mathrm{Sym}(m, \mathbb{Q}_v)$ such that $\mathrm{Sym}(m, \mathbb{Z}_v)$ has volume 1.

For a positive integer M, we define the following congruence subgroups of $\mathrm{Sp}(2m, \mathbb{Z})$.

$$\begin{aligned} \Gamma_{\mathrm{Sp}(2m)}(M) &= \left\{ g \in \mathrm{Sp}(2m, \mathbb{Z}) : \, g \equiv \mathbf{1}_{2m} \mod M \right\}, \\ \Gamma_{\mathrm{Sp}(2m),1}(M) &= \left\{ g \in \mathrm{Sp}(2m, \mathbb{Z}) : \, g \equiv \begin{pmatrix} 1 & \cdots & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & 1 & * & \cdots & * \\ & & & 1 & & \\ & & & \vdots & \ddots & \\ & & & * & \cdots & 1 \end{pmatrix} \mod M \right\}. \end{aligned} \tag{1.0.7}$$

Denote by T_n (resp. T_{n+1}) the standard maximal torus of $\mathrm{GL}(n)$ (resp. $\mathrm{GL}(n+1)$) consisting of diagonal matrices. By (1.0.4), T_n (resp. T_{n+1}) can be identified as the standard maximal torus of G (resp. G'). Via (1.0.6), we view T_n as a subgroup of T_{n+1} and identify T_{n+1} as $\mathbb{G}_m \times T_n$.

The weight space for p-adic forms on G (resp. G') is

$$\mathrm{Hom}_{\mathrm{cont}}\left(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times\right) \qquad \left(\text{resp. } \mathrm{Hom}_{\mathrm{cont}}\left(T_{n+1}(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times\right)\right).$$

A point $(\underline{\tau})$ (resp. $(\underline{\tau}, \kappa)$ of the weight space is called arithmetic if it is a product of algebraic character and a finite order character, and we denote its algebraic part as $\underline{t} = (t_1, \dots, t_n) \in \mathbb{Z}^n$ (resp. $(\underline{t}, k) = \mathbb{Z}^{n+1}$) and its finite order part as $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$ (resp. $(\underline{\epsilon}, \chi)$). An arithmetic point $(\underline{\tau}, \kappa)$ is called admissible if $t_1 \geq \cdots \geq t_n \geq k+1 \geq n+1$.

2. Ordinary families of Klingen Eisenstein series on $\mathrm{Sp}(2n+2)$

2.1. The basic setup for Klingen Eisenstein series and Siegel Eisenstein series. We first recall the definition of Klingen Eisenstein series on G'. Let $P_{G'}$ be the (standard) Klingen parabolic subgroup of G' consisting of elements of the form

$$\begin{array}{c} \begin{array}{cccc} n & 1 & n & 1 \end{array} \\ \begin{pmatrix} a & 0 & b & * \\ * & x & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & x^{-1} \end{pmatrix} \end{array} \begin{array}{c} \\ n \\ 1 \\ n \\ 1 \end{array}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2n)_{/\mathbb{Q}}, \; x \in \mathrm{GL}(1)_{/\mathbb{Q}}.$$

(This $P_{G'}$ is the parabolic subgroup of G' preserving the isoropic subspace space spanned by e'_{n+1} inside $\mathbb{V}'$). Its Levi subgroup is isomorphic to $\mathrm{Sp}(2n) \times \mathrm{GL}(1)$, and its modulus character is

$$\delta_{P_{G'}} : P_{G'}(\mathbb{A}) \longrightarrow \mathbb{C}^\times$$
$$\begin{pmatrix} a & 0 & b & * \\ * & x & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & x^{-1} \end{pmatrix} \longmapsto |x|_{\mathbb{A}}^{2n+2}.$$

Given a Dirichlet character $\xi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ and a complex number s, define $I_{P_{G'}}(s,\xi)$ as the space of smooth functions

$$\Phi(s,\xi) : U_{P_{G'}}(\mathbb{A}) \cdot M_{P_{G'}}(\mathbb{Q}) \backslash G'(\mathbb{A}) \longrightarrow \mathbb{C}$$

such that:

(1) $\Phi(s,\xi)\left(\begin{pmatrix} \mathbf{1}_n & & & \\ & x & & \\ & & \mathbf{1}_n & \\ & & & x^{-1} \end{pmatrix} g'\right) = \xi(x)|x|_{\mathbb{A}}^s \, \delta_{P_{G'}}^{1/2}(x)\, \Phi(s,\xi)(g') = \xi(x)|x|_{\mathbb{A}}^{s+n+1}\, \Phi(s,\xi)(g')$ for all $x \in \mathbb{A}^\times$ and $g' \in G'(\mathbb{A})$,

(2) The space spanned by the right translation of $\mathrm{Sp}(2n+2, \widehat{\mathbb{Z}}) \times K_{G',\infty} \subset G'(\mathbb{A})$ on $\Phi(s,\xi)$ is finite dimensional, where $K_{G',\infty} \subset G'(\mathbb{R})$ is the maximal compact subgroup defined as in (1.0.5),

(3) For any $g' \in G'(\mathbb{A})$ the function $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \Phi(s,\xi)\left(\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g'\right)$ is a cuspidal automorphic form on $G(\mathbb{A})$.

(Note that unlike the degenerate principal series defined below, the space $I_{P_{G'}}(s,\xi)$ does not factorize to a product of local spaces.)

The Klingen Eisenstein series attached to $\Phi(s,\xi) \in I_{P_{G'}}(s,\xi)$ is defined as

$$E^{\mathrm{Kl}}\left(g', \Phi(s,\xi)\right) = \sum_{\gamma \in P_{G'}(\mathbb{Q}) \backslash G'(\mathbb{Q})} \Phi(s,\xi)(\gamma g').$$

The sum converges absolutely for $\mathrm{Re}(s) \gg 0$.

Next we recall the definition of Siegel Eisenstein series on H'. Let $Q_{H'} \subset H'$ be the standard Siegel parabolic subgroup (which is the parabolic subgroup preserving the maximal isotropic subspace W' inside $\mathbb{W}'$ spanned by $e_1, \ldots, e_n,\ e'_1, \ldots, e'_n$). Let $I_{Q_{H'}}(s,\xi)$ be the space consisting of smooth functions $f(s,\xi)$ on $H'(\mathbb{A})$ which are $K_{H',\infty}$-finite and satisfy

$$f(s,\xi)\left(\begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ 0 & \mathfrak{D} \end{pmatrix} h'\right) = \xi(\det \mathfrak{A}) |\det \mathfrak{A}|_{\mathbb{A}}^s \, \delta_{Q_{H'}}^{1/2}(\det \mathfrak{A})\, f(h') = \xi(\det \mathfrak{A}) |\det \mathfrak{A}|_{\mathbb{A}}^{s+n+1}\, f(h').$$

The Siegel Eisenstein series attached to $f(s,\xi) \in I_{Q_{H'}}(s,\xi)$ is defined as

$$\mathfrak{E}^{\mathrm{Si}}\left(h', f(s,\xi)\right) = \sum_{\gamma \in Q_{H'}(\mathbb{Q}) \backslash H'(\mathbb{Q})} f(s,\xi)(\gamma h').$$

Again the sum converges absolutely for $\mathrm{Re}(s) \gg 0$. One defines Siegel Eisenstein series on H in the same way.

We also recall here the definition of intertwining operators which will be frequently used later. The intertwining operator on sections for Klingen Eisenstein series is defined as

$$
\begin{aligned}
M_{P_{G'}}(s,\xi) : I_{P_{G'}}(s,\xi) &\longrightarrow I_{P_{G'}}(-s,\xi^{-1}) \\
\Phi(s,\xi) &\longmapsto \left(M_{P_{G'}}(s,\xi)\Phi(s,\xi)\right)(g) = \int_{U_{P_{G'}}(\mathbb{Q})\backslash U_{P_{G'}}(\mathbb{A})} \Phi(s,\xi)(w_{P_{G'}}ug)\,du.
\end{aligned}
\tag{2.1.1}
$$

Similarly, the intertwining operator on the degenerate principal series is defined as

$$
\begin{aligned}
M_{Q_{H'},v}(s,\xi) : I_{Q_{H'}}(s,\xi)_v &\longrightarrow I_{Q_{H'}}(-s,\xi^{-1})_v \\
f_v(s,\xi) &\longmapsto \left(M_{Q_{H'},v}(s,\xi)f_v(s,\xi)\right)(g) = \int_{U_{Q_{H'}}(\mathbb{Q}_v)} f_v(s,\xi)(w_{Q_{H'}}ug)\,du.
\end{aligned}
\tag{2.1.2}
$$

Here

$$
w_{P_{G'}} = \begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \mathbf{1}_n & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad w_{Q_{H'}} = \begin{pmatrix} 0 & -\mathbf{1}_{2n+1} \\ \mathbf{1}_{2n+1} & 0 \end{pmatrix}.
$$

2.2. **The doubling method formula.** In [Gar89], a slight generalization of the classical doubling method for symplectic groups is introduced and gives rise to an integral representation of the Klingen Eisenstein series on $\mathrm{Sp}(2n+2)$ in terms of the Siegel Eisenstein series on $\mathrm{Sp}(4n+2)$.

In order to state the integral representation formula, we first define the so-called doubling Siegel parabolic subgroup. In addition to $W' \subset \mathbb{W}'$, we introduce another maximal isotropic subspace

$$
W'^{\diamondsuit} = \mathrm{span}\,\{e_1+f'_1,\ldots,e_n+f'_n, f_1+e'_1,\ldots,f_n+e'_n,e'_{n+1}\}.
$$

The doubling Siegel parabolic subgroup $Q^{\diamondsuit}_{H'}$ is defined as the Siegel parabolic subgroup preserving $W'^{\diamondsuit}$. We have

$$
Q^{\diamondsuit}_{H'} = S_{H'}Q_{H'}S^{-1}_{H'}, \qquad S_{H'} = \begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1}_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{1}_n & 0 & \mathbf{1}_n & 0 & 0 \\ \mathbf{1}_n & 0 & 0 & 0 & \mathbf{1}_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Given $f(s,\xi) \in I_{Q_{H'}}(s,\xi)$, define $f^{\diamondsuit}(s,\xi) \in I_{Q^{\diamondsuit}_{H'}}(s,\xi)$ as

$$
f^{\diamondsuit}(s,\xi)(h') = f(s,\xi)(S^{-1}_{H'}h').
$$

For a cuspidal automorphic form $\varphi \in \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$, define the linear functional

$$
\begin{aligned}
\mathscr{L}_{\overline{\varphi}} : \mathcal{A}(H'(\mathbb{Q})\backslash H'(\mathbb{A})) &\longrightarrow \mathcal{A}(G'(\mathbb{Q})\backslash G'(\mathbb{A})) \\
F &\longmapsto \mathscr{L}_{\overline{\varphi}}(F)(g') = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} F(\iota_{H'}(g,g'))\overline{\varphi}(g)\,dg.
\end{aligned}
\tag{2.2.1}
$$

Theorem 2.2.1 ([Gar89, Theorem on p. 255])**.** *For $f(s,\xi) \in I_{Q_{H'}}(s,\xi)$ and $\varphi \in \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$,*

$$
\mathscr{L}_{\overline{\varphi}}\left(\mathfrak{E}^{\mathrm{Si}}(\cdot, f(s,\xi))\right) = E^{\mathrm{Kl}}\left(\cdot, \Phi_{f(s,\xi),\varphi}\right),
$$

where the section $\Phi_{f(s,\xi),\varphi} \in I_{P_{G'}}(s,\xi)$ is given by

$$
\Phi_{f(s,\xi),\varphi}(g') = \int_{G(\mathbb{A})} f^{\diamondsuit}(s,\xi)(\iota_{H'}(g,g'))\overline{\varphi}(g)\,dg.
\tag{2.2.2}
$$

The doubling method formula reduces the construction of families of Klingen Eisenstein series on G' to the selection of families of sections inside the degenerate principal series on H'.

2.3. **Our choices of sections.** There is a simple strategy for selecting sections from the degenerate principal series for p-adic interpolation as explained in [Liu16]. The idea is that, combining the theory of differential operators and theta correspondence, there are natural choices for the archimedean place, and the sections at the place p are determined by the archimedean sections due to the requirement that the Fourier coefficients be p-adically interpolatble. We will not say more about the strategy here, but simply list our choices of the sections. In the following we fix a Dirichlet character $\eta : \mathbb{Q}^\times\mathbb{A}^\times \to \mathbb{C}^\times$ with conductor dividing N. For each admissible point $(\underline{\tau}, \kappa) \in \mathrm{Hom}_{\mathrm{cont}}\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times\right)$ with $\eta\chi(-1) = (-1)^k$, we pick a section $f_{\kappa,\underline{\tau}}(s) \in I_{Q_{H'}}(s, \eta\chi)$.

2.3.1. *The unramified places.* For $v \nmid Np\infty$, set

$$f_{\underline{\tau},\kappa,v}(s) = f_v^{\mathrm{ur}}(s, \eta\chi),$$

the standard unramified section in $I_{Q_{H'}}(s, \eta\chi)$ which takes value 1 on $H'(\mathbb{Z}_v)$.

2.3.2. *The archimedean place.* For an integer k, the canonical section of scalar weight k in $I_{Q_{H'},\infty}(s, \mathrm{sgn}^k)$ is defined as

$$f_\infty^k(s, \mathrm{sgn}^k)\begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix} = \det(\mathfrak{C}\sqrt{-1} + \mathfrak{D})^{-k} |\det(\mathfrak{C}\sqrt{-1} + \mathfrak{D})|^{k-(s+n+1)}.$$

Let $\widehat{\mu}_0^+ = \left(\widehat{\mu}_{0,ij}^+\right)_{1\leq i\leq n,\, 1\leq j\leq n+1}$ be the $n \times (n+1)$ matrix with entries inside $(\mathrm{Lie}H')_\mathbb{C}$ whose (i, j) entry is given as

$$\widehat{\mu}_{0,ij}^+ = J_{H'} \begin{pmatrix} 0 & 0 & 0 & E_{ij} \\ 0 & 0 & {}^{\mathrm{t}}E_{ij} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} J_{H'}^{-1}, \quad J_{H'} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_{2n+1} & \sqrt{-1}\cdot\mathbf{1}_{2n+1} \\ \sqrt{-1}\cdot\mathbf{1}_{2n+1} & \mathbf{1}_{2n+1} \end{pmatrix}, \tag{2.3.1}$$

where E_{ij} is the $n \times (n+1)$ matrix with 1 as the (i, j)-entry and 0 elsewhere. The $\widehat{\mu}_{0,ij}^+$'s act on $\mathcal{A}(H'(\mathbb{Q})\backslash H'(\mathbb{A}))$ by differentiating the right translation of $H'(\mathbb{R})$. Their realizations on the Siegel upper half space are the Maass–Shimura differential operators (see [Liu16, §2.4]).

For admissible $(\underline{\tau}, \kappa)$ with $\eta\chi(-1) = (-1)^k$, set

$$f_{\kappa,\underline{\tau},\infty}(s) = \prod_{j=1}^{n-1} \det{}_j \left(\frac{\widehat{\mu}_0^+}{4\pi\sqrt{-1}}\right)^{t_j - t_{j+1}} \det{}_n \left(\frac{\widehat{\mu}_0^+}{4\pi\sqrt{-1}}\right)^{t_n - k} \cdot f_\infty^k(s, \mathrm{sgn}^k),$$

where $\det_j$ denotes the determinant of the upper left $j \times j$ block of a matrix.

2.3.3. *The places dividing N.* We choose our sections at $v \mid Np$ from a special type of sections, the so-called "big cell" sections. Given a finite place v and a compactly supported locally constant function α_v on $\mathrm{Sym}(2n+1, \mathbb{Q}_v)$, the "big cell" section in $I_{Q_{H'},v}(s, \xi)$ associated to α_v is defined as

$$f^{\alpha_v}(s, \xi)\left(\begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}\right) = \begin{cases} \xi^{-1}(\det\mathfrak{C}) |\det\mathfrak{C}|_v^{-(s+n+1)} \alpha_v(\mathfrak{C}^{-1}\mathfrak{D}), & \text{if } \det\mathfrak{C} \neq 0, \\ 0, & \text{if } \det\mathfrak{C} = 0. \end{cases}$$

We write $\alpha_v = \begin{pmatrix} \alpha_{v,\text{up-left}} & \alpha_{v,\text{off-diag}} \\ {}^{\mathrm{t}}\alpha_{v,\text{off-diag}} & \alpha_{v,\text{low-right}} \end{pmatrix}$ if α_v is factotizable with respect to the embedding $\iota_{H'}$ in the sense that

$$\alpha_v \left(\begin{array}{cc} \begin{array}{cc} n & n+1 \end{array} & \\ \begin{pmatrix} \varsigma & \varsigma_0 \\ {}^{\mathrm{t}}\varsigma_0 & \varsigma' \end{pmatrix} & \begin{array}{c} n \\ n+1 \end{array} \end{array} \right) = \alpha_{v,\text{up-left}}(\varsigma) \cdot \alpha_{v,\text{low-right}}(\varsigma') \cdot \alpha_{v,\text{off-diag}}(\varsigma_0).$$

Later all the α_v's we will use are factorizable.

Define the Schwartz function α_v^{vol} by

(2.3.2)
$$\begin{aligned} &\alpha^{\text{vol}}_{v,\text{up-left}} = \mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}, \qquad\qquad \alpha^{\text{vol}}_{v,\text{low-right}} = \mathbb{1}_{\mathrm{Sym}(n+1,N\mathbb{Z}_v)}, \\ &\alpha^{\text{vol}}_{v,\text{off-diag}} = \text{characteristic function of } \left\{ x_0 \in M_{n,n+1}(\mathbb{Z}_v) : x_0 \equiv -\begin{pmatrix} 1 & * & \cdots & * & * \\ 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \end{pmatrix} \mod N \right\}. \end{aligned}$$

For $v \mid N$, we set $f_{\underline{\tau},\kappa,v}(s) \in I_{Q_{H'},v}(s,\eta\chi)$ to be

(2.3.3)
$$\begin{aligned} f_{\underline{\tau},\kappa,v}(s) =& \gamma_v\left(-s-n, \eta^{-1}\chi^{-1}\right) \prod_{j=1}^{n} \gamma_v(-2s-2n-1+2j, \eta^{-2}\chi^{-2}) \\ & \times M_{Q_{H'},v}(-s,\eta^{-1}\chi^{-1}) f^{\alpha_v^{\text{vol}}}(-s,\eta^{-1}\chi^{-1}), \end{aligned}$$

where the intertwining operator $M_{Q_{H'},v}(-s,\eta^{-1}\chi^{-1})$ is defined in (2.1.2).

Remark 2.3.1. The definition of α_v^{vol} here is slightly different from its analogue in [Liu16] as an intertwining operator is involved and we use a different tame level structure $\Gamma_{G,1}(N)$ rather than the principal level structure $\Gamma_G(N)$ used *loc. cit.* The level structure $\Gamma_{G,1}(N)$ is more convenient for our later computation in §4.6.

2.3.4. *The place p.* For arithmetic $(\underline{\tau},\kappa)$, define $\widehat{\alpha}_{\underline{\tau},\kappa,p} = \begin{pmatrix} \widehat{\alpha}_{\underline{\tau},\kappa,p,\text{up-left}} & \widehat{\alpha}_{\underline{\tau},\kappa,p,\text{off-diag}} \\ {}^{\mathrm{t}}\widehat{\alpha}_{\underline{\tau},\kappa,p,\text{off-diag}} & \widehat{\alpha}_{\underline{\tau},\kappa,p,\text{low-right}} \end{pmatrix}$ with,

$$\widehat{\alpha}_{\underline{\tau},\kappa,p,\text{off-diag}}(\varsigma_0) = \mathbb{1}_{M_{n,n+1}(\mathbb{Z}_p)}(\varsigma_0) \prod_{j=1}^{n} \mathbb{1}_{\mathbb{Z}_p^\times}(\det_j(2\varsigma_0)) \prod_{j=1}^{n-1} \epsilon_j\epsilon_{j+1}^{-1}(\det_j(2\varsigma_0))\ \epsilon_n\chi^{-1}\left(\det_n(2\varsigma_0)\right),$$

$$\widehat{\alpha}_{\underline{\tau},\kappa,p,\text{up-left}} = \mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)^*}, \qquad\qquad \widehat{\alpha}_{\underline{\tau},\kappa,p,\text{low-right}} = \mathbb{1}_{\mathrm{Sym}(n+1,\mathbb{Z}_p)^*}.$$

Let

$$\alpha_{\kappa,\underline{\tau},p}(\varsigma) = \int_{\mathrm{Sym}(2n+1,\mathbb{Q}_p)} \widehat{\alpha}_{\kappa,\underline{\tau},p}(\boldsymbol{\beta})\, \mathbf{e}_p(\mathrm{Tr}\boldsymbol{\beta}\varsigma)\, d\boldsymbol{\beta},$$

the inverse Fourier transform of the above defined $\widehat{\alpha}_{\kappa,\underline{\tau},p}$. Set

$$f_{\underline{\tau},\kappa,p}(s) = f^{\alpha_{\kappa,\underline{\tau},p}}(s,\eta\chi).$$

2.4. **The Fourier coefficients of the adelic Siegel Eisenstein series.** We write $\underline{\epsilon}\cdot k$ to denote an arithmetic element in $\mathrm{Hom}_{\text{cont}}\left(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times\right)$ with finite order part $\underline{\epsilon}$ and algebraic part the scalar weight k. Given $\boldsymbol{\beta} \in \mathrm{Sym}(2n+1,\mathbb{Q})$, we consider the Fourier coefficients

(2.4.1)
$$\mathfrak{E}^{\text{Si}}_{\boldsymbol{\beta}}(h, f_{\underline{\epsilon}\cdot k,\kappa}(s)) := \int_{\mathrm{Sym}(2n+1,\mathbb{Q})\backslash\mathrm{Sym}(2n+1,\mathbb{A})} \mathfrak{E}^{\text{Si}}\left(\begin{pmatrix} \mathbf{1}_{2n+1} & \varsigma \\ 0 & \mathbf{1}_{2n+1} \end{pmatrix} h, f_{\underline{\epsilon}\cdot k,\kappa}(s)\right) \mathbf{e}_{\mathbb{A}}(-\mathrm{Tr}\boldsymbol{\beta}\varsigma)\, d\varsigma,$$

from which we can easily deduce p-adic q-expansions of (2.5.7) for general $\underline{\tau}$, because the action of $\frac{\widehat{\mu}^+_{0,ij}}{4\pi\sqrt{-1}}$ corresponds to the action of algebraic differential operators on global sections of automorphic sheaves over Siegel varieties for H' [Liu16, Proposition 2.3.1, 2.4.1], and those algebraic differential operators admit simple formulas on p-adic q-expansions.

We introduce some notation. Let $b^{Np\infty}_{H'}(s,\xi) = \prod_{v\nmid Np\infty} b_{H',v}(s,\xi)$ with

$$b_{H',v}(s,\xi) = L_v\left(s+n+1,\xi\right)\prod_{j=1}^{n} L_v\left(2s+2n+2-2j,\xi^2\right). \tag{2.4.2}$$

For an integer m, define the gamma function

$$\Gamma_m(s) := \pi^{\frac{m(m-1)}{4}}\prod_{j=0}^{m-1}\Gamma(s-\frac{j}{2}).$$

Given $\boldsymbol{z} = \boldsymbol{x}+\sqrt{-1}\boldsymbol{y} \in \mathbb{H}_{2n+1}$, define

$$h'_{\boldsymbol{z},\infty} = \begin{pmatrix} \sqrt{\boldsymbol{y}} & \boldsymbol{x}\sqrt{\boldsymbol{y}}^{-1} \\ 0 & \sqrt{\boldsymbol{y}}^{-1}\end{pmatrix} \in H'(\mathbb{R}). \tag{2.4.3}$$

Proposition 2.4.1. *Let* $\boldsymbol{\beta} = \begin{array}{cc} & \begin{array}{cc} n & n+1 \end{array} \\ & \begin{pmatrix} \beta & \beta_0 \\ {}^{\mathrm{t}}\beta_0 & \beta' \end{pmatrix} \begin{array}{c} n \\ n+1 \end{array}\end{array} \in \mathrm{Sym}(2n+1,\mathbb{Q})$, *and* $h'_{\boldsymbol{z}} = 1_f\cdot h'_{\boldsymbol{z},\infty} \in H'(\mathbb{A})$, $k^p_f \in \prod_{v\nmid p\infty} H'(\mathbb{Z}_v)$. *The Fourier coefficient* $\mathfrak{E}^{\mathrm{Si}}_{\boldsymbol{\beta}}(k^p_f h'_{\boldsymbol{z}}, f_{\underline{\epsilon}\cdot k,\kappa}(s))\Big|_{s=n+1-k}$ *vanishes unless* $\boldsymbol{\beta}\in N^{-1}\,\mathrm{Sym}(2n+1,\mathbb{Z})^*_{\geq 0}$ *and* $\mathrm{rank}(\boldsymbol{\beta})\geq 2n$. *Moreover, when* $k^p_f = \mathbf{1}_{4n+2}$,

$$b^{Np\infty}_{H'}(s,\eta\chi)\,\mathfrak{E}^{\mathrm{Si}}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}}, f_{\underline{\epsilon}\cdot k,\kappa}(s))\Big|_{s=n+1-k} \tag{2.4.4}$$

is nonvanishing only when $\boldsymbol{\beta}$ *is non-degenerate. For such* $\boldsymbol{\beta}\in N^{-1}\,\mathrm{Sym}(2n+1,\mathbb{Z})^*_{>0}$, *we have*

$$\begin{aligned}(2.4.4) = {} & \frac{\sqrt{-1}^{(2n+1)k}2^{(2n+1)(k-n)}\pi^{(n+1)(2n+1)}}{\Gamma_{2n+1}(n+1)}\, h_{\mathbb{Q}_v}(2\boldsymbol{\beta})\gamma(\overline{\mathbf{e}}_v)^{2n^2+2n}\gamma(\det 2\boldsymbol{\beta}\cdot\overline{\mathbf{e}}_v)^{2n} \\ & \times \prod_{v\mid N}\eta_v^{-1}(\det\boldsymbol{\beta})\,\widehat{\alpha}^{\mathrm{vol}}_v(\boldsymbol{\beta})\,\chi(|\det 2\boldsymbol{\beta}|_v)|\det 2\boldsymbol{\beta}|_v^{k-n-1} \\ & \times \prod_{\substack{v\mid \det 2\boldsymbol{\beta} \\ v\nmid Np\infty}} g_{\boldsymbol{\beta},v}\left(\eta\chi(q_v)q_v^{k-(2n+2)}\right)\,\widehat{\alpha}_{\underline{\epsilon}\cdot k,\kappa,p}(\boldsymbol{\beta})\,(\det\boldsymbol{y})^{\frac{k}{2}}\mathbf{e}_\infty(\mathrm{Tr}\boldsymbol{\beta}\boldsymbol{z}).\end{aligned} \tag{2.4.5}$$

Here $h_{\mathbb{Q}_v}(2\boldsymbol{\beta})$ *is the Hasse invariant and* $\gamma(\det 2\boldsymbol{\beta},\overline{\mathbf{e}}_v)$ *is ratio of Weil indices. For our purpose it suffices to know that they are eight roots of unity. The term* $\widehat{\alpha}^{\mathrm{vol}}_v(\boldsymbol{\beta})$ *is given as*

$$\widehat{\alpha}^{\mathrm{vol}}_v(\boldsymbol{\beta}) = N^{-(n+1)^2}\,\mathbf{e}_v\left(\mathrm{Tr}(2\beta_{0,ij})_{1\leq i,j\leq n}\right)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)^*}(\beta)\,\mathbb{1}_{\mathrm{Sym}(n+1,\mathbb{Z}_v)^*}(N\beta')\,\mathbb{1}_{U_{N,v}}(N\beta_0),$$

where $U_{N,v} = \left\{x_0\in M_{n,n+1}(\mathbb{Z}_v) : x_0 \equiv -\begin{pmatrix} * & 0 & \cdots & 0 & 0 \\ * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \end{pmatrix} \bmod N\right\}$. *The* $g_{\boldsymbol{\beta},v}(\cdot)$*'s are polynomials with coefficients in* $\mathbb{Z}$.

Proof. Since we have chosen the "big cell" section at the place p,

$$\mathfrak{E}^{\mathrm{Si}}_{\boldsymbol{\beta}}(h', f_{\underline{\epsilon}\cdot k,\kappa}(s)) = \prod_v W_{\boldsymbol{\beta},v}(h', f_{\underline{\epsilon}\cdot k,\kappa}(s))$$

with

$$W_{\boldsymbol{\beta},v}(h'_v, f_{\underline{\epsilon}\cdot k,\kappa,v}(s)) = \int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)} f_{\underline{\epsilon}\cdot k,\kappa,v}(s)\left(\begin{pmatrix} 0 & -\mathbf{1}_{2n+1} \\ \mathbf{1}_{2n+1} & \varsigma \end{pmatrix} h_v\right)\mathbf{e}_v(-\mathrm{Tr}\boldsymbol{\beta}\varsigma)\, d_v\varsigma.$$

The proof is about computing $W_{\boldsymbol{\beta},v}(h_v, f_{\underline{\epsilon}\cdot k,\kappa,v}(s))$ place by place.

For $v=p$, an easy computation shows that $W_{\boldsymbol{\beta},p}(\mathbf{1}_{2n+2}, f_{\underline{\tau},\kappa,p}(s)) = \widehat{\alpha}_{\underline{\tau},\kappa,p}(\boldsymbol{\beta})$. Hence the support of our chosen $\alpha_{\underline{\tau},\kappa,p}$ implies that $\mathfrak{E}^{\mathrm{Si}}_{\boldsymbol{\beta}}(k^p_f\cdot h'_{\boldsymbol{z}}, f_{\underline{\epsilon}\cdot k,\kappa}(s))$ vanishes unless $\mathrm{rank}(\boldsymbol{\beta})\geq 2n$. For $v=\infty$ or unramified, the formulae for $W_{\boldsymbol{\beta},v}(h_v, f^{\mathrm{ur}}_v(s,\xi))$ and $W_{\boldsymbol{\beta},v}(h'_{\boldsymbol{z},\infty}, f^k_\infty(s,\mathrm{sgn}^k))$ are computed by Shimura [Shi82, Theorem 4.2] [Shi97, Theorem 13.6, Proposition 14.9]. We omit recalling them here, but simply mention that $W_{\boldsymbol{\beta},v}(h'_{\boldsymbol{z}}, f^k_\infty(s,\mathrm{sgn}^k))\big|_{s=n+1-k}\neq 0$ only if $\boldsymbol{\beta}$ is positive semi-definite. One can look at [LR18, §2.4]) and references there for precise formulas.

For the places dividing N, there is some difference from [Liu16, LR18]. Instead of the "big cell" section in $I_{Q_{H'},v}(s,\eta\chi)$, we have picked the intertwining of the "big cell" section in $I_{Q_{H'},v}(-s,\eta\chi)$. We need the following proposition.

Proposition 2.4.2. *For $v\mid N$ and $\alpha_v\in C^\infty_c(\mathrm{Sym}(2n+1,\mathbb{Q}_v))$, we have*

- *If $\boldsymbol{\beta}$ is of corank 1, then*

$$W_{\boldsymbol{\beta},v}(\mathbf{1}_{4n+2}, M_{Q_{H'},v}(s,\xi)f^{\alpha_v}(s,\xi)) = 0.$$

- *If $\boldsymbol{\beta}$ is non-degenerate, then*

$$W_{\boldsymbol{\beta},v}\left(h'_v, M_{Q_{H'},v}(s,\xi)f_v(s,\xi)\right) = c_v(s,\xi,\boldsymbol{\beta})\cdot W_{\boldsymbol{\beta},v}\left(h'_v, f_v(s,\xi)\right),$$

with

$$\begin{aligned} c_v(s,\xi,\boldsymbol{\beta}) =& h_{\mathbb{Q}_v}(2\boldsymbol{\beta})\gamma(\overline{\mathbf{e}}_v)^{2n^2+2n}\gamma(\det 2\boldsymbol{\beta},\overline{\mathbf{e}}_v)^{2n}\,\xi_v(\det 2\boldsymbol{\beta})^{-1}|\det 2\boldsymbol{\beta}|_v^{-s} \\ &\times\left(\gamma_v\left(s-n,\xi\right)\prod_{j=1}^n\gamma_v\left(2s-2n-1+2j,\xi^2\right)\right)^{-1}. \end{aligned}$$

Proof. The second statement directly follows from the functional equation for non-degenerate Fourier coefficients [LR05, (14)] [Swe95, Proposition 4.8].

Assume that $\boldsymbol{\beta}\in\mathrm{Sym}(2n+1,\mathbb{Q})$ is of corank 1. By definition,

(2.4.6)

$$\begin{aligned} &W_{\boldsymbol{\beta},v}\left(\mathbf{1}_{4n+2}, M_{Q_{H'},v}(s,\xi)f^{\alpha_v}(s,\xi)\right) \\ =&\int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)}\int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)} f^{\alpha_v}\left(w_{H'}\begin{pmatrix}\mathbf{1}_{2n+1} & \tau \\ 0 & \mathbf{1}_{2n+1}\end{pmatrix}w_{H'}\begin{pmatrix}\mathbf{1}_{2n+1} & \sigma \\ 0 & \mathbf{1}_{2n+1}\end{pmatrix}\right)\mathbf{e}_v\left(-\mathrm{Tr}\boldsymbol{\beta}\sigma\right)\,d\tau\,d\sigma \\ =&\int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)}\int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)}\xi^{-1}(\det\tau)|\det\tau|_v^{-s-n-1}\alpha_v(\sigma-\tau^{-1})\,\mathbf{e}_v\left(-\mathrm{Tr}\boldsymbol{\beta}\sigma\right)\,d\tau\,d\sigma. \end{aligned}$$

We can further assume that $\boldsymbol{\beta}=\begin{pmatrix}\boldsymbol{\beta}_1 & 0\\ 0 & 0\end{pmatrix}$ for some $\boldsymbol{\beta}_1=\mathrm{diag}(b_1,\dots,b_n)$ with $b_1,\dots b_n\neq 0$, and that there exists $\alpha_{1,v}\in C^\infty_c(\mathrm{Sym}(2n,\mathbb{Q}_v))$, $\alpha_{2,v}\in C^\infty_c(\mathbb{Q}_v^{2n})$ and $\alpha_{4,v}\in C^\infty_c(\mathbb{Q}_v)$ such that

$$\alpha_v\left(\begin{pmatrix}\tau_1 & \tau_2 \\ {}^{\mathrm{t}}\tau_2 & \tau_4\end{pmatrix}\right) = \alpha_{1,v}(\tau_1)\alpha_{2,v}(\tau_2)\alpha_{4,v}(\tau_4).$$

Then by the change of variable $\tau^{-1}\to\tau$,

$$(2.4.6)=\int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)}\int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)}\xi(\det(\tau_1-\tau_2\tau_4^{-1}{}^{\mathrm{t}}\tau_2))|\det(\tau_1-\tau_2\tau_4^{-1}{}^{\mathrm{t}}\tau_2)|_v^{s-n-1}\xi(\tau_4)|\tau_4|_v^{s-n-1}$$
$$\times\,\mathbf{e}_v(-\mathrm{Tr}\boldsymbol{\beta}_1\sigma_1)\alpha_{1,v}(\sigma_1-\tau_1)\alpha_{2,v}(\sigma_2-\tau_2)\alpha_{4,v}(\sigma_4-\tau_4)\,d\tau\,d\sigma$$
$$=\int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)}\int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)}\xi(\det\tau_1)|\det\tau_1|_v^{s-n-1}\xi(\tau_4)|\tau_4|_v^{s-n-1}\,\mathbf{e}_v(-\mathrm{Tr}\boldsymbol{\beta}_1\sigma_1)$$
$$\times\,\alpha_{1,v}(\sigma_1-\tau_1+\tau_2\tau_4^{-1}{}^{\mathrm{t}}\tau_2)\alpha_{2,v}(\sigma_2-\tau_2)\alpha_{4,v}(\sigma_4-\tau_4)\,d\tau\,d\sigma$$

Since $\boldsymbol{\beta}_1$ is non-degenerate, we can change the order of integration for τ_1,σ_1 and τ_2,σ_2, and get

$$(2.4.6)=\int_{\mathrm{Sym}(2n,\mathbb{Q}_v)}\alpha_{1,v}(\sigma_1)\mathbf{e}_v(-\mathrm{Tr}\boldsymbol{\beta}_1\sigma_1)\,d\sigma_1\int_{\mathrm{Sym}(2n,\mathbb{Q}_v)}\xi(\det\tau_1)|\det\tau_1|_v^{s-n-1}\mathbf{e}_v(-\mathrm{Tr}\boldsymbol{\beta}_1\tau_1)\,d\tau_1$$
$$\times\int_{\mathbb{Q}_v^{2n}}\alpha_{2,v}(\sigma_2)\,d\sigma_2\int_{\mathbb{Q}_v}\int_{\mathbb{Q}_v}\xi(\tau_4)|\tau_4|_v^{s-n-1}\alpha_{4,v}(\sigma_4-\tau_4)\int_{\mathbb{Q}_v^{2n}}\mathbf{e}_v(\mathrm{Tr}\boldsymbol{\beta}_1\tau_2\tau_4^{-1}{}^{\mathrm{t}}\tau_2)\,d\tau_2\,d\tau_4\,d\sigma_4.$$

By [Swe95, Lemma 4.4] we have

$$\int_{\mathbb{Q}_v}\mathbf{e}_v\left(b_j\tau_4^{-1}\tau_{2,j}^2\right)\,d\tau_{2,j}=|2b_j\tau_4^{-1}|_v^{-\frac{1}{2}}\gamma(b_j\tau_4^{-1}\overline{\mathbf{e}}_v).$$

Plugging it into the above equality and putting $\lambda_{\boldsymbol{\beta}_1}=(\cdot,\det\boldsymbol{\beta}_1)_{\mathbb{Q}_v}$ (the Hilbert symbol), we get

$$(2.4.6)=\int_{\mathrm{Sym}(2n,\mathbb{Q}_v)}\alpha_{1,v}(\sigma_1)\mathbf{e}_v(-\mathrm{Tr}\boldsymbol{\beta}_1\sigma_1)\,d\sigma_1\int_{\mathrm{Sym}(2n,\mathbb{Q}_v)}\xi(\det\tau_1)|\det\tau_1|_v^{s-n-1}\mathbf{e}_v(-\mathrm{Tr}\boldsymbol{\beta}_1\tau_1)\,d\tau_1$$
$$\times\int_{\mathbb{Q}_v^{2n}}\alpha_{2,v}(\sigma_2)\,d\sigma_2\cdot|\det2\boldsymbol{\beta}_1|^{-\frac{1}{2}}h_{\mathbb{Q}_v}(2\boldsymbol{\beta}_1)\gamma(\overline{e}_v)^{2n}\gamma(\det2\boldsymbol{\beta}_1,\overline{e}_v)$$
$$\times\int_{\mathbb{Q}_v}\int_{\mathbb{Q}_v}\xi\lambda_{\boldsymbol{\beta}_1}(\tau_4)|\tau_4|_v^{s-1}\alpha_{4,v}(\sigma_4-\tau_4)\,d\tau_4\,d\sigma_4.$$

Now denote by $f^{\alpha_{4,v}}_{\mathrm{SL}(2)}(s,\xi\lambda_{\boldsymbol{\beta}_1})$ the "big cell" section inside $\mathrm{Ind}^{\mathrm{SL}(2,\mathbb{Q}_v)}_{B(\mathbb{Q}_v)}(\xi\lambda_{\boldsymbol{\beta}_1}|\cdot|^s)$. Then

$$\begin{aligned}\int_{\mathbb{Q}_v}\xi\lambda_{\boldsymbol{\beta}_1}(\tau_4)|\tau_4|_v^{s-1}\alpha_{4,v}(\sigma_4-\tau_4)\,d\tau_4&=\left(M_{B,v}(-s,\xi^{-1}\lambda_{\boldsymbol{\beta}_1}^{-1})M_{B,v}(s,\xi\lambda_{\boldsymbol{\beta}_1})f^{\alpha_{4,v}}_{\mathrm{SL}(2)}(s,\xi\lambda_{\boldsymbol{\beta}_1})\right)(\mathbf{1}_2)\\&=\gamma_v(s+1,\xi\lambda_{\boldsymbol{\beta}_1})\gamma_v(-s+1,\xi^{-1}\lambda_{\boldsymbol{\beta}_1}^{-1})\cdot f^{\alpha_{4,v}}_{\mathrm{SL}(2)}(s,\xi\lambda_{\boldsymbol{\beta}_1})(\mathbf{1}_2)\\&=0\end{aligned}$$

which implies that $(2.4.6)=0$ and the proposition is proved. □

Combining this proposition for $v\mid N$ with the fact $W_{\boldsymbol{\beta},p}(\mathbf{1}_{2n+2},f_{\underline{\tau},\kappa,p}(s))=\widehat{\alpha}_{\underline{\tau},\kappa,p}(\boldsymbol{\beta})$ and the formulae in [LR18, §2.4]) for v unramified or archimedean finishes the proof of Proposition 2.4.1. □

Next we normalize the Siegel Eisenstein series $\mathfrak{E}^{\mathrm{Si}}(\cdot,f_{\underline{\tau},\kappa}(s))$ so that its restriction to $G\times G'$ at $s=n+1-k$ is the image of a geometrically defined nearly holomorphic Siegel modular form $\mathcal{E}_{\underline{\tau},\kappa}$ under the embedding (2.5.2). We also compute the p-adic q-expanisons of the $\mathcal{E}_{\underline{\tau},\kappa}$'s at the infinity cusp and see that they admit p-adic interpolation.

2.5. p-adic q-expansion of $\mathcal{E}_{\underline{\tau},\kappa}$.

2.5.1. *Nearly holomorphic Siegel modular forms.* Let $\Gamma \subset G(\mathbb{Z})$ (resp. $\Gamma' \subset G'(\mathbb{Z})$) be a congruence subgroup. Let $X^{\Sigma}_{G,\Gamma}$ (resp. $X^{\Sigma'}_{G',\Gamma'}$) be a smooth toroidal compactification of the Siegel variety (definite over a number field) of level Γ (resp. Γ').

Let $\mathcal{V}_{\underline{t}} = \bigcup_{r\geq 0} \mathcal{V}^r_{\underline{t}}$ (resp. $\mathcal{V}_{\underline{t},k} = \bigcup_{r\geq 0} \mathcal{V}^r_{(\underline{t},k)}$) the locally free sheaf over $X^{\Sigma}_{G,\Gamma}$ (resp. $X^{\Sigma'}_{G',\Gamma'}$) of nearly holomorphic Siegel modular forms of weight $\underline{t}$ (resp. $(\underline{t},k)$). There are canonical embeddings

$$H^0\left(X^{\Sigma}_{G,\Gamma}, \mathcal{V}_{\underline{t}}\right) \longrightarrow \mathcal{A}\left(G(\mathbb{Q})\backslash G(\mathbb{A})\right), \quad H^0\left(X^{\Sigma'}_{G',\Gamma'}, \mathcal{V}_{(\underline{t},k)}\right) \longrightarrow \mathcal{A}\left(G'(\mathbb{Q})\backslash G'(\mathbb{A})\right), \tag{2.5.1}$$

$$H^0\left(X^{\Sigma}_{G,\Gamma} \times X^{\Sigma'}_{G',\Gamma'}, \mathcal{V}_{\underline{t}} \boxtimes \mathcal{V}_{(\underline{t},k)}\right) \longrightarrow \mathcal{A}\left(G(\mathbb{Q}) \times G'(\mathbb{Q})\backslash G(\mathbb{A}) \times G'(\mathbb{A})\right), \tag{2.5.2}$$

as explained at the end of [Liu16, §2.4]. In order to attach an adelic form to a global section of automorphic bundle defined from an irreducible $\mathrm{GL}(n)$-representation σ, one needs to pick a linear functional on σ. We shall always view algebraic irreducible representations of $\mathrm{GL}(m)$ as realized as

$$\left\{ f : \mathrm{GL}(m) \to \mathbb{A}^1 : f(gb) = a_1^{-l_m} \dots a_m^{-l_1} f(g) \text{ for all } b = \begin{pmatrix} a_1 & \cdots & * \\ & \ddots & \vdots \\ & & a_m \end{pmatrix} \in \mathrm{GL}(m) \right\}, \tag{2.5.3}$$

with $(l_1, \dots, l_m)$ being a dominant weight and $\mathrm{GL}(m)$ acting by left inverse translation. By canonical embedding into adelic forms, we mean the linear functional picked as the evaluation at identity.

2.5.2. *p-adic Siegel modular forms and q-expansions.* Let F be a finite extension of $\mathbb{Q}_p$, which is always assumed to be sufficiently large. Let $X^{\Sigma}_G = X^{\Sigma}_{G,\Gamma_G(N)}$ (resp. $X^{\Sigma'}_{G} = X^{\Sigma'}_{G',\Gamma_{G'}(N)}$) be the smooth toroidal compactification of the Seigel variety over $\mathbb{Z}[\zeta_N, 1/N]$ of principal level N, over which there is the semi-abelian scheme $\mathcal{G} \to X^{\Sigma,\mathrm{ord}}_G$ (resp. $\mathcal{G}' \to X^{\Sigma',\mathrm{ord}}_{G'}$) extending the universal principally polarized abelian scheme. Pick a lift E of a certain power of the Hasse invariant, and define

$$X^{\Sigma,\mathrm{ord}}_G = X^{\Sigma}_G[1/E] \qquad\qquad (\text{resp. } X^{\Sigma',\mathrm{ord}}_{G'} = X^{\Sigma'}_{G'}[1/E]).$$

Let $X^{\Sigma,\mathrm{ord}}_{G,m}$ (resp. $X^{\Sigma',\mathrm{ord}}_{G',m}$) be the reduction modulo p^m of $X^{\Sigma,\mathrm{ord}}_G$ (resp. $X^{\Sigma',\mathrm{ord}}_{G'}$), which is independent of the choice of E and is called the ordinary locus.

The Igusa tower over $X^{\Sigma,\mathrm{ord}}_{G,m}$ (resp. $X^{\Sigma',\mathrm{ord}}_{G',m}$) is defined as

$$\mathscr{T}^{\Sigma}_{G,m,l} = \underline{\mathrm{Isom}}_{X^{\Sigma,\mathrm{ord}}_{G,m}}\left((\mathcal{G}[p^m])^{D,\text{ét}}, (\mathbb{Z}/p^m\mathbb{Z})^n\right)$$
$$(\text{resp. } \mathscr{T}^{\Sigma'}_{G',m,l} = \underline{\mathrm{Isom}}_{X^{\Sigma',\mathrm{ord}}_{G',m}}\left((\mathcal{G}'[p^m])^{D,\text{ét}}, (\mathbb{Z}/p^m\mathbb{Z})^{n+1}\right)),$$

where the superscript D means the Cartier dual. There is a natural action of $\mathrm{GL}(n,\mathbb{Z}_p)$ (resp. $\mathrm{GL}(n+1,\mathbb{Z}_p)$) on $\mathscr{T}^{\Sigma}_{G,m,l}$ (resp. $\mathscr{T}^{\Sigma'}_{G',m,l}$). Denote by B_n (resp. B_{n+1}) the standard Borel subgroup of $\mathrm{GL}(n)$ (resp. $\mathrm{GL}(n+1)$) consisting of upper triangular matrices, and by N_n (resp. N_{n+1}) the its unipotent subgroup. Define

$$V_{G,m,l} = H^0\left(\mathscr{T}^{\Sigma}_{G,m,l}, \mathcal{O}_{\mathscr{T}^{\Sigma}_{G,m,l}}\right)^{N_n(\mathbb{Z}_p)}, \qquad V_{G',m,l} = H^0\left(\mathscr{T}^{\Sigma'}_{G',m,l}, \mathcal{O}_{\mathscr{T}^{\Sigma'}_{G',m,l}}\right)^{N_{n+1}(\mathbb{Z}_p)}.$$

The left hand side of (2.5.1)(2.5.2) embeds into p-adic Siegel modular forms [Liu16, §6.2.1], *i.e.*

$$\begin{aligned} &H^0\left(X^{\Sigma}_{G,\Gamma_G(N)\cap\Gamma_{G,1}(p^l)}, \mathcal{V}_{\underline{t}}\right) \longrightarrow \left(\varprojlim_m \varinjlim_l V_{G,m,l}\right)[1/p], \\ &H^0\left(X^{\Sigma'}_{G',\Gamma_{G'}(N)\cap\Gamma_{G',1}(p^l)}, \mathcal{V}_{(\underline{t},k)}\right) \longrightarrow \left(\varprojlim_m \varinjlim_l V_{G',m,l}\right)[1/p], \end{aligned} \tag{2.5.4}$$

(2.5.5)

$$H^0\left(X^{\Sigma}_{G,\Gamma_G(N)\cap\Gamma_{G,1}(N)}\times X^{\Sigma'}_{G',\Gamma_{G'}(N)\cap\Gamma_{G',1}(p^l)},\mathcal{V}_{\underline{t}}\boxtimes\mathcal{V}_{(\underline{t},k)}\right)\longrightarrow\left(\varprojlim_m\varinjlim_l V_{G,m,l}\otimes_{\mathcal{O}_F}V_{G',m,l}\right)[1/p].$$

For each $\gamma_N\in G(\mathbb{Z}/N)$ and $\gamma'_N\in G'(\mathbb{Z}/N)$, by evaluating the elements in $V_{m,l}$ and $V'_{m,l}$ at the Mumford object, one defines the (p-adic) q-expansion maps

$$\epsilon^{\gamma_N}_{q\text{-exp}}:\varprojlim_m\varinjlim_l V_{G,m,l}\longrightarrow\mathcal{O}_F[[N^{-1}\operatorname{Sym}(n,\mathbb{Z})^*_{\geq 0}]]$$

$$\epsilon^{\gamma'_N}_{q\text{-exp}}:\varprojlim_m\varinjlim_l V_{G',m,l}\longrightarrow\mathcal{O}_F[[N^{-1}\operatorname{Sym}(n+1,\mathbb{Z})^*_{\geq 0}]],$$

and

$$\epsilon^{\gamma_N,\gamma'_N}_{q\text{-exp}}:\varprojlim_m\varinjlim_l V_{G,m,l}\otimes_{\mathcal{O}_F}V_{G',m,l}\longrightarrow\mathcal{O}_F[[N^{-1}\operatorname{Sym}(n,\mathbb{Z})^*_{\geq 0}\oplus N^{-1}\operatorname{Sym}(n+1,\mathbb{Z})^*_{\geq 0}]]. \tag{2.5.6}$$

The injectivity follows from the irreducibility of the Igusa tower [FC90, V.7] [Hid02, Theorem 3.1].

2.5.3. *Defining the $\mathcal{E}_{\underline{\tau},\kappa}$'s.* Now we normalize the Siegel Eisenstein series as

$$\mathfrak{E}^{\mathrm{Si},*}(\cdot,f_{\underline{\tau},\kappa}):=\frac{\Gamma_{2n+1}(n+1)}{\sqrt{-1}^{(2n+1)k}2^{(2n+1)(k-n)}\pi^{(n+1)(2n+1)}}\;b^{Np\infty}_{H'}(s,\eta\chi)\,\mathfrak{E}^{\mathrm{Si}}\left(\cdot,f_{\underline{\tau},\kappa}(s)\right)\Big|_{s=n+1-k}. \tag{2.5.7}$$

It is a standard fact that the q-expansions of nearly holomorphic Siegel modular (as global sections over Siegel varieties) can be computed in terms of the Fourier coefficients of their embeddings into the space of adelic automorphic forms. From Proposition 2.4.1 plus the q-expansion principle and the correspondence between algebraic differential operators and the Lie algebra action, we deduce that the automorphic form

$$\mathfrak{E}^{\mathrm{Si},*}(\cdot,f_{\underline{\tau},\kappa})\big|_{G\times G'} \tag{2.5.8}$$

belongs to the image of the embedding (2.5.2).

For admissible $(\underline{\tau},\kappa)$ with $\kappa(-1)=\eta(-1)$, we define

$$\mathcal{E}_{\underline{\tau},\kappa}\in H^0\left(X^{\Sigma}_{G,\Gamma_G(N)\cap\Gamma_{G,1}(p^l)}\times X^{\Sigma'}_{G',\Gamma_{G'}(N)\cap\Gamma_{G',1}(p^l)},\mathcal{V}_{\underline{t}}\boxtimes\mathcal{V}_{(\underline{t},k)}\right)$$

as the element whose image under (2.5.2) equals (2.5.8). We will also view $\mathcal{E}_{\underline{\tau},\kappa}$ as an element in $\left(\varprojlim_m\varinjlim_l V_{G,m,l}\otimes_{\mathcal{O}_F}V'_{G',m,l}\right)[1/p]$ via the embedding (2.5.5).

2.5.4. *The (p-adic) q-expansion of $\mathcal{E}_{\underline{\tau},\kappa}$.* From Proposition 2.4.1 on the Fourier coefficents of the Siegel Eisenstein series $\mathfrak{E}^{\mathrm{Si}}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}},f_{\underline{\epsilon}\cdot k,\kappa}(s))\Big|_{s=n+1-k}$, the same argument as in [Liu16, Proposition 4.4.1] gives the following proposition on the (p-adic) q-expansions of $\mathcal{E}_{\underline{\tau},\kappa}$.

Proposition 2.5.1. *For $(\beta,\beta')\in N^{-1}\operatorname{Sym}(n,\mathbb{Z})^*_{\geq 0}\oplus N^{-1}\operatorname{Sym}(n+1,\mathbb{Z})^*_{\geq 0}$ and $(\gamma_N,\gamma'_N)\in G(\mathbb{Z}/N)\times G'(\mathbb{Z}/N)$, let $\varepsilon^{\gamma_N,\gamma'_N}_{q\text{-exp}}\left(\beta,\beta',\mathcal{E}_{\underline{\tau},\kappa}\right)$ denote the coefficient indexed by (β,β') in the image of $\mathcal{E}_{\underline{\tau},\kappa}$ under the q-expansion map (2.5.6). Then $\varepsilon^{\gamma_N,\gamma'_N}_{q\text{-exp}}\left(\beta,\beta',\mathcal{E}_{\underline{\tau},\kappa}\right)$ vanishes unless $\beta>0$ and $\operatorname{rank}(\beta')\geq n$. If $\gamma_N=\gamma'_N=1$, the coefficient $\left(\beta,\beta',\mathcal{E}_{\underline{\tau},\kappa}\right)$ vanishes unless $\beta,\beta'>0$ and for such (β,β') we have*

$$\varepsilon^{1,1}_{q\text{-exp}}\left(\beta,\beta',\mathcal{E}_{\underline{\tau},\kappa}\right)=\sum_{\boldsymbol{\beta}=\begin{pmatrix}\beta&\beta_0\\ {}^{\mathrm{t}}\beta_0&\beta'\end{pmatrix}\in N^{-1}\operatorname{Sym}(2n+1,\mathbb{Z})^*_{>0}}\mathfrak{c}_{\underline{\tau},\kappa}(\boldsymbol{\beta}), \tag{2.5.9}$$

with

$$
\begin{aligned}
\mathfrak{c}_{\underline{\tau},\kappa}(\boldsymbol{\beta}) = &\prod_{v|N} h_{\mathbb{Q}_v}(2\boldsymbol{\beta})\gamma(\overline{\mathbf{e}}_v)^{2n^2+2n}\gamma(\det 2\boldsymbol{\beta}\cdot\overline{\mathbf{e}}_v)^{2n}\,\eta_v^{-1}(\det\boldsymbol{\beta})\,\widehat{\alpha}_v^{\mathrm{vol}}(\boldsymbol{\beta})\\
&\times\kappa\left(|\det 2\boldsymbol{\beta}|_v\right)|\det 2\boldsymbol{\beta}|_N^{-n-1}\prod_{\substack{v|\det(2\boldsymbol{\beta})\\ v\nmid Np\infty}} g_{\boldsymbol{\beta},v}\left(\eta(q_v)\kappa(q_v)q_v^{-2n-2}\right)\\
&\times\prod_{j=1}^{n}\mathbb{1}_{\mathbb{Z}_p^\times}(\det_j((2\beta_0))\prod_{j=1}^{n-1}\tau_j\tau_{j+1}^{-1}\left(\det_j(2\beta_0)\right)\,\tau_n\kappa^{-1}\left(\det_n(2\beta_0)\right).
\end{aligned}
\tag{2.5.10}
$$

2.6. **Construction of the ordinary family of Klingen Eisenstein series.**

2.6.1. *Recalling some Hida theory.* As our goal is to construct and study p-adic families of Klingen Eisenstein series, the usual Hida theory for cuspidal Siegel modular form [Hid02] [Pil12] does not suffice for our purpose. We need the Hida theory for Siegel modular forms vanishing along the strata with cusp labels of rank >1 as developed in [LR18]. We briefly recall some notation and facts *loc. cit.*.

There is a stratification of the (partial) toroidal compactification $X^{\Sigma',\mathrm{ord}}_{G'),m}$ indexed by cotorsion free isotropic $\mathbb{Z}$-submodules of $\mathbb{V}'_0$, where $\mathbb{V}'_0\subset\mathbb{V}'$ is the $\mathbb{Z}$-lattice spanned by $e'_1,\dots,e'_{n+1},f'_1,\dots,f'_{n+1}$. (Here we do not need the finer stratification indexed by cones in the polyhedral cone decomposition.) The union of all the strata indexed by the $\mathbb{Z}$-submodules of rank strictly larger than 1 is a closed subscheme in $X^{\Sigma',\mathrm{ord}}_{G',m}$, and we denote the corresponding ideal sheaf by $\mathcal{I}^1_{X^{\Sigma',\mathrm{ord}}_{G',m}}$. Write $f_{m,l}:\mathscr{T}^{\Sigma}_{G',m,l}\to X^{\Sigma',\mathrm{ord}}_{G',m}$ for the projection from the Igusa tower to the ordinary locus of Siegel variety. The space of p-adic Siegel modular forms vanishing along the strata with cusp labels of rank >1 is defined as

$$
V^1_{G',m,l}=H^0\left(\mathscr{T}^{\Sigma}_{G',m,l},\,f^*_{m,l}\mathcal{I}^1_{X^{\Sigma',\mathrm{ord}}_{G',m}}\right)^{N_{n+1}(\mathbb{Z}_p)},\qquad \mathcal{V}^1_{G'}=\varinjlim_m\varinjlim_l V^1_{G',m,l},
$$

which are natural $\mathcal{O}_F[\![T_{n+1}(\mathbb{Z}_p)]\!]$-modules.

There are $\mathbb{U}_p$-operators acting on $\mathcal{V}_{G'}$ preserving the subspace $\mathcal{V}^1_{G'}$ [LR18, §1.9]. According to [LR18, Theorem 1.3.1], there exists an ordinary projector $e_{G'}=(e_{G'})^2$ on $\mathcal{V}^1_{G'}$ constructed as limit of powers of $\mathbb{U}_p$-operators. Define

$$
\left(\mathcal{V}^1_{G',\mathrm{ord}}\right)^*=\mathrm{Hom}_{\mathbb{Z}_p}\left(e_{G'}\mathcal{V}^1_{G'},\mathbb{Q}_p/\mathbb{Z}_p\right),\qquad \mathcal{M}^1_{G',\mathrm{ord}}=\mathrm{Hom}_{\Lambda_{n+1}}\left(\left(\mathcal{V}^1_{G',\mathrm{ord}}\right)^*,\Lambda_{n+1}\right),
$$

where $\Lambda_{n+1}=\mathcal{O}_F[[T_{n+1}(1+p\mathbb{Z}_p)]]$. Both $\left(\mathcal{V}^1_{G',\mathrm{ord}}\right)^*$ and $\mathcal{M}^1_{G',\mathrm{ord}}$ are $\mathcal{O}_F[[T_{n+1}(\mathbb{Z}_P)]]$-modules, and they are free of finite rank as Λ_{n+1}-modules. If $\mathcal{P}_{\underline{\tau},\kappa}$ is the ideal attached to an admissible point $(\underline{\tau},\kappa)\in\mathrm{Hom}_{\mathrm{cont}}\left(T_{n+1}(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times\right)$, then there is the Hecke-equivariant embedding

$$
\begin{aligned}
&\varinjlim_l e_{G'}H^0\left(X^{\Sigma'}_{G',\Gamma_{G'}(N)\cap\Gamma_{G',1}(p^l)},\mathcal{V}_{(\underline{t},k)}\otimes\mathcal{I}^1_{X^{\Sigma'}_{G',\Gamma_{G'}(N)\cap\Gamma_{G',1}(p^l)}}\right)_{\epsilon,\chi}\hookrightarrow\left(\varprojlim_m\varinjlim_l eV_{G',m,l}[(\underline{\tau},\kappa)]\right)[1/p]\\
&\qquad\xrightarrow{\sim}\mathcal{M}^1_{G',\mathrm{ord}}\otimes\mathcal{O}_F[\![T_{n+1}(\mathbb{Z}_p)]\!]/\mathcal{P}_{\kappa,\underline{\tau}},
\end{aligned}
\tag{2.6.1}
$$

where the subscript ${}_{\epsilon,\chi}$ means the nebentypus at p.

Replacing G' with G and 1 with 0, we define $V^0_{G,m,l}$, $\mathcal{V}^0_G$, the space of cuspidal p-adic Siegel modular forms of genus n. The cuspidal Hida theory indicates the existence of ordinary projectors and analogous properties as above for $\left(\mathcal{V}^0_{G,\mathrm{ord}}\right)^*$, $\mathcal{M}^0_{G,\mathrm{ord}}$.

Now we define the space which our later constructed measures take values in. Let

$$V^{0,1,\Delta} \subset \varprojlim_m \varinjlim_l V^0_{G,m,l} \otimes_{\mathcal{O}_F} V^1_{G',m,l}$$

be the subspace annihilated by $\gamma \otimes 1 - 1 \otimes (\gamma, 1)$ for all $\gamma \in T_n(\mathbb{Z}_p)$, which admits a natural $\mathcal{O}_F[[T_{n+1}(\mathbb{Z}_p)]]$-module structure. Put

$$V^{0,1,\Delta}_{\mathrm{ord}} = (e_G \times e_{G'}) V^{0,1,\Delta}.$$

Denote by $\mathscr{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, V^{0,1,\Delta}_{\mathrm{ord}}\right)^\natural$ the subspace of $\mathscr{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, V^{0,1,\Delta}_{\mathrm{ord}}\right)$ consisting of measures μ satisfying

$$\delta_{(a_1,\dots,a_n,a_{n+1})} * \mu = \mathrm{diag}(a_1, \dots, a_n, a_{n+1}) \cdot \mu, \qquad (a_1, \dots, a_n, a_{n+1}) \in T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times,$$

where $\delta_{(a_1,\dots,a_n.a_{n+1})}$ is the measure in $\mathscr{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \mathcal{O}_F\right)$ sending a continuous function to its value at $(a_1, \dots, a_n, a_{n+1})$, and

$$* : \mathscr{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \mathcal{O}_F\right) \times \mathscr{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, V^{0,1,\Delta}_{\mathrm{ord}}\right) \longrightarrow \mathscr{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, V^{0,1,\Delta}_{\mathrm{ord}}\right)$$

is the convolution of measures on abelian groups.

The canonical pairing

$$V^{0,1,\Delta}_{\mathrm{ord}} \times \left(\left(\mathcal{V}^0_{G,\mathrm{ord}}\right)^* \times \left(\mathcal{V}^1_{G',\mathrm{ord}}\right)^* \right) \longrightarrow \mathcal{O}_F$$

induces the morphism of $\mathcal{O}_F[[T_{n+1}(\mathbb{Z}_p)]]$-modules

$$\Phi^\Delta : \mathscr{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, V^{0,1,\Delta}_{\mathrm{ord}}\right)^\natural \longrightarrow \mathcal{M}^0_{G,\mathrm{ord}} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]} \mathcal{M}^1_{G',\mathrm{ord}}, \tag{2.6.2}$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathscr{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, V^{0,1,\Delta}_{\mathrm{ord}}\right)^\natural \arrow[r, "\Phi^\Delta"] \arrow[dr, "\mu \mapsto \int_{T_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times} (\underline{\tau},\kappa)\, d\mu"'] & \mathcal{M}^0_{G,\mathrm{ord}} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]} \mathcal{M}^1_{G',\mathrm{ord}} \arrow[d, "\mathscr{s}_{\kappa,\underline{\tau}}"] \\
& \varprojlim_m \varinjlim_l \left(eV^0_{G,m,l}[\underline{\tau}] \otimes_{\mathcal{O}_F} eV^1_{G',m,l}[(\underline{\tau}, \kappa)] \right),
\end{tikzcd}$$

where the specialization map $\mathscr{s}_{\kappa,\underline{\tau}}$ is defined by mod $\mathcal{P}_{\kappa,\underline{\tau}}$ and the isomorphism in (2.6.1).

2.6.2. *Formulae for the adelic $\mathbb{U}_p$-operators.* Via the embedding of (2.5.1) and (2.5.4), the action of the $\mathbb{U}_p$-operators on the space of p-adic Siegel modular forms induces a $\mathbb{U}_p$-action on the automorphic forms on G (resp. G') which belong to the image of (2.5.1) for some $\underline{t}, k$. This action admits the following formulae.

Suppose that φ' (resp. φ) is a nearly holomorphic form of weight $(\underline{t}, k)$ (resp. $\underline{t}$). Given a decreasing $n+1$-tuple (resp. n-tuple) of positive integers $\underline{b}' = (b'_1, \dots, b'_{n+1})$ (resp. $\underline{b} = (b_1, \dots, b_n)$), the action of the $\mathbb{U}_p$-operator $U_{p,\underline{b}'}$ (resp. $U_{p,\underline{b}}$) on φ' (resp. φ) is

$$\left(U_{p,\underline{b}'}\varphi'\right)(g') = p^{\left\langle (\underline{t},k)+2\rho_{G',c},\, \underline{b}' \right\rangle} \int_{U_{B_{G'}}(\mathbb{Z}_p)} \varphi(g' u p^{\underline{b}'})\, du$$

$$\Big(\text{resp. } \left(U_{p,\underline{b}}\varphi\right)(g) = p^{\left\langle \underline{t}+2\rho_{G,c},\, \underline{b} \right\rangle} \int_{U_{B_G}(\mathbb{Z}_p)} \varphi(g u p^{\underline{b}})\, du\Big).$$

Here $B_{G'}$ is the standard Borel subgroup of G' consisting of $\begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ 0 & \mathfrak{d} \end{pmatrix} \in G'$ with $\mathfrak{a}$ upper triangular and $U_{B_{G'}}$ is its unipotent radical. Similar definition applies to B_G, U_{B_G}. The weight

$$2\rho_{G',c} = (n, n-2, \dots, -n) \qquad \text{(resp. } 2\rho_{G,c} = (n-1, n-3, \dots, -n+1))$$

is the sum of compact roots of $\operatorname{Lie} G'$ (resp. $\operatorname{Lie} G$), and

$$p^{\underline{b}'} = \operatorname{diag}(p^{b_1}, \dots, p^{b_{n+1}}, p^{-b_1}, \dots, p^{-b_{n+1}}) \qquad \text{(resp. } p^{\underline{b}} = \operatorname{diag}(p^{b_1}, \dots, p^{b_n}, p^{-b_1}, \dots, p^{-b_n})).$$

Up to scalar the operator is purely local, but the normalization crucially depends on the archimedean weight of the automorphic form.

2.6.3. *The ordinary family $\mathcal{E}_{\mathrm{ord}}$ on $G \times G'$.* It follows from the vanishing of the coefficients indexed by (β, β') with $\operatorname{rank}(\beta) < n$ or $\operatorname{rank}(\beta') < n$ in the q-expansion of $\mathcal{E}_{\underline{\tau},\kappa}$ at all p-adic cusps that

$$\mathcal{E}_{\underline{\tau},\kappa} \in \varprojlim_m \varinjlim_l V^0_{G,m,l}[\underline{\tau}] \otimes_{\mathcal{O}_F} V^1_{G',m,l}[(\underline{\tau},\kappa)] \subset V^{0,1,\Delta}.$$

Proposition 2.6.1. *There exists an ordinary family $\mathcal{E}_{\mathrm{ord}} \in \mathcal{M}^0_{G,\mathrm{ord}} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]} \mathcal{M}^1_{G',\mathrm{ord}}$ such that*

$$\jmath_{\underline{\tau},\kappa}(\mathcal{E}_{\mathrm{ord}}) = \begin{cases} (e_G \times e_{G'})\mathcal{E}_{\underline{\tau},\kappa}, & \text{if } (\underline{\tau},\kappa) \text{ is admissible and } \eta\chi(-1) = (-1)^k, \\ 0, & \text{if } \eta(-1) \neq \kappa(-1), \end{cases} \tag{2.6.3}$$

where $\mathcal{E}_{\underline{\tau},\kappa}$ is the p-adic Siegel modular form defined in §2.5.3.

Proof. First, a simple examination of the terms in (2.5.10) plus the theorem on the existence of Kubota–Leopoldt p-adic L-functions [Hid93, Theorem 4.4.1] verifies the existence of

$$\mu_{\mathcal{E},q\text{-exp}} \in \mathscr{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \mathcal{O}_F[\![N^{-1}\operatorname{Sym}(n,\mathbb{Z})^*_{\geq 0} \oplus N^{-1}\operatorname{Sym}(n+1,\mathbb{Z})^*_{\geq 0}]\!]\right)$$

with properties

$$\int_{T_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times} (\underline{\tau},\kappa)\, d\mu_{\mathcal{E},q\text{-exp}} = \begin{cases} \varepsilon^{1,1}_{q\text{-exp}}\left(\mathcal{E}_{\underline{\tau},\kappa}\right), & \text{if } (\underline{\tau},\kappa) \text{ is admissible and } \eta\chi(-1) = (-1)^k, \\ 0, & \text{if } \eta(-1) \neq \kappa(-1). \end{cases}$$

The p-adic density of admissible points implies that the measure $\mu_{\mathcal{E},q\text{-exp}}$ takes values inside the subspace of $\mathcal{O}_F[\![N^{-1}\operatorname{Sym}(n,\mathbb{Z})^*_{\geq 0} \oplus N^{-1}\operatorname{Sym}(n+1,\mathbb{Z})^*_{\geq 0}]\!]$ consisting of the q-expansions of p-adic forms. Hence, there exists $\mu_{\mathcal{E}} \in \mathscr{M}eas\left(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, V^{0,1,\Delta}\right)^\natural$ such that

$$\varepsilon^{1,1}_{q\text{-exp}}\left(\mu_{\mathcal{E}}\right) = \mu_{\mathcal{E},q\text{-exp}}.$$

Define the ordinary family $\mathcal{E} \in \mathcal{M}^0_{G,\mathrm{ord}} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]} \mathcal{M}^1_{G',\mathrm{ord}}$ as

$$\mathcal{E}_{\mathrm{ord}} = \Phi^\Delta\left((e \times e')\mu_{\mathcal{E}}\right).$$

It is easily checked that the specializations of $\mathcal{E}_{\mathrm{ord}}$ satisfy (2.6.3). $\square$

2.6.4. *Projecting $\mathcal{E}_{\mathrm{ord}}$ to $\mathscr{C}$ on the first factor.* Denote by $\mathbb{T}^N_{\mathrm{ord}}$ the $\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]$-algebra generated by the unramified Hecke operators away from Np and the $\mathbb{U}_p$-operators acting on $\mathcal{M}^0_{G,\mathrm{ord}}$. The algebra $\mathbb{T}^N_{\mathrm{ord}}$ is reduced and finite torsion free over Λ_n. Let $\mathscr{C}$ be a geometrically irreduible component $\operatorname{Spec}\left(\mathbb{T}^N_{\mathrm{ord}} \otimes_{\mathcal{O}_F} F\right)$ with function field $F_{\mathscr{C}}$. Define $\mathbb{I}_{\mathscr{C}}$ as the integral closure of Λ_n in $F_{\mathscr{C}}$. Attached to $\mathscr{C}$, there is a homomorphism $\lambda_{\mathscr{C}} : \mathbb{T}^N_{\mathrm{ord}} \to \mathbb{I}_{\mathscr{C}}$ of Λ_n-algebras, and an isomorphism of $F_{\mathscr{C}}$-algebras

$$\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} F_{\mathscr{C}} = F_{\mathscr{C}} \oplus R_{\mathscr{C}}$$

such that the projection onto the first factor coincides with $\lambda_{\mathscr{C}}$. Define $\mathbb{1}_{\mathscr{C}} \in \mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} F_{\mathcal{C}}$ to be the element corresponding to $(1,0) \in F_{\mathscr{C}} \oplus R_{\mathscr{C}}$.

Define the $(n+1)$-variable ordinary family of Klingen Eisenstein series attached to $\mathscr{C}$ as

$$\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}} = (\mathbb{1}_{\mathscr{C}} \times 1)\,\mathcal{E}_{\mathrm{ord}} \in F_{\mathscr{C}} \otimes_{\Lambda_n} \mathcal{M}^0_{G,\mathrm{ord}} \otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]} \mathcal{M}^1_{G',\mathrm{ord}}.$$

From the construction of $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}$ and the doubling method formula recorded in Theorem 2.2.1, we deduce the following theorem.

Theorem 2.6.2. *There exists an ordinary family $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}} \in F_{\mathscr{C}} \otimes_{\Lambda_n} \mathcal{M}^0_{G,\mathrm{ord}} \otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]} \mathcal{M}^1_{G',\mathrm{ord}}$ whose specialization at $(x,\kappa) \in \mathscr{C}(\overline{\mathbb{Q}}_p) \times \mathrm{Hom}_{\mathrm{cont}}\left(\mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times\right)$ satisfies:*

- *If $\eta \neq \kappa(-1)$, then the specialization is 0.*
- *If $(\kappa, \underline{\mathfrak{t}}_x)$ is admissible with $\eta = \kappa(-1)$, and the weight projection is étale at x, then $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}(x,\kappa)$ is the p-adic form attached to the Klingen Eisenstein series*

$$\frac{\Gamma_{2n+1}(n+1)\, b^{Np\infty}_{H'}(s,\eta\chi)}{\sqrt{-1}^{(2n+1)k}2^{(2n+1)(k-n)}\pi^{(n+1)(2n+1)}} \sum_{\varphi \in \mathfrak{S}_x} \frac{\varphi \otimes E^{\mathrm{Kl}}\left(\cdot, \Phi_{f_{\underline{\mathfrak{t}}_x,\kappa}(s),\varphi}\right)}{\langle \varphi, \overline{\varphi}\rangle}\Bigg|_{s=n+1-k},$$

where $\Phi_{f_{\underline{\mathfrak{t}}_x,\kappa}(s),\varphi} \in I_{P_{G'}}(s,\eta\chi)$ is defined from $f_{\underline{\mathfrak{t}}_x,\kappa}(s)$ and φ as (2.2.2), and $\mathfrak{S}_x$ consists of an orthogonal basis of the space spanned by cuspidal ordinary holomorphic Siegel modular forms on $G(\mathbb{A})$ of weight $\underline{t}_x$ and tame level $\Gamma_{G,1}(N)$ on which the Hecke operators act through the eigensystem parameterized by x.

Note that although we construct the family $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}$ inside the space of Hida families of tame principal level N, our choice of sections $f_{\underline{\mathfrak{t}},\kappa,v}(s)$ in fact implies that the specializations are of tame level $\Gamma_{G,1}(N) \times \Gamma^{\mathrm{opp}}_{G',1}(N)$, where $\Gamma^{\mathrm{opp}}_{G',1}(N) = \{g \in G'(\mathbb{Z}) : {}^{\mathrm{t}}g \in \Gamma_{G',1}(N)\}$.

In order to apply the above constructed Klingen family $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}$ to study the Klingen Eisenstein congruence ideal for G', one needs very precise information on its image under the map the Siegel operator $\mathscr{P}_{\mathrm{deg}}$, as well as its non-degenerate Fourier coefficients. We discuss these two problems in the next two sections.

3. The image of the Klingen Eisenstein family under the Siegel operator

3.1. **The Siegel operator and the short exact sequence.** According to the Hida theory for non-cuspidal Siegel modular forms established in [LR18, Theorem 1.3.1], there is a short exact sequence

$$(3.1.1)\quad 0 \longrightarrow \mathcal{M}^0_{G',\mathrm{ord}} \longrightarrow \mathcal{M}^1_{G',\mathrm{ord}} \xrightarrow{\mathscr{P}_{\mathrm{deg}}} \bigoplus_{\substack{L\in\mathfrak{C}_{\mathbb{V}'}/\Gamma_{G'}(N)\\ \mathrm{rk}L=1}} \mathcal{M}^0_{L,\mathrm{ord}} \otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]} \mathcal{O}_F[\![T_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times]\!] \longrightarrow 0.$$

Here $\mathfrak{C}_{V'}$ stands for the set of cotorsion free isotropic $\mathbb{Z}$-submodules of $\mathbb{V}'_0$. There is a natural $G(\mathbb{Z})$-action on it. The space $\mathcal{M}^0_{L,\mathrm{ord}}$ is isomorphic to $\mathcal{M}^0_{G,\mathrm{ord}}$. This short exact sequence is sometimes called the fundamental exact sequence in the study of Eisenstein congruences. If λ is an eigensystem valued in $\mathbb{I}_\lambda$ of the Hecke algebra acting on $\mathcal{M}^1_{G',\mathrm{ord}}$, and if $\mathcal{F} \in \mathcal{M}^1_{G',\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_{n+1}(\mathbb{Z}_p)]\!]}\mathbb{I}_\lambda$ is a primitive eigenfamily for λ, then the image of $\mathcal{F}$ under the Siegel operator measures the congruences between λ and cuspidal Hecke eigensystems.

We define the following Siegel operator

$$\mathscr{P}_{\mathrm{deg}} : \mathcal{M}^0_{G,\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]}\mathcal{M}^1_{G',\mathrm{ord}} \longrightarrow \bigoplus_{\substack{L\in\mathfrak{C}_{\mathbb{V}'}/\Gamma_{G'}(N)\\ \mathrm{rk}L=1}} \mathcal{M}^0_{G,\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]}\mathcal{M}^0_{L,\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]}\mathcal{O}_F[\![T_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times]\!]$$

as the identity (on the first factor) tensored with the quotient map in (3.1.1) (by abuse of notation we denote this operator still by $\mathscr{P}_{\mathrm{deg}}$).

3.2. **The q-expansions of $\mathscr{P}_{\deg}\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)$ and p-adic L-functions for $\mathscr{C}$.** As explained in [Liu16, §6.1.5], for $\beta \in N^{-1}\operatorname{Sym}(n,\mathbb{Z})^*_{>0}$ and $\gamma_N \in \mathbf{G}(\mathbb{Z}/N)$ (resp. $\beta' \in N^{-1}\operatorname{Sym}(n+1,\mathbb{Z})^*_{\geq 0}$ and $\gamma'_N \in G'(\mathbb{Z}/N)$) there is the $\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]$-linear (resp. $\mathcal{O}_F[\![\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p)]\!]$-linear) map of taking the coefficient indexed by β (resp. β') in the p-adic q-expansion at the cusp indexed by γ_N (resp. γ'_N),

$$\varepsilon^{\gamma_N}_{q\text{-exp}}(\beta,\cdot):\mathcal{M}^0_{G,\mathrm{ord}}\longrightarrow\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!],\quad(\text{resp. }\varepsilon^{\gamma'_N}_{q\text{-exp}}(\beta',\cdot):\mathcal{M}^1_{G',\mathrm{ord}}\longrightarrow\mathcal{O}_F[\![T_{n+1}(\mathbb{Z}_p)]\!]).\tag{3.2.1}$$

We denote by

$$\varepsilon^{\gamma_N,\gamma'_N}_{q\text{-exp}}(\beta,\beta',\cdot):\mathcal{M}^0_{G,\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]}\mathcal{M}^1_{G',\mathrm{ord}}\longrightarrow\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]$$

the map combining the ones in (3.2.1). Similarly, for $\beta_1,\beta_2 \in N^{-1}\operatorname{Sym}(n,\mathbb{Z})^*_{>0}$ and $\gamma_{N,1},\gamma_{N,2} \in G(\mathbb{Z}/N)$ we have the map

$$\varepsilon^{\gamma_{N,1},\gamma_{N,2}}_{q\text{-exp}}(\beta_1,\beta_2,\cdot):\mathcal{M}^0_{G,\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]}\mathcal{M}^0_{G,\mathrm{ord}}\longrightarrow\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!].$$

Given $L \in \mathfrak{C}_{\mathbb{V}'}/\Gamma(N)$ of rank 1, we fix a basis of $L^\perp/L$ so that we can define the q-expansion map for $\mathcal{M}^0_{L,\mathrm{ord}}$ like (3.2.1). Take $\gamma'_L \in G'(\mathbb{Z})$ such that the basis

$$(\widetilde{e}'_1,\dots,\widetilde{e}'_{n+1},\widetilde{f}'_1,\dots,\widetilde{f}'_{n+1})\gamma'_L=(e'_1,\dots,e'_{n+1},f'_1,\dots,f'_{n+1})$$

satisfies that $\widetilde{e}'_{n+1}$ spans L, and $\widetilde{e}'_1,\dots,\widetilde{e}'_n,\widetilde{f}'_1,\dots,\widetilde{f}'_n \mod L$ is our fixed basis of $L^\perp/L$. These properties determines γ'_L up to left multiplication by an element in $G'(\mathbb{Z})$ of the form $\left(\begin{smallmatrix}\mathbf{1}_n&0&0&*\\ *&\pm1&*&*\\ 0&0&\mathbf{1}_n&*\\ 0&0&0&\pm1\end{smallmatrix}\right)$. Define

$$\begin{aligned}\iota_L:G(\mathbb{Z})&\longrightarrow G'(\mathbb{Z})\\ \begin{pmatrix}a&b\\c&d\end{pmatrix}&\longmapsto\gamma'_L\begin{pmatrix}a&0&b&0\\0&1&0&0\\c&0&d&0\\0&0&0&1\end{pmatrix}.\end{aligned}$$

Proposition 3.2.1. *Let $\mathscr{P}_{\deg,L}$ be the projection to the direct summand indexed by L in the right most term in (3.1.1). For $\beta_1,\beta_2 \in N^{-1}\operatorname{Sym}(n,\mathbb{Z})^*_{>0}$ and $\gamma_{N,1},\gamma_{N,2} \in G(\mathbb{Z}/N)$, we have the following commutative diagram*

$$\begin{tikzcd}
\mathcal{M}^0_{G,\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]}\mathcal{M}^1_{G',\mathrm{ord}} \arrow[r, "\mathscr{P}_{\deg,L}"] \arrow[dr, "\varepsilon^{\gamma_{N,1},\iota_L(\gamma_{N,2})}_{q\text{-exp}}\left(\beta_1,\left(\begin{smallmatrix}\beta_2&0\\0&0\end{smallmatrix}\right),\cdot\right)"'] & \mathcal{M}^0_{G,\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]}\mathcal{M}^0_{L,\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]}\mathcal{O}_F[\![T_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times]\!] \arrow[d, "\varepsilon^{\gamma_{N,1},\gamma_{N,2}}_{q\text{-exp}}(\beta_1,\beta_2,\cdot)"] \\
 & \mathcal{O}_F[\![T_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times]\!]
\end{tikzcd}$$

Therefore, in order to detect information on $\mathscr{P}_{\deg}\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)$, it suffices to look at $\varepsilon^{\gamma_N,\gamma'_N}_{q\text{-exp}}\left(\beta_1,\left(\begin{smallmatrix}\beta_2&0\\0&0\end{smallmatrix}\right),\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)$ for all $\gamma_N \in G(\mathbb{Z}/N)$, $\gamma'_N \in G'(\mathbb{Z}/N)$ and $\beta_1,\beta_2 \in N^{-1}\operatorname{Sym}(n,\mathbb{Z})^*_{>0}$. The rest of this section is devoted to prove Theorem 3.6.1, which essentially says that the image under the Siegel operator of our constructed family $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}$ is the p-adic L-function attached to $\mathscr{C}$.

3.3. **The degenerate coefficients in the q-expansions.** We deduce information on the degenerate coefficient $\varepsilon^{\gamma_N,\gamma'_N}_{q\text{-exp}}\left(\beta_1,\left(\begin{smallmatrix}\beta_2&0\\0&0\end{smallmatrix}\right),\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)$ by computing the $\left(\begin{smallmatrix}\beta_2&0\\0&0\end{smallmatrix}\right)$-th Fourier coefficient of the Siegel modular form

$$\mathscr{L}_{\overline{\varphi}}\left(\mathfrak{E}^{\mathrm{Si}}(\cdot,f_{\underline{\tau},\kappa})\big|_{G\times G'}\right)\tag{3.3.1}$$

for admissible $(\underline{\tau}, \kappa)$. For given $\beta \in N^{-1}\operatorname{Sym}(n,\mathbb{Z})^*_{>0}$, we compute the Fourier coefficient

$$E^{\mathrm{Kl}}_{\left(\begin{smallmatrix}\beta & 0\\ 0 & 0\end{smallmatrix}\right)}\left(g'_N g'_{z',\infty}, \Phi_{f_{\underline{\tau},\kappa}(s),\varphi}\right) = \int_{\operatorname{Sym}(n+1,\mathbb{Q}\backslash\mathbb{A})} E^{\mathrm{Kl}}\left(u(\varsigma) g'_N g'_{\infty,z'}, \Phi_{f_{\underline{\tau},\kappa}(s),\varphi}\right)\mathbf{e}_{\mathbb{A}}\left(-\operatorname{Tr}\left(\begin{smallmatrix}\beta & 0\\ 0 & 0\end{smallmatrix}\right)\varsigma\right)d\varsigma,$$

where $g'_N \in \prod_{v|N} G'(\mathbb{Q}_v)$, $g'_{z',\infty} \in Q_{G'}(\mathbb{R})$ is the element attached to $z' \in \mathbb{H}_{n+1}$ in the same way as (2.4.3), $u(\varsigma) = \left(\begin{smallmatrix}\mathbf{1}_{n+1} & \varsigma\\ 0 & \mathbf{1}_{n+1}\end{smallmatrix}\right)$, and $\Phi_{f_{\underline{\tau},\kappa}(s),\varphi} \in I_{P_{G'}}(s,\eta\chi)$ is the section attached to our chosen section $f_{\underline{\tau},\kappa}(s) \in I_{Q_{H'}}(s,\eta\chi)$ and a cuspidal Siegel modular $\varphi \in \pi$ as in (2.2.2). It follows from [MgW94, II.1.7] that

(3.3.2)

$$E^{\mathrm{Kl}}_{\left(\begin{smallmatrix}\beta & 0\\ 0 & 0\end{smallmatrix}\right)}\left(g', \Phi(s,\xi)\right) = \int_{\operatorname{Sym}(n,\mathbb{Q}\backslash\mathbb{A})}\Big(\Phi(s,\xi) + M_{P_{G'}}(s,\xi)\Phi(s,\xi)\Big)\left(\begin{pmatrix}\mathbf{1}_{n+1} & \left(\begin{smallmatrix}\varsigma & 0\\ 0 & 0\end{smallmatrix}\right)\\ 0 & \mathbf{1}_{n+1}\end{pmatrix} g'\right)\mathbf{e}_{\mathbb{A}}\left(-\operatorname{Tr}\beta\varsigma\right)d\varsigma.$$

The proof of Theorem 3.6.1 reduces to computing the right hand side of (3.3.2) for $g' = g'_N g'_{z',\infty}$ and $\Phi(s,\xi) = \Phi_{f_{\underline{\tau},\kappa}(s),\varphi}$ as above.

3.4. **The vanishing of the first term in (3.3.2).**

Proposition 3.4.1. *By our choice of $f_{\underline{\tau},\kappa}(s) \in I_{Q_{H'}}(s,\eta\chi)$, for $g'_N \in \prod_{v|N} G'(\mathbb{Q}_v)$ and $z' \in \mathbb{H}_{n+1}$,*

$$\int_{\operatorname{Sym}(n,\mathbb{Q}\backslash\mathbb{A})}\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}\left(\begin{pmatrix}\mathbf{1}_{n+1} & \left(\begin{smallmatrix}\varsigma & 0\\ 0 & 0\end{smallmatrix}\right)\\ 0 & \mathbf{1}_{n+1}\end{pmatrix} g'_N g'_{z',\infty}\right)\mathbf{e}_{\mathbb{A}}\left(-\operatorname{Tr}\beta\varsigma\right)d\varsigma = 0.$$

Proof. It suffices to show that the projection to $G'(\mathbb{Q}_p)$ of the support of $\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}$ intersects trivially with $Q_{G'}(\mathbb{Q}_p)$, and this will be implied by that the support of $f_{\underline{\tau},\kappa,p}(s)$ intersects trivially with $S_{H'}^{-1}\cdot\iota_{H'}\left(G(\mathbb{Q}_p)\times Q_{G'}(\mathbb{Q}_p)\right)$ because

$$\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}(g') = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} f_{\underline{\tau},\kappa}(s)\left(S_{H'}^{-1}\iota_{H'}(g,g')\right)\overline{\varphi}(g)\,dg.$$

The lower left $(2n+1)\times(2n+1)$ block of

$$S_{H'}^{-1}(\iota_{H'}(g,g')) = \begin{pmatrix} a & 0 & b & 0\\ 0 & \mathfrak{a} & 0 & \mathfrak{b}\\ c & \left(\mathbf{1}_n\ 0\right)\mathfrak{a} & d & \left(\mathbf{1}_n\ 0\right)\mathfrak{b}\\ \left(\begin{smallmatrix}\mathbf{1}_n\\ 0\end{smallmatrix}\right)a & 0 & \left(\begin{smallmatrix}\mathbf{1}_n\\ 0\end{smallmatrix}\right)b & \mathfrak{d}\end{pmatrix},\qquad g = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\ g' = \begin{pmatrix}\mathfrak{a} & \mathfrak{a}\\ 0 & \mathfrak{d}\end{pmatrix},$$

is not invertible. By definition the support of $f_{\underline{\tau},\kappa,p}(s)$ lies inside the "big cell", so its intersection with $S_{H'}^{-1}\cdot\iota_{H'}\left(G(\mathbb{Q}_p)\times Q_{G'}(\mathbb{Q}_p)\right)$ is empty. $\square$

3.5. **Computing $M_{P_{G'}}(s,\eta\chi)\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}$.**

3.5.1. *The unramified places.* At $v \nmid Np\infty$, we have chosen $f_{\underline{\tau},\kappa,v}(s) \in I_{Q_{H'}}(s,\eta\chi)$ to be the standard unramified section and we assume that $\varphi \in \pi$ is fixed by $G(\mathbb{Z}_v)$. Hence $\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}$ is spherical at $v \nmid Np\infty$.

Before moving on, we introduce some notation. For $f(s,\xi) \in I_{Q_{H'},v}(s,\xi)$ and $g'_v \in G'(\mathbb{Q}_v)$, we define the operator $T^{M_{P_{G'}}}_{f_v(s,\xi),g'_v} : \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A})) \longrightarrow \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A})$ as

$$\left(T^{M_{P_{G'}}}_{f_v(s,\xi),g'_v}\varphi\right)(g_1) = \int_{U_{P_{G'}}(\mathbb{Q}_v)}\int_{G(\mathbb{Q}_v)} f_v^{\diamondsuit}(s,\xi)\left(\iota_{H'}(g_v, w_{P_{G'}} u g'_v)\right)\varphi(g_1 g_v)\,dg_v\,du, \tag{3.5.1}$$

which can be viewed as a combination of the doubling zeta integral and the intertwining operator. Similarly as (2.4.2), we define

$$b_{H,v}(s,\xi)=L_v\left(s+\frac{2n+1}{2},\xi\right)\prod_{j=1}^{n}L_v\left(2s+2n+1-2j,\xi^2\right),$$

and it is easily checked that $b_{H,v}(s+\frac{1}{2},\xi)=b_{H',v}(s,\xi)$.

Proposition 3.5.1. *For $g'_N=\otimes g'_v\in\prod_{v|N}G'(\mathbb{Q}_v)$ and $g_1\in G(\mathbb{A})$, we have*

$$\begin{aligned}\left(M_{P_{G'}}(s,\eta\chi)\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}\right)(g_1\,g'_N)&=b_H^{Np\infty}(s+\frac{1}{2},\eta\chi)^{-1}L^{Np\infty}(s,\pi\times\eta\chi)\\&\times\left(T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,\infty}(s),\mathbf{1}_{2n+2}}T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,p}(s),\mathbf{1}_{2n+2}}\prod_{v|N}T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,v}(s),g'_v}\overline{\varphi}\right)(g_1^{\vartheta}).\end{aligned}$$

Here $g_1^{\vartheta}=\begin{pmatrix}&\mathbf{1}_n\\\mathbf{1}_n&\end{pmatrix}g_1\begin{pmatrix}&\mathbf{1}_n\\\mathbf{1}_n&\end{pmatrix}$ is the MVW involution of $g_1\in G(\mathbb{A})$.

Proof. It follows from the doubling method [Gar84, PSR87] that for $g_1,g^{Np\infty}g_{Np\infty}\in G(\mathbb{A})$,

$$\begin{aligned}&\int_{G(\mathbb{A}^{Np\infty})}f^{\diamondsuit}_{\underline{\tau},\kappa}(s,\eta\chi)\left(\iota_{H'}(g^{Np\infty}g_{Np\infty},g_1g'_N)\right)\overline{\varphi}(g)\,dg^{Np\infty}\\&=f^{\diamondsuit}_{\underline{\tau},\kappa,Np\infty}(s,\eta\chi)\left(\iota_{H'}(g_{Np\infty},g'_N)\right)\int_{G(\mathbb{A}^{Np\infty})}\prod_{v\nmid Np\infty}f^{\mathrm{ur},\diamondsuit}_{H,v}(s+\frac{1}{2},\eta\chi)\left(\iota_H(g^{Np\infty},\mathbf{1}_{2n})\right)\overline{\varphi}(g_1^{\vartheta}g)\,dg\\&=b_H^{Np\infty}(s+\frac{1}{2},\eta\chi)^{-1}L^{Np\infty}(s+1,\pi\times\eta\chi)\cdot f^{\diamondsuit}_{\underline{\tau},\kappa,Np\infty}(s,\eta\chi)\left(\iota_{H'}(g_{Np\infty},g'_N)\right)\overline{\varphi}(g_1^{\vartheta}),\end{aligned}$$

Then by the definition of $\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}\in I_{P_{G'}}(s,\eta\chi)$ as in (2.2.2), we get

(3.5.2)

$$\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}(g_1g'_N)=b_H^{Np\infty}(s+\frac{1}{2},\eta\chi)^{-1}L^{Np\infty}(s+1,\pi\times\eta\chi)\int_{G(\mathbb{A}_{Np\infty})}f^{\diamondsuit}_{\underline{\tau},\kappa,Np\infty}(s,\eta\chi)\left(\iota_{H'}(g,g'_N)\right)\overline{\varphi}(g_1^{\vartheta}g)\,dg$$

Next, by [Sha10, Proposition 4.3.1] we have

$$\left(M^{Np\infty}_{P_{G'}}(s,\eta\chi)\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}\right)(g_1g'_N)=\frac{L^{Np\infty}(s,\pi\times\eta\chi)}{L^{Np\infty}(s+1,\pi\times\eta\chi)}\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}(g_1g'_N),$$

which combining with (3.5.2) proves the proposition. ☐

3.5.2. *The archimedean place.*

Proposition 3.5.2. *Assumed that $\pi\cong\mathcal{D}_{\underline{t}}$ and $\varphi\in\pi$ is holomorphic of weight $\underline{t}$. Then*

$$\begin{aligned}T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,\infty}(s),\mathbf{1}_{2n+2}}\overline{\varphi}\Big|_{s=n+1-k}&=\frac{\sqrt{-1}^{(2n+1)k}2^{(2n+1)(k-n)}\pi^{(n+1)(2n+1)}}{\Gamma_{2n+1}(n+1)}\\&\times\frac{\sqrt{-1}^{2nk+\frac{n^2-n}{2}}2^{-\sum t_j+\frac{n^2-n}{2}}}{\dim(\mathrm{GL}(n),\underline{t})}E^-_{\infty}(n+1-k,\pi\times\eta\chi)\,\overline{\varphi},\end{aligned}$$

where $\dim(\mathrm{GL}(n),\underline{t})$ is the dimension of the irreducible algebraic representation of $\mathrm{GL}(n)$ of highest weight $\underline{t}$, and $E^-_{\infty}(s,\pi\times\eta\chi)$ is the modified archimedean Euler factor conjectured by Coates–Perrin-Riou for p-adic interpolations [Coa91].

Remark 3.5.3. According to [Coa91], the modified archimedean Euler factor for p-adic interpolation of critical values of $L(s,\pi\times\eta\chi)$ to the left (resp. right) of the center is

$$E_\infty^-(s,\pi\times\eta\chi)=e^{(1-s)\frac{\pi\sqrt{-1}}{2}}\prod_{j=1}^{n}\Gamma_{\mathbb{C}}(s+t_j-j)$$
$$\text{(resp. } E_\infty^+(s,\pi\times\eta\chi)=e^{s\frac{\pi\sqrt{-1}}{2}}\frac{\Gamma_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(1-s)}\prod_{j=1}^{n}\Gamma_{\mathbb{C}}(s+t_j-j)),$$

where $\Gamma_{\mathbb{C}}(s)=2(2\pi)^{-s}\Gamma(s)$, $\Gamma_{\mathbb{R}}=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$.

Before starting proving this proposition, we show a proposition in the flavor of functional equations. For $f_v(s,\xi)\in I_{Q_{H'},v}(s,\xi)$ and $g'_v\in G'(\mathbb{Q}_v)$, define the operator

$$\begin{aligned}T_{f_v(s,\xi),g'_v}:\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))&\longrightarrow\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))\\ \varphi&\longmapsto\left(T_{f_v(s,\xi),g'_v}\varphi\right)(g_1)=\int_{G(\mathbb{Q}_v)}f_v^{\diamondsuit}(s,\xi)\left(\iota_{H'}(g_v,g'_v)\right)\varphi(g_1g_v)\,dg_v.\end{aligned}$$

Proposition 3.5.4. *For $\varphi\in\pi$ and a place v, we have*

$$T^{M_{P_{G'}}}_{f_v(s,\xi),g'_v}\varphi=\Gamma_v\left(s-\frac{1}{2},\pi\times\xi\right)^{-1}T_{M_{Q_{H'},v}(s,\xi)f_v(s,\xi),g'_v}\varphi,$$

where

$$\Gamma_v\left(s-\frac{1}{2},\pi,\xi\right)=\pi_v(-1)\,\gamma_v\left(s,\pi\times\xi\right)\left(\gamma_v\left(s-n,\xi\right)\prod_{j=1}^{n}\gamma_v\left(2s-2n-1+2j,\xi^2\right)\right)^{-1}. \tag{3.5.3}$$

Proof. We first show that for all $h'\in H'(\mathbb{Q}_v)$, the function

$$\begin{aligned}H(\mathbb{Q}_v)&\longrightarrow\mathbb{C}\\ h&\longmapsto\int_{U_{P_{G'}}(\mathbb{Q}_v)}f_v^{\diamondsuit}(s,\xi)\left(\iota_{H'}(\mathbf{1}_{2n},w_{P_{G'}}u)S_{H'}hh'\right)\,du\end{aligned}\tag{3.5.4}$$

belongs to $I_{Q_H,v}(s-\frac{1}{2},\xi)$. For $u=\begin{pmatrix}\mathbf{1}_n&0&0&y\\ {}^{\mathrm{t}}x&1&{}^{\mathrm{t}}y&z+{}^{\mathrm{t}}xy\\ 0&0&\mathbf{1}_n&-x\\ 0&0&0&1\end{pmatrix}\in U_{P_{G'}(\mathbb{Q}_v)}$ and $q_H=\begin{pmatrix}A&B\\ 0&{}^{\mathrm{t}}A^{-1}\end{pmatrix}\in Q_H(\mathbb{Q}_v)$, we have

$$\begin{aligned}S_{H'}^{-1}\iota_{H'}(\mathbf{1}_{2n},w_{P_{G'}}u)S_{H'}&=\left(\begin{array}{ccc|ccc}\mathbf{1}_n&0&0&0&0&0\\ 0&\mathbf{1}_n&0&0&0&y\\ 0&0&0&0&0&-1\\ \hline 0&0&0&\mathbf{1}_n&0&-y\\ 0&0&0&0&\mathbf{1}_n&-x\\ {}^{\mathrm{t}}y&{}^{\mathrm{t}}x&1&0&{}^{\mathrm{t}}y&z+{}^{\mathrm{t}}xy\end{array}\right)\\ &=\left(\begin{array}{ccc|ccc}\mathbf{1}_n&0&0&0&0&0\\ 0&\mathbf{1}_n&-y&0&0&0\\ 0&0&1&0&0&0\\ \hline 0&0&0&\mathbf{1}_n&0&0\\ 0&0&0&0&\mathbf{1}_n&0\\ 0&0&0&0&{}^{\mathrm{t}}y&1\end{array}\right)\left(\begin{array}{ccc|ccc}\mathbf{1}_n&0&0&0&0&0\\ 0&\mathbf{1}_n&0&0&0&0\\ 0&0&0&0&0&-1\\ \hline 0&0&0&\mathbf{1}_n&0&-y\\ 0&0&0&0&\mathbf{1}_n&-x\\ {}^{\mathrm{t}}y&{}^{\mathrm{t}}x&1&0&0&z+{}^{\mathrm{t}}xy+{}^{\mathrm{t}}yx\end{array}\right),\end{aligned}$$

and

$$S_{H'}^{-1}\iota_{H'}(\mathbf{1}_{2n}, w_{P_{G'}}u)S_{H'}q_H = \left(\begin{array}{ccc:ccc} \mathbf{1}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1}_n & -y & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \mathbf{1}_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_n & 0 \\ 0 & 0 & 0 & 0 & {}^{\mathrm{t}}y & 1 \end{array}\right) \left(\begin{array}{cc:cc} A & -AB\left(\begin{smallmatrix} y \\ x \end{smallmatrix}\right) & B & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & {}^{\mathrm{t}}A^{-1} & 0 \\ 0 & 0 & \left(\begin{smallmatrix} {}^{\mathrm{t}}y & {}^{\mathrm{t}}x \end{smallmatrix}\right) B & 1 \end{array}\right)$$

$$\times \left(\begin{array}{cc:cc} \mathbf{1}_{2n} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \hdashline 0 & 0 & \mathbf{1}_{2n} & -{}^{\mathrm{t}}A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} {}^{\mathrm{t}}y & {}^{\mathrm{t}}x \end{smallmatrix}\right) A & 1 & 0 & z + {}^{\mathrm{t}}xy + {}^{\mathrm{t}}yx + \left(\begin{smallmatrix} {}^{\mathrm{t}}y & {}^{\mathrm{t}}x \end{smallmatrix}\right) {}^{\mathrm{t}}B {}^{\mathrm{t}}A \left(\begin{smallmatrix} y \\ x \end{smallmatrix}\right) \end{array}\right).$$

Therefore,

$$\begin{aligned} &\int_{U_{P_{G'}}(\mathbb{Q}_v)} f_v^{\diamondsuit}(s,\xi)\left(\iota_{H'}(\mathbf{1}_{2n}, w_{P_{G'}}u)S_{H'}q_H h'\right)\, du \\ =\,&\xi(\det A)|\det A|^{s+n+1-1}\int_{U_{P_{G'}}(\mathbb{Q}_v)} f_v^{\diamondsuit}(s,\xi)\left(\iota_{H'}(\mathbf{1}_{2n}, w_{P_{G'}}u)S_{H'}h'\right)\, du, \end{aligned}$$

and the function (3.5.4) belongs to $I_{Q_H,v}(s-\frac{1}{2},\xi)$. Now applying the functional equation for the local doubling zeta integrals [LR05, (19)(25)], we get

(3.5.5)

$$\begin{aligned} &\Gamma_v\left(s-\frac{1}{2},\pi,\xi\right)\left(T^{M_{P_{G'}}}_{f_v(s,\xi),g_v'}\varphi\right)(g_1) \\ =&\int_{G(\mathbb{Q}_v)}\int_{U_H(\mathbb{Q}_v)}\int_{U_{P_{G'}}(\mathbb{Q}_v)} f_v^{\diamondsuit}(s,\xi)\left(\iota_{H'}(\mathbf{1}_{2n}, w_{P_{G'}}u)\cdot S_{H'}w_{Q_H}u_1S_{H'}^{-1}\cdot\iota_{H'}(g_v,g_v')\right)\varphi(g_1g_v)\,du\,du_1\,dg_v \\ =&\int_{G(\mathbb{Q}_v)}\int_{U_H(\mathbb{Q}_v)}\int_{U_{P_{G'}}(\mathbb{Q}_v)} f_v(s,\xi)\left(\iota_{H'}(\mathbf{1}_{2n}, w_{P_{G'}})w_{Q_H}\cdot w_{Q_H}^{-1}S_{H'}^{-1}\iota_{H'}(\mathbf{1}_{2n},u)S_{H'}w_{Q_H}u_1\cdot S_{H'}^{-1}\iota_{H'}(g_v,g_v')\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\varphi(g_1g_v)\,du\,du_1\,dg_v. \end{aligned}$$

The formula (3.5.3) for $\Gamma_v\left(s-\frac{1}{2},\pi,\xi\right)$ is given in [LR05, (14)(19)(25)], where for determining the factor $c_v(s,\xi,A)$, one can use the formulas in [Swe95, Proposition 4.8] for finite places and compute by definition with formulas in [Shi82] for the archimedean place.

Since

$$w_{Q_H}^{-1}S_{H'}^{-1}\left(\begin{array}{ccc:ccc} \mathbf{1}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1}_n & 0 & 0 & 0 & y \\ 0 & {}^{\mathrm{t}}x & 1 & 0 & {}^{\mathrm{t}}y & z+{}^{\mathrm{t}}xy \\ \hdashline 0 & 0 & 0 & \mathbf{1}_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_n & -x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)S_{H'}w_{Q_H} = \left(\begin{array}{ccc:ccc} \mathbf{1}_n & 0 & 0 & 0 & 0 & y \\ 0 & \mathbf{1}_n & 0 & 0 & 0 & x \\ 0 & -{}^{\mathrm{t}}y & 1 & {}^{\mathrm{t}}y & {}^{\mathrm{t}}x & z+{}^{\mathrm{t}}xy \\ \hdashline 0 & 0 & 0 & \mathbf{1}_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_n & y \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

we obtain

$$\begin{aligned} (3.5.5) &= \int_{G(\mathbb{Q}_v)}\int_{U_{H'}(\mathbb{Q}_v)} f_v(s,\xi)\left(w_{Q_{H'}}u'\cdot S_{H'}^{-1}\iota_{H'}(g_v,g_v')\right)\varphi(g_1g_v)\,du'dg_v \\ &= \int_{G(\mathbb{Q}_v)}\left(M_{Q_{H'},v}(s,\xi)f_v(s,\xi)\right)^{\diamondsuit}\left(\iota_{H'}(g_v,g_v')\right)\varphi(g_1g_v)\,dg_v \\ &= \left(T_{M_{Q_{H'},v}(s,\xi)f_v(s,\xi),g_v'}\varphi\right)(g_1). \end{aligned}$$

$\square$

Proof of Proposition 3.5.2. First we observe that $T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,\infty}(s),g'_{z',\infty}}\overline{\varphi}$ is a multiple of $\overline{\varphi}$, because it lies inside $\overline{\pi}$, and by definition of $f_{\underline{\tau},\kappa,\infty}$, it is of the same weight as $\overline{\varphi}$. Hence it suffices to compute the inner product $\left\langle T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,\infty}(s),g'_{z',\infty}}\overline{\varphi},\varphi\right\rangle$.

It follows from [Shi82, (1.31)(4.34K)] and the definition of $f_{\underline{\tau},\kappa,\infty}(s)\in I_{Q_{H'}}(s,\mathrm{sgn}^k)$ that

$$M_{Q_{H'},\infty}(s,\mathrm{sgn}^k)f_{\underline{\tau},\kappa,\infty}(s)=c_{H',k}(s)\,f_{\underline{\tau},\kappa,\infty}(-s) \tag{3.5.6}$$

with

$$c_{H',k}(s)=\sqrt{-1}^{(2n+1)k}\,2^{(2n+1)(1-s)}\,\pi^{(n+1)(2n+1)}\,\frac{\Gamma_{2n+1}(s)}{\Gamma_{2n+1}\left(\frac{1}{2}(s+n+1)+\frac{k}{2}\right)\Gamma_{2n+1}\left(\frac{1}{2}(s+n+1)-\frac{k}{2}\right)}. \tag{3.5.7}$$

By Proposition 3.5.4,

$$\begin{aligned}\frac{\left\langle T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,\infty}(s),\mathbf{1}_{2n+2}}\overline{\varphi},\varphi\right\rangle}{\langle\varphi,\overline{\varphi}\rangle}&=\Gamma_\infty\left(s-\frac{1}{2},\pi\times\eta\chi\right)^{-1}\frac{\left\langle T_{M_{Q_{H'},\infty}(s,\mathrm{sgn}^k)f_{\underline{\tau},\kappa,\infty}(s),\mathbf{1}_{2n+2}}\overline{\varphi},\varphi\right\rangle}{\langle\varphi,\overline{\varphi}\rangle}\\&=\Gamma_\infty\left(s-\frac{1}{2},\pi\times\eta\chi\right)^{-1}c_{H',k}(s)\frac{\left\langle T_{f_{\underline{\tau},\kappa,\infty}(-s),\mathbf{1}_{2n+2}}\overline{\varphi},\varphi\right\rangle}{\langle\varphi,\overline{\varphi}\rangle}\\&=\Gamma_\infty\left(s-\frac{1}{2},\pi\times\eta\chi\right)^{-1}c_{H',k}(s)\frac{\left\langle T_{f^H_{\underline{\tau},\kappa,\infty}(-s+\frac{1}{2})}\overline{\varphi},\varphi\right\rangle}{\langle\varphi,\overline{\varphi}\rangle}\\&=\Gamma_\infty\left(s-\frac{1}{2},\pi\times\eta\chi\right)^{-1}c_{H',k}(s)\frac{Z_\infty\left(f^H_{\underline{\tau},\kappa,\infty}(-s+\frac{1}{2}),v_{\underline{t}},v^\vee_{\underline{t}}\right)}{\left\langle v^\vee_{\underline{t}},v_{\underline{t}}\right\rangle},\end{aligned} \tag{3.5.8}$$

where the operator $T_{f^H_{\underline{\tau},\kappa,\infty}(-s)}$ is defined as in [Liu16, §4.1] and its connection with the standard doubling zeta integral Z_∞ (with definition recalled *loc. cit*) is obvious by definition. Plugging the formula for Z_∞ [Liu19, Theorem 2.4.1] into (3.5.8) proves the proposition.

$\square$

3.5.3. *The places dividing N.* For a finite place v, we consider the "big cell" (with respect to the Klingen parabolic) in G',

$$P_{G'}w_{P'_G}U_{P_{G'}}=\left\{g'=\begin{array}{cc}\begin{array}{cccc}n&1&n&1\end{array}&\\\begin{pmatrix}\mathfrak{a}_1&\mathfrak{a}_2&\mathfrak{b}_1&\mathfrak{b}_2\\\mathfrak{a}_3&\mathfrak{a}_4&\mathfrak{b}_3&\mathfrak{b}_4\\\mathfrak{c}_1&\mathfrak{c}_2&\mathfrak{d}_1&\mathfrak{d}_2\\\mathfrak{c}_3&\mathfrak{c}_4&\mathfrak{d}_3&\mathfrak{d}_4\end{pmatrix}&\begin{array}{c}n\\1\\n\\1\end{array}\end{array}\in G':\mathfrak{c}_4\in\mathrm{GL}(1)\right\}. \tag{3.5.9}$$

By the decomposition

$$g' = \begin{pmatrix} \mathfrak{a}_1 - \mathfrak{a}_2\mathfrak{c}_4^{-1}\mathfrak{c}_3 & 0 & \mathfrak{b}_1 - \mathfrak{a}_2\mathfrak{c}_4^{-1}\mathfrak{d}_3 & 0 \\ 0 & 1 & 0 & 0 \\ \mathfrak{c}_1 - \mathfrak{c}_2\mathfrak{c}_4^{-1}\mathfrak{c}_3 & 0 & \mathfrak{d}_1 - \mathfrak{c}_2\mathfrak{c}_4^{-1}\mathfrak{d}_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & 0 & 0 & ({}^{\mathrm{t}}\mathfrak{b}_3 - {}^{\mathrm{t}}\mathfrak{d}_3{}^{\mathrm{t}}\mathfrak{c}_4^{-1}{}^{\mathrm{t}}\mathfrak{a}_4)\mathfrak{c}_4 \\ \mathfrak{a}_3 - \mathfrak{a}_4\mathfrak{c}_4^{-1}\mathfrak{c}_3 & {}^{\mathrm{t}}\mathfrak{c}_4^{-1} & \mathfrak{b}_3 - \mathfrak{a}_4\mathfrak{c}_4^{-1}\mathfrak{d}_3 & \mathfrak{a}_4 \\ 0 & 0 & \mathbf{1}_n & (-{}^{\mathrm{t}}\mathfrak{a}_3 + {}^{\mathrm{t}}\mathfrak{c}_3{}^{\mathrm{t}}\mathfrak{c}_4^{-1}{}^{\mathrm{t}}\mathfrak{a}_4)\mathfrak{c}_4 \\ 0 & 0 & 0 & \mathfrak{c}_4 \end{pmatrix}$$
$$\times \begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \mathbf{1}_n & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & 0 & 0 & {}^{\mathrm{t}}\mathfrak{d}_3{}^{\mathrm{t}}\mathfrak{c}_4^{-1} \\ \mathfrak{c}_4^{-1}\mathfrak{c}_3 & 1 & \mathfrak{c}_4^{-1}\mathfrak{d}_3 & \mathfrak{c}_4^{-1}\mathfrak{d}_4 \\ 0 & 0 & \mathbf{1}_n & -{}^{\mathrm{t}}\mathfrak{c}_3{}^{\mathrm{t}}\mathfrak{c}_4^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we define the map

(3.5.10)
$$(p_G, p_{GL(1)}, p_U) : P_{G'} w_{P'_G} U_{P_{G'}} \longrightarrow G \times \mathrm{GL}(1) \times U_{P'_G}$$
$$g' \longmapsto \left(\begin{pmatrix} \mathfrak{a}_1 - \mathfrak{a}_2\mathfrak{c}_4^{-1}\mathfrak{c}_3 & \mathfrak{b}_1 - \mathfrak{a}_2\mathfrak{c}_4^{-1}\mathfrak{d}_3 \\ \mathfrak{c}_1 - \mathfrak{c}_2\mathfrak{c}_4^{-1}\mathfrak{c}_3 & \mathfrak{d}_1 - \mathfrak{c}_2\mathfrak{c}_4^{-1}\mathfrak{d}_3 \end{pmatrix}, \mathfrak{c}_4, \begin{pmatrix} \mathbf{1}_n & 0 & 0 & {}^{\mathrm{t}}\mathfrak{d}_3{}^{\mathrm{t}}\mathfrak{c}_4^{-1} \\ \mathfrak{c}_4^{-1}\mathfrak{c}_3 & 1 & \mathfrak{c}_4^{-1}\mathfrak{d}_3 & \mathfrak{c}_4^{-1}\mathfrak{d}_4 \\ 0 & 0 & \mathbf{1}_n & -{}^{\mathrm{t}}\mathfrak{c}_3{}^{\mathrm{t}}\mathfrak{c}_4^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix} \right).$$

Proposition 3.5.5. *For $v \mid N$ and $g'_v \in G'(\mathbb{Q}_v)$, we have*

- *If g'_v belongs to the "big cell" with $p_U(g'_v) \in U_{P_{G'}}(\mathbb{Z}_v)$, $p_U(g'_v) \equiv \begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_n & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \mod N$, then*

$$\left(T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,v}(s),g'_v}\overline{\varphi}\right)(g_1) = (\eta\chi)_v^{-1}\left(p_{\mathrm{GL}(1)}(g'_v)\right)|p_{\mathrm{GL}(1)}(g'_v)|_v^{s-n-1}$$
$$\times \pi_v(-1)(\eta\chi)_v(-1)^n \mathrm{vol}\left(\Gamma_{G,1}(N)_v\right) \gamma_v(s, \pi \times \eta\chi)^{-1} \cdot \overline{\varphi}\left(g_1 p_G(g'_v)^{\vartheta}\right).$$

- *If g'_v does not satisfy the conditions above, then*

$$T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,v}(s),g'_v}\overline{\varphi} = 0.$$

Proof. According to Proposition 3.5.4,

$$T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,v}(s),g'_v}\overline{\varphi} = \pi_v(-1) \cdot \gamma_v(s, \pi \times \eta\chi)^{-1}$$
$$\times \gamma_v(s-n, \eta\chi) \prod_{j=1}^{n} \gamma_v(2s - 2n - 1 + 2j, \eta^2\chi^2) \cdot T_{M_{Q_{H'},v}(s,\eta\chi) f_{\underline{\tau},\kappa,v}(s), g'_v}\overline{\varphi}.$$

By the definition of $f_{\underline{\tau},\kappa,v}$, we have

$$\gamma_v(s-n, \eta\chi) \prod_{j=1}^{n} \gamma_v(2s - 2n - 1 + 2j, \eta^2\chi^2)\, M_{Q_{H'},v}(s, \eta\chi) f_{\underline{\tau},\kappa,v}(s) = f^{\alpha_v^{\mathrm{vol}}}(-s, \eta^{-1}\chi^{-1}). \tag{3.5.11}$$

Thus,

$$T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,v}(s),g'_v}\overline{\varphi} = \pi_v(-1)\,\gamma_v(s, \pi \times \eta\chi)^{-1}\, T_{f^{\alpha_v^{\mathrm{vol}}}(-s,\eta^{-1}\chi^{-1}),g'_v}\overline{\varphi},$$

and we need to compute

$$\left(T_{f^{\alpha_v^{\mathrm{vol}}}(-s,\eta^{-1}\chi^{-1}),g'_v}\overline{\varphi}\right)(g_1) = \int_{G(\mathbb{Q}_v)} f^{\alpha_v^{\mathrm{vol}}}(-s, \eta^{-1}\chi^{-1})\left(S_{H'}^{-1}\iota_{H'}(g_v, g'_v)\right)\overline{\varphi}(g_1 g_v)\, dg_v. \tag{3.5.12}$$

For $g_v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Q}_v)$ and $g'_v \in G'(\mathbb{Q}_v)$ written in blocks as in (3.5.9),

$$S_{H'}^{-1}\iota_{H'}(g_v, g'_v) = \left(\begin{array}{ccc:ccc} a & 0 & 0 & b & 0 & 0 \\ 0 & \mathfrak{a}_1 & \mathfrak{a}_2 & 0 & \mathfrak{b}_1 & \mathfrak{b}_2 \\ 0 & \mathfrak{a}_3 & \mathfrak{a}_4 & 0 & \mathfrak{b}_3 & \mathfrak{b}_4 \\ \hdashline c & -\mathfrak{a}_1 & -\mathfrak{a}_2 & d & -\mathfrak{b}_1 & -\mathfrak{b}_2 \\ -a & \mathfrak{c}_1 & \mathfrak{c}_2 & -b & \mathfrak{d}_1 & \mathfrak{d}_2 \\ 0 & \mathfrak{c}_3 & \mathfrak{c}_4 & 0 & \mathfrak{d}_3 & \mathfrak{d}_4 \end{array}\right).$$

The support of $f^{\alpha_v^{\mathrm{vol}}}(-s, \eta^{-1}\chi^{-1})$ contains $S_{H'}^{-1}\iota_{H'}(g_v, g'_v)$ only if

$$\begin{pmatrix} c & -\mathfrak{a}_1 & -\mathfrak{a}_2 \\ -a & \mathfrak{c}_1 & \mathfrak{c}_2 \\ 0 & \mathfrak{c}_3 & \mathfrak{c}_4 \end{pmatrix}^{-1} \begin{pmatrix} d & -\mathfrak{b}_1 & -\mathfrak{b}_2 \\ -b & \mathfrak{d}_1 & \mathfrak{d}_2 \\ 0 & \mathfrak{d}_3 & \mathfrak{d}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n & {}^{\mathrm{t}}b\mathfrak{a}_1 - {}^{\mathrm{t}}d\mathfrak{c}_1 & {}^{\mathrm{t}}b\mathfrak{a}_2 - {}^{\mathrm{t}}d\mathfrak{c}_2 \\ 0 & -{}^{\mathrm{t}}a\mathfrak{a}_1 + {}^{\mathrm{t}}c\mathfrak{c}_1 & -{}^{\mathrm{t}}a\mathfrak{a}_2 + {}^{\mathrm{t}}c\mathfrak{c}_2 \\ 0 & \mathfrak{c}_3 & \mathfrak{c}_4 \end{pmatrix}^{-1} \begin{pmatrix} 0 & {}^{\mathrm{t}}b\mathfrak{b}_1 - {}^{\mathrm{t}}d\mathfrak{d}_1 & {}^{\mathrm{t}}b\mathfrak{b}_2 - {}^{\mathrm{t}}d\mathfrak{d}_2 \\ \mathbf{1}_n & -{}^{\mathrm{t}}a\mathfrak{b}_1 + {}^{\mathrm{t}}c\mathfrak{d}_1 & -{}^{\mathrm{t}}a\mathfrak{b}_2 + {}^{\mathrm{t}}c\mathfrak{d}_2 \\ 0 & \mathfrak{d}_3 & \mathfrak{d}_4 \end{pmatrix}$$

belongs to $\mathrm{Sym}(2n+1, \mathbb{Z}_v)$ and is congruent to

$$\left(\begin{array}{cccc:ccccc} * & * & \cdots & * & -1 & * & \cdots & * & * \\ * & * & \cdots & * & 0 & -1 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 & 0 & \cdots & -1 & * \\ \hdashline -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ * & * & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 \\ * & * & \cdots & * & 0 & 0 & \cdots & 0 & 0 \end{array}\right) \mod N,$$

so only if $\begin{pmatrix} \mathfrak{c}_3 & \mathfrak{c}_4 \end{pmatrix} \neq 0$ and there exists $a \in \mathrm{GL}(n, \mathbb{Z}_v)$, $a \equiv \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \bmod N$, $x \in M_{1,n}(\mathbb{Z}_v)$ and $\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \in \mathrm{Sym}(n+1, \mathbb{Z}_v)$ such that

$$\begin{pmatrix} \mathbf{1}_n & * & * \\ 0 & * & * \\ 0 & \mathfrak{c}_3 & \mathfrak{c}_4 \end{pmatrix} \begin{pmatrix} * & * & * \\ -{}^{\mathrm{t}}a & Ny_{11} & Ny_{12} \\ x & Ny_{21} & Nx_{22} \end{pmatrix} = \begin{pmatrix} 0 & * & * \\ \mathbf{1}_n & * & * \\ 0 & \mathfrak{d}_3 & \mathfrak{d}_4 \end{pmatrix}.$$

From

$$a \in \mathrm{GL}(n, \mathbb{Z}_v), \quad \begin{pmatrix} \mathfrak{c}_3 & \mathfrak{c}_4 \end{pmatrix} \neq 0, \quad \mathfrak{c}_3{}^{\mathrm{t}}a = \mathfrak{c}_4 x, \quad \mathfrak{d}_3 = N(\mathfrak{c}_3 y_{11} + \mathfrak{c}_4 y_{21}), \quad \mathfrak{d}_4 = N(\mathfrak{c}_3 y_{21} + \mathfrak{c}_4 y_{22}),$$

we deduce that the necessary conditions for $f^{\alpha_v^{\mathrm{vol}}}(-s, \eta^{-1}\chi^{-1}) \left(S_{H'}^{-1}\iota_{H'}(g_v, g'_v)\right) \neq 0$ include

$$\mathfrak{c}_4 \neq 0, \quad \text{and} \quad \mathfrak{c}_4^{-1} \begin{pmatrix} \mathfrak{c}_3 & \mathfrak{d}_3 & \mathfrak{d}_4 \end{pmatrix} \in M_{1,n}(\mathbb{Z}_v) \times M_{1,n+1}(N\mathbb{Z}_v). \tag{3.5.13}$$

This proves the vanishing statement in the proposition.

Now suppose that (3.5.13) is satisfied. Then

$$\begin{aligned} (3.5.12) &= |p_{\mathrm{GL}(1)}(g'_v)|_v^{s-n-1} \eta_v \chi_v \left(p_{\mathrm{GL}(1)}(g'_v)\right) \int_{G(\mathbb{Q}_v)} f^{\alpha_v^{\mathrm{vol}}}(-s, \eta^{-1}\chi^{-1}) \left(S_{H'}^{-1}\iota_{H'}(g_v, p_G(g'_v) w_{P_{G'}})\right) \overline{\varphi}(g_1 g_v)\, dg_v \\ &= |p_{\mathrm{GL}(1)}(g'_v)|_v^{s-n-1} \eta_v \chi_v \left(p_{\mathrm{GL}(1)}(g'_v)\right) \int_{G(\mathbb{Q}_v)} f^{\alpha_v^{\mathrm{vol}}}(-s, \eta^{-1}\chi^{-1}) \left(S_{H'}^{-1}\iota_{H'}(g_v, w_{P_{G'}})\right) \overline{\varphi}\left(g_1 p_G(g'_v)^{\vartheta} g_v\right) dg_v. \end{aligned}$$

Meanwhile, since

$$S_{H'}^{-1}\iota_{H'}(g_v, w_{P_{G'}}) = \left(\begin{array}{ccc:ccc} a & 0 & 0 & b & 0 & 0 \\ 0 & \mathbf{1}_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \hdashline c & -\mathbf{1}_n & 0 & d & 0 & 0 \\ -a & 0 & 0 & -b & \mathbf{1}_n & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad \begin{pmatrix} c & -\mathbf{1}_n & 0 \\ -a & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} d & 0 & 0 \\ -b & \mathbf{1}_n & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a^{-1}b & -a^{-1} & 0 \\ -{}^{\mathrm{t}}a^{-1} & ca^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we see that $f^{\alpha_v^{\mathrm{vol}}}(-s, \eta^{-1}\chi^{-1})\left(S_{H'}^{-1}\iota_{H'}(g_v, w_{P_{G'}})\right) \neq 0$ if and only if $g_v \in \Gamma_{G,1}(N)_v$. Therefore,

$$(3.5.12) = |p_{\mathrm{GL}(1)}(g'_v)|_v^{s-n-1}\eta_v\chi_v\left(p_{\mathrm{GL}(1)}(g'_v)\right) \cdot \eta_v\chi_v(-1)^n \mathrm{vol}\left(\Gamma_{G,1}(N)_v\right) \cdot \overline{\varphi}\left(g_1 p_G(g'_v)^{\vartheta}\right).$$

$\square$

3.5.4. *The place p.* In order to obtain information on $\mathscr{P}_{\deg}\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)$, one needs to consider (3.3.1) with the action of $\mathbb{U}_p$-operators being taken into account. In this subsection, we compute the p-adic limit

$$\lim_{m\to\infty}\left(T^{M_{P_{G'}}}_{(U^{m!}_{p,G}\times U^{m!}_{p,G'})f_{\underline{\tau},\kappa,p}(s),\mathbf{1}_{2n+2}}\varphi\Big|_{s=n+1-k}\right),$$

where $U_{p,G}$ (resp. $U_{p,G'}$) is the adelic $\mathbb{U}_p$-operator attached to $(n, n-1, \dots, 1)$ (resp. $(n+1, n, \dots, 1)$) normalized by weight $\underline{t}$ (resp. $(\underline{t}, k)$). (The formula for its action is given in §2.6.2).

In the same way as defining the "big cell" section in §2.3.3, one can define the "big cell" section associated to a Schwartz function α_v^H on $\mathrm{Sym}(2n, \mathbb{Q}_v)$ inside $I_{Q_H}(s, \xi)$ as

$$f^{\alpha_v^H}(s,\xi)\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{cases} \xi^{-1}(\det C)|\det C|^{-(s+\frac{2n+1}{2})}\alpha_v(C^{-1}D), & \text{if } \det C \neq 0, \\ 0, & \text{if } \det C = 0. \end{cases}$$

Proposition 3.5.6. *For $\underline{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ and $\underline{b}' = (b_1, \dots, b_n, b_{n+1}) \in \mathbb{Z}^{n+1}$ with $a_1 \geq \cdots \geq a_n \geq 0$ and $b_1 \geq \cdots \geq b_{n+1} \geq 0$,*

$$\begin{aligned}(3.5.14)\qquad \left(T^{M_{P_{G'}}}_{(U_{p,\underline{a}}\times U_{p,\underline{b}'})f_{\underline{\tau},\kappa,p}(s),\mathbf{1}_{2n+2}}\varphi\right)(g_1) &= \eta_p(p)^{-b_{n+1}-\sum_{j=1}^n a_j+b_j}\, p^{(s-(n+1-k))b_{n+1}} \\ &\quad\times\left(T_{(U_{p,\underline{a}}\times U_{p,\underline{b}})f^{\alpha^H_{\underline{\tau},\kappa,p}}(s-\frac{1}{2})}\varphi\right)(g_1),\end{aligned}$$

where $\underline{b} = (b_1, \dots, b_n)$ and the Schwartz function $\widehat{\alpha}^H_{\underline{\tau},\kappa,p}$ on $\mathrm{Sym}(n, \mathbb{Q}_p)$ is defined as

$$\alpha^H_{\underline{\tau},\kappa,p}(\varsigma) = \alpha_{\underline{\tau},\kappa,}\begin{pmatrix} \varsigma & 0 \\ 0 & 0 \end{pmatrix},$$

and the normalization of $U_{p,\underline{a}}$, $U_{p,\underline{b}}$ (resp. $U_{p,\underline{b}'}$) is with respect to weight $\underline{t}$ (resp. $(\underline{t}, k)$).

Proof. For $u_B \in U_{B_G}(\mathbb{Z}_p)$, $u'_B \in U_{B_{G'}}(\mathbb{Z}_p)$ and $u'_P \in U_{P_{G'}}(\mathbb{Q}_p)$, we write

$$u_B = \begin{pmatrix} u_1^{-1} & \sigma_1{}^{\mathrm{t}}u_1 \\ 0 & {}^{\mathrm{t}}u_1 \end{pmatrix}, \qquad u'_B = \begin{pmatrix} u_2^{-1} & 0 & \sigma_2{}^{\mathrm{t}}u_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & {}^{\mathrm{t}}u_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{1}_n & -m & -m{}^{\mathrm{t}}v & v-mw \\ 0 & 1 & {}^{\mathrm{t}}v & w \\ 0 & 0 & \mathbf{1}_n & 0 \\ 0 & 0 & {}^{\mathrm{t}}m & 1 \end{pmatrix}, \qquad u'_P = \begin{pmatrix} \mathbf{1}_n & 0 & 0 & y \\ {}^{\mathrm{t}}x & 1 & {}^{\mathrm{t}}y & z+{}^{\mathrm{t}}xy \\ 0 & 0 & \mathbf{1}_n & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $u_1, u_2 \in B_n(\mathbb{Z}_p)$, $\sigma_1, \sigma_2 \in \mathrm{Sym}(n, \mathbb{Z}_p)$, $m, v \in \mathbb{Z}_p^n$, $w \in \mathbb{Z}_p$, $x, y \in \mathbb{Q}_p^n$, $z \in \mathbb{Q}_p$. By definition, the left hand side of (3.5.14) equals

(3.5.15)

$$\begin{aligned}&p^{\langle \underline{t}+2\rho_{G,c},\underline{a}\rangle+\langle(\underline{t},k)+2\rho_{G',c},\underline{b}'\rangle}\int_{U_{P_{G'}}(\mathbb{Q}_p)}\int_{G(\mathbb{Q}_p)}\int_{U_{B_G}(\mathbb{Z}_p)\times U_{B_{G'}}(\mathbb{Z}_p)} \\ &\qquad f_{\underline{\tau},\kappa,p}(s)\left(S_{H'}^{-1}\iota_{H'}\left(g_1u_B\begin{pmatrix} p^{\underline{a}} & \\ & p^{-\underline{a}}\end{pmatrix}, w_{P'_G}u'_Pu'_B\begin{pmatrix} p^{\underline{b}'} & \\ & p^{-\underline{b}'}\end{pmatrix}\right)\right)\varphi(g_1)\,du_B du'_B\,dg_1\,du'_P.\end{aligned}$$

Let

$$g_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad h' = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix} = S_{H'}^{-1}\iota_{H'}\left(g_1 u_B \begin{pmatrix} p^{\underline{a}} & \\ & p^{-\underline{a}} \end{pmatrix}, w_{P'_G} u'_P u'_B \begin{pmatrix} p^{\underline{b}'} & \\ & p^{-\underline{b}'} \end{pmatrix}\right).$$

We consider the integration of (3.5.15) over $U_{P_{G'}}(\mathbb{Q}_p) \times U_{B_G}(\mathbb{Z}_p) \times U_{B_{G'}}(\mathbb{Z}_p)$. (Interchanging the integration order for g_1 and u'_P can be justified when $s \gg 0$.) We have

$$C^{-1}\mathfrak{D} = \begin{bmatrix} p^{-\underline{a}}\, u_1 & & \\ & p^{-\underline{b}}\, u_2 & \\ & & 1 \end{bmatrix} \left[\begin{array}{c|c|c} -a^{-1}b + \sigma_1 & -a^{-1} & 0 \\ \hline {}^{\mathrm{t}}a^{-1} & -ca^{-1} + \sigma_2 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} {}^{\mathrm{t}}u_1 p^{-\underline{a}} & & \\ & {}^{\mathrm{t}}u_2 p^{-\underline{b}} & \\ & & 1 \end{bmatrix}$$
$$+ \begin{bmatrix} p^{-\underline{a}} & & \\ & p^{-\underline{b}} & \\ & & p^{-b_{n+1}} \end{bmatrix} \left[\begin{array}{c|c|c} 0 & u_1 a^{-1} x {}^{\mathrm{t}}m & u_1 a^{-1} x \\ \hline m {}^{\mathrm{t}}x {}^{\mathrm{t}}a^{-1} {}^{\mathrm{t}}u_1 & \begin{array}{c} m {}^{\mathrm{t}}y {}^{\mathrm{t}}u_2 + u_2 y {}^{\mathrm{t}}m + \\ m {}^{\mathrm{t}}x c a^{-1} u_2 + u_2 c a^{-1} x {}^{\mathrm{t}}m \\ + m z {}^{\mathrm{t}}m - m {}^{\mathrm{t}}x c a^{-1} x {}^{\mathrm{t}}m \end{array} & \begin{array}{c} v + u_2(y + ca^{-1}x) \\ + m(z - {}^{\mathrm{t}}x c a^{-1} x) \end{array} \\ \hline {}^{\mathrm{t}}x {}^{\mathrm{t}}a^{-1} {}^{\mathrm{t}}u_1 & \begin{array}{c} {}^{\mathrm{t}}v + ({}^{\mathrm{t}}y + {}^{\mathrm{t}}x c a^{-1}) {}^{\mathrm{t}}u_2 \\ + (z - {}^{\mathrm{t}}x c a^{-1} x) {}^{\mathrm{t}}m \end{array} & w + z - {}^{\mathrm{t}}x c a^{-1} x \end{array}\right] \begin{bmatrix} p^{-\underline{a}} & & \\ & p^{-\underline{b}} & \\ & & p^{-b_{n+1}} \end{bmatrix}.$$

Applying the change of variables $\widetilde{x} = u_1 a^{-1} x$, $\widetilde{y} = u_2(my + ca^{-1}x) + m(z - {}^{\mathrm{t}}xca^{-1})$, $\widetilde{z} = z - {}^{\mathrm{t}}xca^{-1}$, we obtain

$$\mathfrak{C}^{-1}\mathfrak{D} = \begin{bmatrix} p^{-\underline{a}} & & \\ & p^{-\underline{b}} & \\ & & p^{-b_{n+1}} \end{bmatrix} \left[\begin{array}{c|c|c} u_1(-a^{-1}b + \sigma_1){}^{\mathrm{t}}u_1 & -u_1 a^{-1} {}^{\mathrm{t}}u_2 + \widetilde{x} {}^{\mathrm{t}}m & \widetilde{x} \\ \hline u_2 {}^{\mathrm{t}}a^{-1} {}^{\mathrm{t}}u_1 + m {}^{\mathrm{t}}\widetilde{x} & \begin{array}{c} u_2(-ca^{-1} + \sigma_2){}^{\mathrm{t}}u_2 + \\ m {}^{\mathrm{t}}\widetilde{y} + \widetilde{y} {}^{\mathrm{t}}m - m \widetilde{z} {}^{\mathrm{t}}m \end{array} & \widetilde{y} + w \\ \hline {}^{\mathrm{t}}\widetilde{x} & {}^{\mathrm{t}}\widetilde{y} + {}^{\mathrm{t}}w & \widetilde{z} + v \end{array}\right] \begin{bmatrix} p^{-\underline{a}} & & \\ & p^{-\underline{b}} & \\ & & p^{-b_{n+1}} \end{bmatrix}.$$

If $\mathfrak{C}^{-1}\mathfrak{D}$ lies inside the support of $\alpha_{\underline{\tau},\kappa,p}$, then $(p^{-\underline{b}}\,\widetilde{y} p^{-b_{n+1}}, p^{-2b_{n+1}}\widetilde{z}) \in \mathbb{Z}_p^n \times \mathbb{Z}_p$, so $m {}^{\mathrm{t}}\widetilde{y} + \widetilde{y} {}^{\mathrm{t}}m - m \widetilde{z} {}^{\mathrm{t}}m$ belongs to $\mathrm{Sym}(n, \mathbb{Z}_p)$ and we can apply the change of variable $\widetilde{\sigma}_2 = \sigma_2 + u_2^{-1}(m {}^{\mathrm{t}}\widetilde{y} + \widetilde{y} {}^{\mathrm{t}}m - m \widetilde{z} {}^{\mathrm{t}}m) {}^{\mathrm{t}}u_2^{-1}$. Setting $(\eta\chi)_{v,s} = (\eta\chi)_v \,|\cdot|_v^s$, we have

(3.5.16)

$$\begin{aligned} & p^{\langle \underline{t} + 2\rho_{G,c}, \underline{a} \rangle + \langle (\underline{t},k) + 2\rho_{G',c}, \underline{b}' \rangle}\, f_{\underline{\tau},\kappa,p}(s)(h') \\ = & \, p^{\langle \underline{t} + 2\rho_{G,c}, \underline{a} + \underline{b} \rangle + \sum_{j=1}^n b_j + (k-n) b_{n+1}} (\eta\chi)_{p,s+n+1}^{-1} \left(\det(-a)\, p^{b_{n+1} + \sum_{j=1}^n a_j + b_j}\right) \cdot \alpha_{\underline{\tau},\kappa,p}(\mathfrak{C}^{-1}\mathfrak{D}) \\ = & \, \eta_p(p)^{-b_{n+1}} p^{\langle \underline{t} + 2\rho_{G,c}, \underline{a} + \underline{b} \rangle + \sum_{j=1}^n b_j + (k+s+1) b_{n+1}} (\eta\chi)_{p,s+n+1}^{-1} \left(\det(-a)\, p^{\sum_{j=1}^n a_j + b_j}\right) \cdot \alpha_{\underline{\tau},\kappa,p}(\mathfrak{C}^{-1}\mathfrak{D}). \end{aligned}$$

Integrating (3.5.16) with respect to $v, w, \widetilde{y}, \widetilde{z}$, we obtain

$$\begin{aligned} & \eta_p(p)^{-b_{n+1} - \sum_{j=1}^n a_j + b_j} \cdot p^{\langle \underline{t} + 2\rho_{G,c}, \underline{a} + \underline{b} \rangle} (\eta\chi)_{p,s+n}^{-1} \left(\det(-a)\, p^{\sum_{j=1}^n a_j + b_j}\right) \\ & \times \alpha^H_{\underline{\tau},\kappa,p,\text{up-left}}\left(p^{-\underline{a}}\, u_1(-a^{-1}b + \sigma_1){}^{\mathrm{t}}u_1 p^{-\underline{a}}\right) \alpha^H_{\underline{\tau},\kappa,p,\text{low-right}}\left(p^{-\underline{b}}\, u_2(-ca^{-1} + \sigma_2){}^{\mathrm{t}}u_2 p^{-\underline{b}}\right) \\ & \times p^{(k-n-1+s)b_{n+1} + \sum_{j=1}^n a_j + bj}\, \alpha^H_{\underline{\tau},\kappa,p,\text{off-diag}}\left(p^{-\underline{a}}(-u_1 a^{-1} {}^{\mathrm{t}}u_2 + \widetilde{x} {}^{\mathrm{t}}m) p^{-\underline{b}}\right) \mathbb{1}_{\mathbb{Z}_p^n}(p^{-\underline{a}}\widetilde{x} p^{-b_{n+1}}). \end{aligned} \tag{3.5.17}$$

Next we integrate (3.5.17) over $\widetilde{x}, m$.

(3.5.18)

$$\begin{aligned} & \int_{\mathbb{Q}_p^n}\int_{\mathbb{Z}_p^n} \alpha^H_{\underline{\tau},\kappa,p,\text{off-diag}}\left(p^{-\underline{a}}(-u_1 a^{-1} {}^{\mathrm{t}}u_2 + \widetilde{x} {}^{\mathrm{t}}m) p^{-\underline{b}}\right) \mathbb{1}_{\mathbb{Z}_p^n}(p^{-\underline{a}}\widetilde{x} p^{-b_{n+1}})\, dm\, d\widetilde{x} \\ = & \int_{\mathbb{Z}_p^n \times \mathbb{Q}_p^n \times \mathrm{Sym}(n,\mathbb{Z}_p)} \widehat{\alpha}^H_{\underline{\tau},\kappa,p,\text{off-diag}}(\varsigma)\, \mathbb{1}_{\mathbb{Z}_p^n}(p^{-\underline{a}}\widetilde{x} p^{-b_{n+1}})\, \mathbf{e}_p\Big(\mathrm{Tr}(-\varsigma p^{-\underline{a}} u_1 a^{-1} {}^{\mathrm{t}}u_2 p^{-\underline{b}} + \varsigma p^{-\underline{a}} \widetilde{x} {}^{\mathrm{t}}m p^{-\underline{b}})\Big)\, dm\, d\widetilde{x}\, d\varsigma. \end{aligned}$$

After another change of variable $\widetilde{\widetilde{x}} = \varsigma p^{-\underline{a}}\widetilde{x}$ and integrating with respect to m,

$$(3.5.18) = p^{-\sum_{j=1}^n a_j} \int_{\mathbb{Q}_p^n\times \mathrm{Sym}(n,\mathbb{Z}_p)} \widehat{\alpha}^H_{\underline{\tau},\kappa,p,\text{off-diag}}(\varsigma)\, \mathbb{1}_{\mathbb{Z}_p^n}(\varsigma^{-1}\widetilde{\widetilde{x}}p^{-b_{n+1}})\, \mathbb{1}_{\mathbb{Z}_p^n}(p^{-\underline{b}}\,\widetilde{\widetilde{x}})$$
$$\mathbf{e}_p(-\mathrm{Tr}\,\varsigma p^{-\underline{a}}u_1a^{-1}{}^{\mathrm{t}}u_2p^{-\underline{b}})\, d\widetilde{\widetilde{x}}\, d\varsigma.$$

By our definition of $\widehat{\alpha}^H_{\underline{\tau},\kappa,p,\text{off-diag}}$ (§2.3.4), it is supported on $\mathrm{GL}(n,\mathbb{Z}_p)$ and for $\varsigma\in\mathrm{GL}(n,\mathbb{Z}_p)$, we have $\mathbb{1}_{\mathbb{Z}_p^n}(\varsigma^{-1}\widetilde{\widetilde{x}}p^{-b_{n+1}}) = \mathbb{1}_{\mathbb{Z}_p^n}(\widetilde{\widetilde{x}}p^{-b_{n+1}})$. Hence

$$(3.5.18) = p^{-\sum_{j=1}^n a_j+b_j}\, \alpha^H_{\underline{\tau},\kappa,p,\text{off-diag}}(p^{-\underline{a}}u_1a^{-1}{}^{\mathrm{t}}u_2p^{-\underline{b}}).$$

Plugging this into (3.5.17), we obtain

$$\int_{U_{P_{G'}}(\mathbb{Q}_p)}\int_{U_{B_G}(\mathbb{Z}_p)\times U_{B_{G'}}(\mathbb{Z}_p)} (3.5.16)\, du_B\, du'_B\, du'_P$$
$$= \eta_p(p)^{-b_{n+1}-\sum_{j=1}^n a_j+b_j}\, p^{(k-n-1+s)b_{n+1}}\cdot p^{\langle \underline{t}+2\rho_{G,c},\underline{a}+\underline{b}\rangle}\int_{U_{B_G}(\mathbb{Z}_p)\times U_{B_G}(\mathbb{Z}_p)} (\eta\chi)^{-1}_{p,s+n}\left(\det(-a)\,p^{\sum_{j=1}^n a_j+b_j}\right)$$
$$\times\, \alpha^H_{\underline{\tau},\kappa,p,\text{off-diag}}\left(\begin{pmatrix} p^{-\underline{a}}\,u_1 & \\ & p^{-\underline{b}}\,u_2\end{pmatrix}\begin{pmatrix} -a^{-1}b+\sigma_1 & a^{-1}\\ {}^{\mathrm{t}}a^{-1} & -ca^{-1}+\sigma_2\end{pmatrix}\begin{pmatrix} {}^{\mathrm{t}}u_1p^{-\underline{a}} & \\ & {}^{\mathrm{t}}u_2p^{-\underline{b}}\end{pmatrix}\right) d\sigma_1du_1\,d\sigma_2du_2$$
$$= \eta_p(p)^{-b_{n+1}-\sum_{j=1}^n a_j+b_j}\, p^{(k-n-1+s)b_{n+1}}\cdot\left(U_{p,\underline{a}}\times U_{p,\underline{b}}\right) f^H_{\underline{\tau},\kappa,p}(s-\frac{1}{2})(g_1),$$

which combined with (3.5.15) proves the proposition. $\square$

Now combining the above proposition and [LR18, §2.8], we deduce

Proposition 3.5.7. *Suppose that $\varphi\in\pi$ is an ordinary adelic holomorphic Siegel modular form of weight $\underline{t}$ and nebentypus $\underline{\epsilon}$ invariant under $\widehat{\Gamma}_{G,1}(N)^p$. Then*

$$\lim_{m\to\infty}\left(T^{M_{P_{G'}}}_{(U^{m!}_{p,G}\times U^{m!}_{p,G'})f_{\underline{\tau},\kappa,p}(s),\mathbf{1}_{2n+2}}\overline{\varphi}\Big|_{s=n+1-k}\right)^{\vartheta}$$
$$= p^{2n^2}(p-1)^v\mathrm{vol}(\Gamma_{G,1}(p)_p)\chi(-1)^nE_p^-(n+1-k,\pi\times\eta\chi)\cdot e_G\mathscr{W}(\varphi)$$

where the operator $\mathscr{W}:\pi\to\pi$ is defined as

$$\mathscr{W}(\varphi)(g) = \int_{U_{B_G}(\mathbb{Z}_p)}\overline{\varphi}^{\vartheta}(gu)\,du,\quad g\in G(\mathbb{A}), \tag{3.5.19}$$

and the factor $E_p^-(s,\pi\times\eta\chi)$ is the modified Euler factor at p for p-adic interpolation as defined in [Coa91].

Remark 3.5.8. For the convenience of the reader, we briefly describe the modified Euler factor at p in our case. The condition in the above proposition implies that there exist continuous characters $\theta_1,\dots,\theta_n:\mathbb{Q}_p^\times\to\mathbb{C}^\times$ such that $\mathrm{val}_p(\theta_j(p)) = -t_j+j$, $\theta_j|_{\mathbb{Z}_p^\times} = \epsilon_j^{-1}$ and $\pi_p\hookrightarrow \mathrm{Ind}^{G(\mathbb{Q}_p)}_{B_G(\mathbb{Q}_p)}(\theta_1,\dots,\theta_n)$. Then for p-adically interpolating the critical values to the left (resp. right) of the center, the modified Euler factor is given as

$$E_p^-(s,\pi\times\eta\chi) = \prod_{j=1}^n\gamma_p(s,\theta_j\eta_p\chi_p)^{-1},\quad (\text{resp }.\ E_p^+(s,\pi\times\eta\chi) = \gamma_p(s,\eta_p\chi_p)^{-1}\prod_{j=1}^n\gamma_p(s,\theta_j\eta_p\chi_p)^{-1}).$$

We refer to [LR18, §2.3] for more details.

3.6. **The image of $\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}$ under the Siegel operator and the p-adic L-function for $\mathscr{C}$.** For all $L \in \mathfrak{C}_{\mathbb{V}'}/\Gamma_{G'}(N)$, as in §3.2, we can choose a basis of $L^{\perp}/L$ and use it to define the (p-adic) q-expansion map for $\mathcal{M}^{0}_{L,\mathrm{ord}}$. The basis gives an element $\gamma'_L \in G'(\mathbb{Z})$. (Whether γ'_L belongs to the "big cell", the image $p_G(\gamma'_L) \in G(\mathbb{Q})$ and the image $p_U(\gamma'_L) \in U_{P_{G'}}(\mathbb{Q})$ up to $U_{P_{G'}}(N\mathbb{Z})$ do not depend the choice.)

By the doubling method (Theorem 2.2.1), the coefficient indexed by $(\beta_1,\beta_2) \in \mathrm{Sym}(n, N^{-1}\mathbb{Z})^{*\oplus 2}_{>0}$ in the p-adic q-expansion of $\Phi_{\mathrm{deg},L}\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)$ at the cusp $(\gamma_{N,1},\gamma_{N,2})$ equals (up to necessary normalizations) the product of $\varepsilon^{\gamma_{N,1}}_{q\text{-exp}}(\beta_1,\varphi)$ and

$$
\begin{aligned}
& b_H^{Np\infty}(s+\frac{1}{2},\eta\chi)\, e_{G'} E^{\mathrm{Kl}}_{\left(\begin{smallmatrix}\beta_2 & 0\\ 0 & 0\end{smallmatrix}\right)}\left(\gamma'_L\gamma_{N,2}g'_{z',\infty}, \Phi_{f_{\underline{\tau},\kappa}(s),\varphi}\right)\Big|_{s=n+1-k}\\
&= \lim_{m\to\infty} L^{Np\infty}(s,\pi\times\eta\chi)\left(T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,\infty}(s),\mathbf{1}_{2n+2}} T^{M_{P_{G'}}}_{(U^{m!}_{p,G}\times U^{m!}_{p,G'})f_{\underline{\tau},\kappa,p}(s),\mathbf{1}_{2n+2}} \prod_{v\mid N} T^{M_{P_{G'}}}_{f_{\underline{\tau},\kappa,v}(s),\gamma'_L\gamma_{N,2}}\overline{\varphi}\right)^{\vartheta}_{\beta_2}\Bigg|_{s=n+1-k}.
\end{aligned}
$$

Recall that (3.3.2) says that the left hand side is computed by the sum $\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}+M_{P_{G'}}(s,\eta\chi)\Phi_{f_{\underline{\tau},\kappa}(s),\varphi}$, Proposition 3.4.1 says that the first term vanishes at $\gamma'_L\gamma_{N,2}g'_{z',\infty}$ due to our choice of $f_{\underline{\tau},\kappa,p}(s)$, and Proposition 3.5.1 says that the contribution from second term is the right hand side, which is computed in Proposition 3.5.2, 3.5.5, 3.5.6.

Theorem 3.6.1. *Given $L \in \mathfrak{C}_{\mathbb{V}'}/\Gamma_{G'}(N)$ of rank 1, $\gamma_{N,1},\gamma_{N,2} \in G(\mathbb{Z}/N)$ and $\beta_1,\beta_2 \in N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*}_{>0}$, the meromorphic function $\varepsilon^{\gamma_{N,1},\gamma_{N,2}}_{q\text{-exp}}\left(\beta_1,\beta_2,\Phi_{\mathrm{deg},L}\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)\right)$ inside $\mathbb{I}_{\mathscr{C}}[[\mathbb{Z}_p^{\times}]]\otimes_{\mathbb{I}_{\mathscr{C}}} F_{\mathscr{C}}$ satisfies the following interpolation properties.*

Let $x : \mathbb{I}_{\mathscr{C}} \to \overline{\mathbb{Q}}_p$ be an $\overline{\mathbb{Q}}_p$-point of $\mathscr{C}_P$. Suppose that the weight projection $\Lambda_n \to \mathbb{T}^N_{ord}$ is étale at x and maps x to an admissible point $\underline{\tau} \in \mathrm{Hom}_{\mathrm{cont}}\left(T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^{\times}\right)$. For $\kappa \in \mathrm{Hom}_{\mathrm{cont}}\left(\mathbb{Z}_p^{\times},\overline{\mathbb{Q}}_p^{\times}\right)$ such that $(\underline{\tau},\kappa)$ is admissible, we have

- *If $\eta\kappa(-1) = 1$ and γ'_L belongs to the "big cell" with $p_U(\gamma'_L) \in U_{P_{G'}}(\mathbb{Z}_v)$, congruent to $\left(\begin{smallmatrix}\mathbf{1}_n & 0 & 0 & 0\\ * & 1 & 0 & 0\\ 0 & 0 & \mathbf{1}_n & *\\ 0 & 0 & 0 & 1\end{smallmatrix}\right)$ modulo N for all $v \mid N$ (see (3.5.10) for the definition of the map $(p_G, p_{\mathrm{GL}(1)}, p_U) : P_{G'}w_{P_{G'}}U_{P_{G'}} \to G\times\mathrm{GL}(1)\times U_{P_{G'}}$), then*

$$
\begin{aligned}
&\varepsilon^{\gamma_{N,1},\gamma_{N,2}}_{q\text{-exp}}\left(\beta_1,\beta_2,\Phi_{\mathrm{deg},L}\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)\right)(x,\kappa)\\
&= p^{2n^2}(p-1)^n\,\mathrm{vol}(\widehat{\Gamma}_{G,1}(Np))\,(\eta\kappa)_N\left(p_{\mathrm{GL}(1)}(\gamma'_L)\right)\prod_{j=1}^{n}\tau_j(-1)\prod_{v\mid N}\gamma_v(n+1-k,\pi\times\eta\chi)^{-1}\\
&\quad\times\frac{\sqrt{-1}^{\frac{n^2-n}{2}}2^{-\sum t_i+\frac{n^2-n}{2}}}{\dim\left(\mathrm{GL}(n),\underline{t}_x\right)}\sum_{\varphi\in\mathfrak{s}_x}\frac{\varepsilon^{\gamma_{N,1}}_{q\text{-exp}}(\beta_1,\varphi)\,\varepsilon^{p_G(\gamma'_L),\gamma_{N,2}}_{q\text{-exp}}(\beta_2,e_G\mathcal{W}(\varphi))}{\langle\varphi,\overline{\varphi}\rangle}\\
&\quad\times E_p^{-}(n+1-k,\pi\times\eta\chi)\,E_{\infty}^{-}(n+1-k,\pi\times\eta\chi)\,L^{Np\infty}(n+1-k,\pi\times\eta\chi).
\end{aligned}
$$

- *Otherwise,*

$$
\varepsilon^{\gamma_{N,1},\gamma_{N,2}}_{q\text{-exp}}\left(\beta_1,\beta_2,\Phi_{\mathrm{deg},L}\left(\mathcal{E}^{\mathrm{Kl}}_{\mathscr{C}}\right)\right)(x,\kappa) = 0.
$$

Here $(\eta\kappa)_N(p_{\mathrm{GL}(1)}(\gamma'_L)) = \prod_{v\mid N}(\eta\chi)_v(p_{\mathrm{GL}(1)}(\gamma'_L))\,|p_{\mathrm{GL}(1)}(\gamma'_L|_v^{-k}$ and $\mathfrak{s}_x$ is a finite set consisting of an orthogonal basis of the eigenspace for the Hecke eigensystem parameterized by x inside the space of ordinary cuspidal Siegel modular form of genus g, weight $\underline{t}$, p-nebentypus $\underline{\epsilon}$ and tame level $\Gamma_{G,1}(N)$. If $\mathfrak{s}_x$ is empty, then the evaluation is 0.

Remark 3.6.2. By applying the functional equation for the L-function $L(s,\pi\times\eta\chi)$ [LR05, (33)], we can also write the formula on $\varepsilon_{q\text{-exp}}^{\gamma_{N,1},\gamma_{N,2}}\left(\beta_1,\beta_2,\mathcal{P}_{\deg,L}\left(\mathcal{E}_{\mathscr{C}}^{\mathrm{Kl}}\right)\right)(x,\kappa)$ in terms of the critical L-values to the right of the center, *i.e.*

$$\begin{aligned}
&\varepsilon_{q\text{-exp}}^{\gamma_{N,1},\gamma_{N,2}}\left(\beta_1,\beta_2,\mathcal{P}_{\deg,L}\left(\mathcal{E}_{\mathscr{C}}^{\mathrm{Kl}}\right)\right)(x,\kappa)\\
=&\,p^{2n^2}(p-1)^n\operatorname{vol}\big(\widehat{\Gamma}_{G,1}(pN)\big)(\eta\kappa)_N\left(p_{\mathrm{GL}(1)}(\gamma_L')\right)\\
&\times\frac{\sqrt{-1}^{\frac{n^2-n}{2}}2^{-\sum t_i+\frac{n^2-n}{2}}}{\dim\left(\mathrm{GL}(n),\underline{t}_x\right)}\sum_{\varphi\in\mathfrak{s}_x}\frac{\varepsilon_{q\text{-exp}}^{\gamma_{N,1}}\left(\beta_1,\varphi\right)\varepsilon_{q\text{-exp}}^{p_G(\gamma_L'),\gamma_{N,2}}\left(\beta_2,e_G\mathcal{W}(\varphi)\right)}{\langle\varphi,\overline{\varphi}\rangle}\\
&\times E_p^+(k-n,\pi\times\eta^{-1}\chi^{-1})\,E_\infty^+(k-n,\pi\times\eta^{-1}\chi^{-1})\,L^{Np\infty}(k-n,\pi\times\eta^{-1}\chi^{-1}).
\end{aligned}$$

The interpolation properties of $\varepsilon_{q\text{-exp}}^{\gamma_{N,1},\gamma_{N,2}}\left(\beta_1,\beta_2,\mathcal{P}_{\deg,L}\left(\mathcal{E}_{\mathscr{C}}^{\mathrm{Kl}}\right)\right)(x,\kappa)$ justify that it can be viewed as the p-adic L-function attached to $\mathscr{C}$. The theorem essentially says that the image of our constructed family $\mathcal{E}_{\mathscr{C}}^{\mathrm{Kl}}$ of Klingen Eisenstein series under the Siegel operator

$$\mathcal{P}_{\deg}:\mathcal{M}^0_{G,\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]}\mathcal{M}^1_{G',\mathrm{ord}}\longrightarrow\bigoplus_{\substack{L\in\mathfrak{C}_{\mathbb{V}'}/\Gamma_{G'}(N)\\ \mathrm{rk}L=1}}\mathcal{M}^0_{G,\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]}\mathcal{M}^0_{L,\mathrm{ord}}\otimes_{\mathcal{O}_F[\![T_n(\mathbb{Z}_p)]\!]}\mathcal{O}_F[\![T_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times]\!]$$

is given by the p-adic L-functions attached to $\mathscr{C}$.

4. Non-degenerate Fourier coefficients of the Klingen Eisenstein family

In order to relate the "constant term" $\mathcal{P}_{\deg}\left(\mathcal{E}_{\mathscr{C}}^{\mathrm{Kl}}\right)$ to the congruence ideal associated to the Hecke eigensystem of $\mathcal{E}_{\mathscr{C}}^{\mathrm{Kl}}$, one needs to verify the primitivity of the Klingen Eisenstein family $\mathcal{E}_{\mathscr{C}}^{\mathrm{Kl}}$. One strategy (as used in [Urb06, SU14]) is to show the coprimeness of the non-degenerate Fourier coefficients and (the Fourier coefficients of) $\mathcal{P}_{\deg}\left(\mathcal{E}_{\mathscr{C}}^{\mathrm{Kl}}\right)$.

In this section, we study the non-degenerate coefficients $\varepsilon_{q\text{-exp}}^{1,\gamma_N'}\left(\beta,\beta',\mathcal{E}_{\mathscr{C}}^{\mathrm{Kl}}\right)$. More precisely, we compute the Fourier coefficients

(4.0.1)
$$(e_{G'}E^{\mathrm{Kl}})_{\beta'}\left(m(\mathfrak{a}_f)g'_{z',\infty},\Phi_{f_{\underline{\tau},\kappa},\varphi}\right)=\int_{\mathrm{Sym}(n+1,\mathbb{Q}\backslash\mathbb{A})}e_{G'}E^{\mathrm{Kl}}\left(\begin{pmatrix}\mathbf{1}_{n+1}&\varsigma\\0&\mathbf{1}_{n+1}\end{pmatrix}m(\mathfrak{a}_f)g'_{z',\infty},\Phi_{f_{\underline{\tau},\kappa},\varphi}\right)\mathbf{e}_{\mathbb{A}}(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma$$

for a certain collection of $\beta'\in\mathrm{Sym}(n+1,\mathbb{Q})_{>0}$ and $\mathfrak{a}_f\in\mathrm{GL}(n+1,\mathbb{A}_f)$. The results are recorded in Theorem 2.6.2, which expresses (4.0.1) as the Petersson inner product of $\overline{\varphi}$ with the product of a Siegel Eisenstein series and a theta series. For the purpose of further studying of $\varepsilon_{q\text{-exp}}^{1,1}\left(\beta,\beta',\mathcal{E}_{\mathscr{C}}^{\mathrm{Kl}}\right)$ and verifying the primitivity of $\mathcal{E}_{\mathscr{C}}^{\mathrm{Kl}}$, it is crucial to obtain precise formulas for the local sections giving rise to the Seigel Eisenstein series and the Schwartz functions giving rise to the theta series.

We compute (4.0.1) by first computing

$$\begin{aligned}
&\mathfrak{E}^{\mathrm{Si}}\big|_{G\times G',\beta'}\left(g,m(\mathfrak{a}_f)g'_{z',\infty},(U_{p,G}^m\times U_{p,G'}^m)f_{\kappa,\underline{\tau}}(s)\right)\\
=&\int_{\mathrm{Sym}(n+1,\mathbb{Q}\backslash\mathbb{A})}\mathfrak{E}^{\mathrm{Si}}\left(\iota_{H'}\left(g,\begin{pmatrix}\mathbf{1}_{n+1}&\varsigma\\0&\mathbf{1}_{n+1}\end{pmatrix}m(\mathfrak{a}_f)g'_{z'}\right),(U_{p,G}^m\times U_{p,G'}^m)f_{\kappa,\underline{\tau}}(s)\right)\mathbf{e}_{\mathbb{A}}(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma,
\end{aligned}$$

and then pairing it with $\overline{\varphi}$.

Compared to the analogous computation for unitary groups in [Wan15], one major improvement of our computation pertains to the computation at the place p. By handling the intertwining operator in a more effective way, we do not need to assume the condition that the nebentypus is sufficiently ramified. This is important especially when one cannot identify the non-degenerate coefficients in the q-expansion of the family with a known p-adic L-function. Moreover, we include a discussion of expressing the local $FJ_{\beta',v}$'s by using the Siegel–Weil sections, which on one hand can be useful for considering the seesaw diagram, and on the other hand allows us to obtain a better expression for the place 2 when n is even and deal with the vector weight case for the archimedean place. (Only the scalar weight case is computed in [Wan15].)

4.1. **The unfolding.** Define the parabolic subgroup $P_{H'}$ of H' as the subgroup consisting of elements of the following form

$$\begin{array}{cccccc} n & n+1 & n & n+1 & & \\ \end{array}$$
$$\begin{pmatrix} A & 0 & B & * \\ * & \mathfrak{A} & * & * \\ C & 0 & D & * \\ 0 & 0 & 0 & {}^{\mathrm{t}}\mathfrak{A}^{-1} \end{pmatrix} \begin{matrix} n \\ n+1 \\ n \\ n+1 \end{matrix} \in H', \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G,\, \mathfrak{A} \in \mathrm{GL}(n+1).$$

It is not difficult to check that the image of $W_{Q_{G'}}\backslash W_{G'}/W_{Q_{G'}}$ under $\iota_{H'}$ constitutes a set of representatives of $W_{Q_{H'}}\backslash W_{H'}/W_{P_{H'}}$. Given $f(s,\xi) \in I_{Q_{H'}}(s,\xi)$ and $\beta' \in \mathrm{Sym}(n+1,\mathbb{Q})_{>0}$,

$$\begin{aligned} &\mathfrak{E}^{\mathrm{Si}}\big|_{G\times G',\beta'}(g,g',f(s,\xi)) \\ &= \int_{\mathrm{Sym}(n+1,\mathbb{Q}\backslash\mathbb{A})} \sum_{\gamma\in(Q_{H'}\backslash H')(\mathbb{Q})} f(s,\xi)\left(\gamma\iota_{H'}\left(g, \begin{pmatrix} \mathbf{1}_{n+1} & \varsigma \\ 0 & \mathbf{1}_{n+1} \end{pmatrix} g'\right)\right) \mathbf{e}_{\mathbb{A}}(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma \\ &= \int_{\mathrm{Sym}(n+1,\mathbb{Q}\backslash\mathbb{A})} \sum_{w\in W_{Q_{H'}}\backslash W_{H'}/W_{P_{H'}}} \sum_{\gamma\in\left(Q^w_{H'}\cap P_{H'}\backslash P_{H'}\right)(\mathbb{Q})} f(s,\xi)\left(w\gamma\iota_{H'}\left(g, \begin{pmatrix} \mathbf{1}_{n+1} & \varsigma \\ 0 & \mathbf{1}_{n+1} \end{pmatrix} g'\right)\right) \mathbf{e}_{\mathbb{A}}(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma \\ &= \int_{\mathrm{Sym}(n+1,\mathbb{Q}\backslash\mathbb{A})} \sum_{w\in W_{Q_{G'}}\backslash W_{G'}/W_{Q_{G'}}} \sum_{\gamma\in\left(Q^w_{H'}\cap P_{H'}\backslash P_{H'}\right)(\mathbb{Q})} \\ &\qquad\qquad f(s,\xi)\left(\iota_{H'}\left(1, w\begin{pmatrix} \mathbf{1}_{n+1} & \varsigma \\ 0 & \mathbf{1}_{n+1} \end{pmatrix}\right)\gamma\iota_{H'}(g,g')\right) \mathbf{e}_{\mathbb{A}}(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma, \end{aligned}$$

where for $w \in W_{H'}$ (resp. $w \in W_{G'}$), $Q^w_{H'} = w^{-1}Q_{H'}w$ (resp. $Q^w_{H'} = \iota_{H'}(1,w)^{-1}Q_{H'}\iota_{H'}(1,w)$). Because of the non-degeneracy of β', the only nonvanishing term in the sum over $W_{Q_{G'}}\backslash W_{G'}/W_{Q_{G'}}$ is the one attached to $w_{Q'_G} = \begin{pmatrix} 0 & -\mathbf{1}_{n+1} \\ \mathbf{1}_{n+1} & 0 \end{pmatrix}$. Hence,

$$\begin{aligned} \mathfrak{E}^{\mathrm{Si}}\big|_{G\times G',\beta'}(g,g',f(s,\xi)) &= \sum_{\gamma\in Q_G(\mathbb{Q})\backslash G(\mathbb{Q})} \sum_{x\in M_{n,n+1}(\mathbb{Q})} \int_{\mathrm{Sym}(n+1,\mathbb{A})} \\ & f(s,\xi)\left(\iota_{H'}(\mathbf{1}_{2n}, w_{Q_{G'}})\begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ {}^{\mathrm{t}}x & \mathbf{1}_{n+1} & 0 & 0 \\ 0 & 0 & \mathbf{1}_n & -x \\ 0 & 0 & 0 & \mathbf{1}_{n+1} \end{pmatrix}\iota_{H'}\left(\gamma g, \begin{pmatrix} \mathbf{1}_{n+1} & \varsigma \\ 0 & \mathbf{1}_{n+1} \end{pmatrix} g'\right)\right) \mathbf{e}_{\mathbb{A}}(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma. \end{aligned}$$

Denoting the term in the summand as $FJ_{\beta'}(\gamma g, g', x, f(s,\xi))$, the above identity is written as

$$\mathfrak{E}^{\mathrm{Si}}\big|_{G\times G',\beta'}(g,g',f(s,\xi)) = \sum_{\gamma\in Q_G(\mathbb{Q})\backslash G(\mathbb{Q})} \sum_{x\in M_{n,n+1}(\mathbb{Q})} FJ_{\beta'}(\gamma g, g', x, f(s,\xi)). \tag{4.1.1}$$

If $f_v(s,\xi)$ factorizes, then $FJ_{\beta'}(g,g',x,f(s,\xi)) = \prod_v FJ_{\beta',v}(g,g',x,f_v(s,\xi))$ with

(4.1.2)

$$\begin{aligned} &FJ_{\beta',v}(g,g',x,f_v(s,\xi)) \\ &= \int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_v(s,\xi)\left(\iota_{H'}(\mathbf{1}_{2n}, w_{Q_{G'}})\begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ {}^{\mathrm{t}}x & \mathbf{1}_{n+1} & 0 & 0 \\ 0 & 0 & \mathbf{1}_n & -x \\ 0 & 0 & 0 & \mathbf{1}_{n+1} \end{pmatrix}\iota_{H'}\left(g, \begin{pmatrix} \mathbf{1}_{n+1} & \varsigma \\ 0 & \mathbf{1}_{n+1} \end{pmatrix} g'\right)\right) \mathbf{e}_v(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma. \end{aligned}$$

By the functional equation for Siegel Eisenstein series, we also have

$$\mathfrak{E}^{\mathrm{Si}}\big|_{G\times G',\beta'}(g,g',f(s,\xi)) = \sum_{\gamma\in Q_G(\mathbb{Q})\backslash G(\mathbb{Q})}\sum_{x\in M_{n,n+1}(\mathbb{Q})} FJ_{\beta'}\left(\gamma g, g', x, M_{Q_{H'}}(s,\xi)f(s,\xi)\right). \tag{4.1.3}$$

In the following we will work with (4.1.3) rather than (4.1.1)

4.2. **Some remarks on $FJ_{\beta',v}$ and its computation.** The next few sections are devoted to computing $FJ_{\beta',v}\left(g,g',x,M_{Q_{H'}}(s,\xi)f_{\underline{\tau},\kappa,v}(s)\right)$ place by place. Before launching the involved computation, we give a rough description on how it should look like and in what format we would like to express it.

It is not very difficult to observe that as a function on $(g,x)\in G(\mathbb{Q}_v)\times M_{n,n+1}(\mathbb{Q}_v)$, the term $FJ_{\beta',v}\left(g,g',x,M_{Q_{H'}}(s,\xi)f_{\underline{\tau},\kappa,v}(s)\right)$ takes the form of (a linear combination of) the product

$$(*)\cdot f_{\widetilde{G},v}\left(-s,(\xi_v\widetilde{\lambda}_{2\beta',v})^{-1}\right)(g,\epsilon)\cdot\omega_{2\beta'}(g,\epsilon)\phi_v(x), \tag{4.2.1}$$

where $(*)$ is a constant independent of g,x, the pair (g,ϵ) with $\epsilon\in\{\pm1\}$ denotes an element inside the metaplectic group $\widetilde{G}(\mathbb{Q}_v)=\widetilde{\mathrm{Sp}}(2n,\mathbb{Q}_v)$, $f_{\widetilde{G},v}$ is a section inside the degenerate principal series on $\widetilde{G}(\mathbb{Q}_v)$, and $\omega_{2\beta'}(g,\epsilon)\phi_v$ is the action of (g,ϵ) on the Schwartz function $\phi_v\in\mathcal{S}(M_{n,n+1}(\mathbb{Q}_v),\mathbb{C})$ via the Weil representation (with respect to the symmetric form $2\beta'$).

Therefore, as a function on $g\in G(\mathbb{Q})\backslash G(\mathbb{A})$, the above $\mathfrak{E}^{\mathrm{Si}}\big|_{G\times G',\beta'}(g,g',f(s,\xi))$ is a (linear combination of) product of a Siegel Eisenstein series and a theta series associated to the orthogonal group $\mathrm{O}(2\beta')$. The β'-th Fourier coefficient of the Klingen Eisenstein series $E^{\mathrm{Kl}}\left(\cdot,\Phi_{f_{\underline{\tau},\kappa},\varphi}\right)$ is (a linear combination of) the Petersson inner product of $\varphi\in\pi$ with the product of a Siegel Eisenstein series and a theta series associated to $\mathrm{O}(2\beta')$. Right now it is not clear if this Petersson inner product is related to an integral representation of certain L-functions (unless when $n=1$ it is the usual integral representation of Rankin-Selberg L-function).

When evaluated at $s=n+1-k$, if ξ_v is a quadratic character, by applying the Siegel–Weil formula, one may write the (linear combination of) (4.2.1) in terms of theta lifts from an orthogonal group of size $2k$, and attempt to transfer the Petersson inner product on $G(\mathbb{Q})\backslash G(\mathbb{A})$ to an integral on orthogonal groups via a seesaw diagram. This is discussed in §4.9. In fact the main reason that we choose to compute $FJ_{\beta',v}\left(g,g',x,M_{Q_{H'}}(s,\xi)f_{\underline{\tau},\kappa,v}(s)\right)$ instead of $FJ_{\beta',v}\left(g,g',x,f_{\underline{\tau},\kappa,v}(s)\right)$ is that when evaluated at $s=n+1-k$ with k relatively small, their images in $I_{Q_G,v}(-s,\xi)$ under the intertwining operator are more conveniently related to Siegel–Weil sections associated to an orthogonal group of size $2k-n-1$.

4.3. **Basics on Weil representation.** We recall some basic facts about the Weil representation of metaplectic groups, which will be needed in our upcoming computation. Let $K=\mathbb{Q}_v$, $V=\mathbb{V}\otimes K$ and W be a finite dimensional symmetric space. Fix a polarization $V=X\oplus Y$ with $X=\mathrm{span}_K\{e_1,\dots,e_n\}$ and $Y=\mathrm{span}_K\{f_1,\dots,f_n\}$.

4.3.1. *The metaplectic group.* Let $\widetilde{\mathrm{Sp}}(V)=\mathrm{Sp}(V)\ltimes\{\pm1\}$ be the metaplectic group. The group law is given by

$$(g_1,\epsilon_1)\cdot(g_2,\epsilon_2)=(g_1g_2,\epsilon_1\epsilon_2c(g_1,g_1)),$$

where $c(\ ,\ )$ is a 2-cocycle on $\mathrm{Sp}(V)$ valued in $\{\pm1\}$. The covering splits uniquely over U_{Q_X} which can be viewed canonically as a subgroup of $\widetilde{\mathrm{Sp}}(V)$. Let $Q_X\subset\mathrm{Sp}(V)$ be the Siegel parabolic subgroup preserving $X\subset V$ and $\widetilde{Q}_X$ be the inverse image of Q_X in $\widetilde{\mathrm{Sp}}(V)$. Then $\widetilde{Q}_X$ admits the Levi decomposition

$$\widetilde{Q}_X=\widetilde{M}_X\ltimes U_{Q_X},$$

with $\widetilde{M}_X = \mathrm{GL}(X) \ltimes \{\pm 1\}$ equipped with the group law

$$(a_1, \epsilon_1) \cdot (a_2, \epsilon_2) = (a_1 a_2, \epsilon_1 \epsilon_2 (\det a_1, \det a_2)_K)\,,$$

where $(\cdot,\cdot)_K$ is the Hilbert symbol of K.

4.3.2. *Formulas for the Weil representation of* $\widetilde{\mathrm{Sp}}(V) \times \mathrm{O}(W)$. Denote by $\mathcal{S}(Y \otimes_K W)$ the space of Schwartz functions on $Y \otimes_K W$. The symmetric bilinear form on W and the skew-symmetric form on V induces a skew-symmetric form on $W \otimes_K V$ which we denote by $\langle \cdot, \cdot \rangle$. The Weil representation depends on the choice of an additive character of $K = \mathbb{Q}_v$. We fix our choice as $\mathbf{e}_v$.

Define the characters

$$\begin{aligned} \lambda_{W,v} : K^\times &\longrightarrow \mathbb{C}^\times \\ (t,z) &\longmapsto \left(t, (-1)^{\frac{\dim W(\dim W-1)}{2}} \det W\right)_K, \end{aligned}$$

and

$$\tag{4.3.1} \begin{aligned} \widetilde{\lambda}_{W,v} : K^\times \times \mathbb{C}^\times &\longrightarrow \mathbb{C}^\times \\ (t,z) &\longmapsto \lambda_{W,v}(t) \cdot \begin{cases} 1, & \text{if } \dim W \text{ is even,} \\ z \cdot \gamma_F(t, \mathbf{e}_v)^{-1}, & \text{if } \dim W \text{ is odd.} \end{cases} \end{aligned}$$

Denote by $\omega_{V,W}$ the Weil representation of $\widetilde{\mathrm{Sp}}(V) \times \mathrm{O}(W)$ on $\mathcal{S}(Y \otimes_K W)$. Then for $\phi \in \mathcal{S}(Y \otimes_K W)$, the action of $\mathrm{O}(W)$ is by inverse translation on W, and the action of $\mathrm{Sp}(V)$ is by

- $\omega_{V,W}\left(\begin{pmatrix} a & b\,{}^{\mathrm{t}}a^{-1} \\ 0 & {}^{\mathrm{t}}a^{-1} \end{pmatrix}, \epsilon\right)\phi(y) = \widetilde{\lambda}_W(\det a, \epsilon) |\det a|_v^{\frac{\dim V}{2}}\, \mathbf{e}_v\left(\frac{1}{2}\langle y, by\rangle\right) \phi({}^{\mathrm{t}}ay)$,
- $\omega_{V,W}\left(\begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}, 1\right)\phi(y) = \gamma(\mathbf{e}_v \circ W)^{-\dim V} \int_{X \otimes_K V} \phi(x)\, \mathbf{e}_v(\langle y, x\rangle)\, dx$.

Here we identify functions on $X \otimes_K W$ and $Y \otimes_K W$ via our fixed basis. The Haar measure on $X \otimes_K W$ is the unique one such that the above formulas define a group action. The constant $\gamma(\mathbf{e}_v \circ W)$ is the Weil index associated to $\mathbf{e}_v$ and W.

4.4. **The unramified places.** First we define some notation. Given a section $f_{\widetilde{G},v}(s, \xi\widetilde{\lambda}_{2\beta',v}) \in I_{Q_{\widetilde{G}},v}(s, \xi\widetilde{\lambda}_{2\beta',v})$ and a Schwartz function $\phi_{2\beta',v} \in \mathcal{S}\left(M_{n,n+1}(\mathbb{Q}_v), \mathbb{C}\right)$, we define the following function on $G(\mathbb{Q}_v) \times M_{n,n+1}(\mathbb{Q}_v)$,

$$\tag{4.4.1} S\left(g, x; f_{\widetilde{G},v}(s, \xi\widetilde{\lambda}_{2\beta',v}), \phi_{2\beta' v}\right) = f_{\widetilde{G},v}(s, \xi\widetilde{\lambda}_{2\beta',v})(g,\epsilon) \cdot \omega_{2\beta',v}(g,\epsilon)\phi_{2\beta',v}(x),$$

where the right hand side does not depend on the choice of $\epsilon \in \{\pm 1\}$. Like $b_{H',v}(s,\xi)$ (defined in (2.4.2)), define

$$b_{G',v}(s,\xi) = L_v\left(s + \frac{n+1}{2}, \xi\right) \prod_{j=1}^{\left[\frac{n+1}{2}\right]} L_v\left(2s + n + 1 - 2j, \xi^2\right).$$

Proposition 4.4.1. *Let v be a finite place where ξ_v is unramified. For given $\beta' \in \mathrm{Sym}(n+1, \mathbb{Q})$ and $\mathfrak{a}_v \in \mathrm{GL}(n+1, \mathbb{Q}_v)$ such that ${}^{\mathrm{t}}\mathfrak{a}_v \beta' \mathfrak{a}_v \in \mathrm{Sym}(n+1, \mathbb{Z}_v)^*$ and $2{}^{\mathrm{t}}\mathfrak{a}_v \beta' \mathfrak{a}_v \in \mathrm{GL}(n+1, \mathbb{Z}_v)$, we have*

$$\begin{aligned} FJ_{\beta',v}\left(g, m(\mathfrak{a}_v), x, f_v^{\mathrm{ur}}(s,\xi)\right) = {} & \xi(\det \mathfrak{a}_v)^{-1} |\det \mathfrak{a}_v|_v^{-s+1}\, b_{G',v}\left(s + \frac{n}{2}, \xi\right)^{-1} \begin{cases} 1 & n \text{ even} \\ L_v\left(s + \frac{n+1}{2}, \xi\lambda_{2\beta'}\right) & n \text{ odd} \end{cases} \\ & \times S\left(g, x; f^{\mathrm{ur}}_{\widetilde{G},v}(s, \xi\widetilde{\lambda}^{-1}_{2\beta'}), R({}^{\mathrm{t}}\mathfrak{a}_v^{-1})\mathbb{1}_{M_{n,n+1}(\mathbb{Z}_v)}\right), \end{aligned}$$

and

$$
\begin{aligned}
&b_{H',v}(s,\xi)\, FJ_{\beta',v}\left(g, m(\mathfrak{a}_v), x, M_{Q_{H'}}(s,\xi) f_v^{\mathrm{ur}}(s,\xi)\right)\\
=&\,\xi(\det\mathfrak{a}_v)|\det\mathfrak{a}_v|_v^{s+1}\,\gamma_v(s-n,\xi)^{-1}\prod_{j=1}^{\left[\frac{n+1}{2}\right]}\gamma_v\left(2s-2n-1+2j,\xi^2\right)^{-1}\prod_{j=\left[\frac{n+1}{2}\right]+1}^{n} L_v(2s-2n-1+2j,\xi^2)\\
&\times\begin{cases}1 & n\ \mathit{even}\\ L_v\left(-s+\frac{n+1}{2},\xi^{-1}\lambda_{2\beta'}\right) & n\ \mathit{odd}\end{cases}\cdot S\left(g, x; f^{\mathrm{ur}}_{\widetilde{G},v}(-s,\xi\widetilde{\lambda}_{2\beta'}^{-1}), R({}^{\mathrm{t}}\mathfrak{a}_v^{-1})\mathbb{1}_{M_{n,n+1}(\mathbb{Z}_v)}\right),
\end{aligned}
$$

where $R({}^{\mathrm{t}}\mathfrak{a}_v^{-1})$ denotes the right translation by ${}^{\mathrm{t}}\mathfrak{a}_v^{-1}$.

Remark 4.4.2. Note that when n is even and $v=2$, one cannot find positive definite β' and $\mathfrak{a}_v$ such that the conditions in the above proposition is satisfied. In this case the $FJ_{\beta',v}$ cannot be written as a simple product as (4.4.1). Instead, it is a linear combination of such products. See §4.9 for more discussion.

Proof. The function $g\mapsto FJ_{\beta',v}\left(g, m(\mathfrak{a}_v), x, f_v^{\mathrm{ur}}(s,\xi)\right)$ is invariant under the right translation by $G(\mathbb{Z}_v)$. Meanwhile, the conditions in the above proposition imply the existence of an unramified section inside the degenerate principal series on $\widetilde{G}(\mathbb{Q}_v)$ attached to the character $\xi_v\widetilde{\lambda}_{2\beta'}$, as well as the invariance of the Schwartz function $\mathbb{1}_{M_{n,n+1}(\mathbb{Z}_v)}$ under the action of $\widetilde{G}(\mathbb{Z}_v)$ by the Weil representation. Thus it suffices to check the identities in the proposition for $g=\begin{pmatrix} a & b{}^{\mathrm{t}}a^{-1}\\ 0 & {}^{\mathrm{t}}a^{-1}\end{pmatrix}\in Q_G(\mathbb{Q}_v)$. The left hand side is computed as follows.

$$
\begin{aligned}
&FJ_{\beta',v}\left(\left(\begin{smallmatrix} a & b{}^{\mathrm{t}}a^{-1}\\ 0 & {}^{\mathrm{t}}a^{-1}\end{smallmatrix}\right), m(\mathfrak{a}_v), x, f_v^{\mathrm{ur}}(s,\xi)\right)\\
=&\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_v^{\mathrm{ur}}(s,\xi)\left(\begin{pmatrix}\mathbf{1}_n & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathbf{1}_{n+1}\\ 0 & 0 & \mathbf{1}_n & 0\\ 0 & \mathbf{1}_{n+1} & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{1}_n & 0 & 0 & 0\\ {}^{\mathrm{t}}x & \mathbf{1}_{n+1} & 0 & 0\\ 0 & 0 & \mathbf{1}_n & -x\\ 0 & 0 & 0 & \mathbf{1}_{n+1}\end{pmatrix}\begin{pmatrix} a & 0 & b{}^{\mathrm{t}}a^{-1} & 0\\ 0 & \mathfrak{a}_v & 0 & \varsigma{}^{\mathrm{t}}\mathfrak{a}_v^{-1}\\ 0 & 0 & {}^{\mathrm{t}}a^{-1} & 0\\ 0 & 0 & 0 & {}^{\mathrm{t}}\mathfrak{a}_v^{-1}\end{pmatrix}\right)\mathbf{e}_v(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma\\
=&\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_v^{\mathrm{ur}}(s,\xi)\left(\begin{pmatrix} a & -{}^{\mathrm{t}}bx{}^{\mathrm{t}}\mathfrak{a}_v^{-1} & b{}^{\mathrm{t}}a^{-1} & 0\\ 0 & {}^{\mathrm{t}}\mathfrak{a}_v^{-1} & 0 & 0\\ 0 & 0 & {}^{\mathrm{t}}a^{-1} & 0\\ 0 & 0 & xb{}^{\mathrm{t}}a^{-1} & \mathfrak{a}_v\end{pmatrix}\begin{pmatrix}\mathbf{1}_n & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathbf{1}_{n+1}\\ 0 & 0 & \mathbf{1}_n & -{}^{\mathrm{t}}ax{}^{\mathrm{t}}\mathfrak{a}_v^{-1}\\ \mathfrak{a}_v^{-1}{}^{\mathrm{t}}xa & \mathbf{1}_{n+1} & 0 & \mathfrak{a}_v^{-1}\varsigma{}^{\mathrm{t}}\mathfrak{a}_v^{-1}+\mathfrak{a}_v^{-1}{}^{\mathrm{t}}xbx{}^{\mathrm{t}}\mathfrak{a}_v^{-1}\end{pmatrix}\right)\mathbf{e}_v(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma\\
=&\,\xi(\det a)|\det a|_v^{s+n+1}\mathbf{e}_v\left(\mathrm{Tr}\beta'{}^{\mathrm{t}}xbx\right)\,\xi(\det\mathfrak{a}_v)^{-1}|\det\mathfrak{a}_v|_v^{-s+1}\\
&\times\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_v^{\mathrm{ur}}(s,\xi)\left(\begin{pmatrix}\mathbf{1}_n & 0 & 0 & 0\\ 0 & \mathbf{1}_{n+1} & 0 & 0\\ 0 & {}^{\mathrm{t}}ax{}^{\mathrm{t}}\mathfrak{a}_v^{-1} & \mathbf{1}_n & 0\\ \mathfrak{a}_v^{-1}{}^{\mathrm{t}}xa & \varsigma & 0 & \mathbf{1}_{n+1}\end{pmatrix}\right)\mathbf{e}_v(\mathrm{Tr}{}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v\varsigma)\,d\varsigma.
\end{aligned}
$$

Put

$$
\phi_{2{}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v,v}(x)=\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_v^{\mathrm{ur}}(s,\xi)\left(\begin{pmatrix}\mathbf{1}_n & 0 & 0 & 0\\ 0 & \mathbf{1}_{n+1} & 0 & 0\\ 0 & x & \mathbf{1}_n & 0\\ {}^{\mathrm{t}}x & \varsigma & 0 & \mathbf{1}_{n+1}\end{pmatrix}\right)\mathbf{e}_v(\mathrm{Tr}{}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v\varsigma)\,d\varsigma.
$$

Then by the formulas of Weil representations recalled in last section,

$$
\begin{aligned}
FJ_{\beta',v}\left(\left(\begin{smallmatrix} a & b{}^{\mathrm{t}}a^{-1}\\ 0 & {}^{\mathrm{t}}a^{-1}\end{smallmatrix}\right), m(\mathfrak{a}_v), x, f_v^{\mathrm{ur}}(s,\xi)\right)=&\,\xi(\det\mathfrak{a}_v)^{-1}|\det\mathfrak{a}_v|_v^{-s+1}\cdot f^{\mathrm{ur}}_{\widetilde{G},v}(s,\xi\widetilde{\lambda}_{2\beta',v}^{-1})\left(\left(\begin{smallmatrix} a & b{}^{\mathrm{t}}a^{-1}\\ 0 & {}^{\mathrm{t}}a^{-1}\end{smallmatrix}\right)\right)\\
&\times\omega_{2\beta'}\left(\left(\begin{pmatrix} a & b{}^{\mathrm{t}}a^{-1}\\ 0 & {}^{\mathrm{t}}a^{-1}\end{pmatrix},1\right)\right)R({}^{\mathrm{t}}\mathfrak{a}_v^{-1})\cdot\phi_{2{}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v,v}(x).
\end{aligned}
$$

Therefore, the desired identity follows from the following proposition.

Proposition 4.4.3. *Let v be a finite place where ξ_v is unramified and β'_v be an element inside $\mathrm{Sym}(n+1,\mathbb{Z}_v)^*$ such that $\mathrm{val}_v(\det 2\beta'_v) \leq 1$. Then*

$$\begin{aligned}\phi_{2\beta'_v,v}(x) &= \int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_v^{\mathrm{ur}}(s,\xi)\left(\begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & \mathbf{1}_{n+1} & 0 & 0 \\ 0 & x & \mathbf{1}_n & 0 \\ {}^{\mathrm{t}}x & \varsigma & 0 & \mathbf{1}_{n+1}\end{pmatrix}\right)\mathbf{e}_v(\mathrm{Tr}\beta'_v\varsigma)\,d\varsigma \\ &= W_{\beta'_v,v}\left(\mathbf{1}_{n+1}, f_{G',v}^{\mathrm{ur}}(s+\frac{n}{2},\xi)\right)\cdot \mathbb{1}_{M_{n,n+1}(\mathbb{Z}_v)}(x),\end{aligned}$$

where $f_{G',v}^{\mathrm{ur}}(s+\frac{n}{2},\xi) \in I_{Q_{G'},v}(s+\frac{n}{2},\xi)$ is the standard unramified section in the degenerate principal series $I_{Q_{G'},v}(s,\xi)$ on $G'(\mathbb{Q}_v)$, and $W_{\beta'_v,v}$ is the local Fourier coefficient for $I_{Q_{G'},v}(s,\xi)$ defined as

$$W_{\beta'_v,v}\left(g, f_{G',v}(s,\xi)\right) = \int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_{G',v}(s,\xi)\left(\begin{pmatrix} \mathbf{1}_{n+1} & 0 \\ \varsigma & \mathbf{1}_{n+1}\end{pmatrix}\right)\mathbf{e}_v(\mathrm{Tr}\beta'_v\varsigma)\,d\varsigma.$$

Moreover, by [Shi97, Theorem 13.6, Proposition 14.9], we have

$$W_{\beta'_v,v}\left(\mathbf{1}_{n+1}, f_{G',v}^{\mathrm{ur}}(s,\xi)\right) = g_{{}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v}\left(\xi(q_v)q_v^{-(s+n+1)}\right) b_{G',v}(s)^{-1}\begin{cases} 1, & n \text{ even} \\ L_v\left(s+\frac{1}{2},\xi\lambda_{2\beta'_v}\right) & n \text{ odd}\end{cases},$$

with $g_{{}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v}(T) \in \mathbb{Z}[T]$ of degree less or equal to $4n\cdot\mathrm{val}_v(\det 2\beta'_v)$ and constant term 1.

Proof. Given $\varsigma \in M_{n,n+1}(\mathbb{Q}_v)$, pick $a \in \mathrm{GL}(n+1,\mathbb{Z}_v)$ and $a' \in \mathrm{GL}(n+1,\mathbb{Z}_v)$ such that

$$ax{}^{\mathrm{t}}a' = \begin{array}{cc} \begin{array}{cc} r & n+1-r \end{array} & \\ \begin{pmatrix} q_v^{-\underline{m}} & 0 \\ 0 & * \end{pmatrix} & \begin{array}{c} r \\ n-r \end{array}\end{array} = \begin{pmatrix} q_v^{-m_1} & & & \\ & \ddots & & \\ & & q_v^{-m_r} & \\ & & & * \end{pmatrix} \tag{4.4.2}$$

with $\underline{m} = (m_1,\ldots,m_r) \in \mathbb{Z}_{>0}^r$ and $* \in M_{n-r,n+1-r}(\mathbb{Z}_v)$. Then

$$\begin{aligned}\phi_{2\beta'_v,v}(x) &= \int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_v^{\mathrm{ur}}(s,\xi)\left(\begin{array}{cc:cc} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & \mathbf{1}_{n+1} & 0 & 0 \\ \hdashline 0 & \begin{pmatrix} q_v^{-\underline{m}} & 0 \\ 0 & * \end{pmatrix} & \mathbf{1}_n & 0 \\ \begin{pmatrix} q_v^{-\underline{m}} & 0 \\ 0 & {}^{\mathrm{t}}* \end{pmatrix} & a'\varsigma{}^{\mathrm{t}}a' & 0 & \mathbf{1}_{n+1}\end{array}\right)\mathbf{e}_v(\mathrm{Tr}\beta'_v\varsigma)\,d\varsigma \\ &= \int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_v^{\mathrm{ur}}(s,\xi)\left(\begin{array}{cc:cc} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & \mathbf{1}_{n+1} & 0 & 0 \\ \hdashline 0 & \begin{pmatrix} q_v^{-\underline{m}} & 0 \\ 0 & 0 \end{pmatrix} & \mathbf{1}_n & 0 \\ \begin{pmatrix} q_v^{-\underline{m}} & 0 \\ 0 & 0 \end{pmatrix} & \varsigma & 0 & \mathbf{1}_{n+1}\end{array}\right)\mathbf{e}_v(\mathrm{Tr}{}^{\mathrm{t}}a'^{-1}\beta'_v a'^{-1}\varsigma)\,d\varsigma.\end{aligned} \tag{4.4.3}$$

Write

$$\varsigma = \begin{array}{cc} \begin{array}{cc} r & n+1-r \end{array} & \\ \begin{pmatrix} \varsigma_{11} & \varsigma_{12} \\ \varsigma_{21} & \varsigma_{22} \end{pmatrix} & \begin{array}{c} r \\ n+1-r \end{array}\end{array}.$$

Then

$$\left(\begin{array}{cc|cc} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & \mathbf{1}_{n+1} & 0 & 0 \\ \hline 0 & \left(\begin{smallmatrix} q_v^{-\underline{m}} & 0 \\ 0 & 0 \end{smallmatrix}\right) & \mathbf{1}_n & 0 \\ \left(\begin{smallmatrix} q_v^{-\underline{m}} & 0 \\ 0 & 0 \end{smallmatrix}\right) & \varsigma & 0 & \mathbf{1}_{n+1} \end{array}\right) = \left(\begin{array}{cccc|cccc} q_v^{\underline{m}} & 0 & q_v^{\underline{m}}\varsigma_{11}q_v^{\underline{m}} & -q_v^{\underline{m}}\varsigma_{12} & 0 & 0 & \mathbf{1}_r & 0 \\ 0 & \mathbf{1}_{n-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_v^{\underline{m}} & 0 & \mathbf{1}_r & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n+1-r} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & q_v^{-\underline{m}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{n-r} & 0 & 0 \\ 0 & 0 & 0 & 0 & \varsigma_{11} & 0 & q_v^{-\underline{m}} & 0 \\ 0 & 0 & 0 & 0 & \varsigma_{21} & 0 & 0 & \mathbf{1}_{n+1-r} \end{array}\right)$$

$$\times \left(\begin{array}{cccc|cccc} \mathbf{1}_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1}_{n-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n+1-r} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbf{1}_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{n-r} & 0 & 0 \\ 0 & 0 & q_v^{\underline{m}}\varsigma_{11}q_v^{\underline{m}} & q_v^{\underline{m}}\varsigma_{12} & 0 & 0 & \mathbf{1}_r & 0 \\ 0 & 0 & \varsigma_{21}q_v^{\underline{m}} & \varsigma_{22} & 0 & 0 & 0 & \mathbf{1}_{n+1-r} \end{array}\right) \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{1}_r & 0 \\ 0 & \mathbf{1}_{n-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{1}_r & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n+1-r} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{1}_r & 0 & q_v^{\underline{m}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{n-r} & 0 & 0 \\ \mathbf{1}_r & 0 & 0 & 0 & 0 & 0 & q_v^{\underline{m}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{n+1-r} \end{array}\right).$$

Therefore, by change of variable $\varsigma \mapsto \left(\begin{smallmatrix} q_v^{-\underline{m}} & 0 \\ 0 & \mathbf{1}_{n+1-r} \end{smallmatrix}\right) \varsigma \left(\begin{smallmatrix} q_v^{-\underline{m}} & 0 \\ 0 & \mathbf{1}_{n+1-r} \end{smallmatrix}\right)$,

$$\begin{aligned} (4.4.3) =& \xi(\det q_v^{\underline{m}})^2 |\det q_v^{\underline{m}}|^{2s+n} \\ & \times \int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_v^{\mathrm{ur}}(s,\xi)\left(\left(\begin{smallmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & \mathbf{1}_{n+1} & 0 & 0 \\ 0 & 0 & \mathbf{1}_n & 0 \\ 0 & \varsigma & 0 & \mathbf{1}_{n+1} \end{smallmatrix}\right)\right) \mathbf{e}_v\left(\mathrm{Tr}\left(\begin{smallmatrix} q_v^{-\underline{m}} & 0 \\ 0 & \mathbf{1}_{n+1-r} \end{smallmatrix}\right) {}^{\mathrm{t}}a'^{-1}\widetilde{\beta}' a'^{-1} \left(\begin{smallmatrix} q_v^{-\underline{m}} & 0 \\ 0 & \mathbf{1}_{n+1-r} \end{smallmatrix}\right)\varsigma\right) d\varsigma. \end{aligned}$$

This integral vanishes unless

$$\left(\begin{matrix} q_v^{-\underline{m}} & 0 \\ 0 & \mathbf{1}_{n+1-r} \end{matrix}\right) {}^{\mathrm{t}}a'^{-1}\beta_v' a'^{-1} \left(\begin{matrix} q_v^{-\underline{m}} & 0 \\ 0 & \mathbf{1}_{n+1-r} \end{matrix}\right) \in \mathrm{Sym}(n+1,\mathbb{Z}_v)^*. \tag{4.4.4}$$

Meanwhile, if $\mathrm{val}_v(2\det\beta_v') \leq 1$, then $2\left(\begin{matrix} q_v^{-\underline{m}} & 0 \\ 0 & \mathbf{1}_{n+1-r} \end{matrix}\right) {}^{\mathrm{t}}a'^{-1}\beta_v' a'^{-1} \left(\begin{matrix} q_v^{-\underline{m}} & 0 \\ 0 & \mathbf{1}_{n+1-r} \end{matrix}\right)$ cannot be integral unless $\underline{m} = 0$ and $r = 0$, *i.e.* (4.4.4) holds only if $\underline{m} = 0$ and $r = 0$. Hence by (4.4.2), $\phi(x) \neq 0$ only if $x \in M_{n,n+1}(\mathbb{Z}_v)$. For $x \in M_{n,n+1}(\mathbb{Z}_v)$, we have

$$\begin{aligned} \phi(x) &= \int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_v^{\mathrm{ur}}(s,\xi)\left(\left(\begin{smallmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & \mathbf{1}_{n+1} & 0 & 0 \\ 0 & 0 & \mathbf{1}_n & 0 \\ 0 & \varsigma & 0 & \mathbf{1}_{n+1} \end{smallmatrix}\right)\right) \mathbf{e}_v(\mathrm{Tr}\beta_v'\varsigma)\, d\varsigma \\ &= \int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f_{G',v}^{\mathrm{ur}}(s+\frac{n}{2},\xi)\left(\left(\begin{smallmatrix} \mathbf{1}_{n+1} & 0 \\ \varsigma & \mathbf{1}_{n+1} \end{smallmatrix}\right)\right) \mathbf{e}_v(\mathrm{Tr}\beta_v'\varsigma)\, d\varsigma = W_{\beta_v'}\left(\mathbf{1}_{n+1}, f_{G',v}^{\mathrm{ur}}(s+\frac{n}{2},\xi)\right). \end{aligned}$$

The proposition is proved. ∎

∎

4.5. **The archimedean place.** In this section, we compute $FJ_{\beta',\infty}$ evaluated at $s = n+1-k$ for the special case $\underline{t} = (k,\dots,k)$. The general case will be discussed in §4.9.5.

Denote by $\widetilde{\mathrm{sgn}}^{k-\frac{n+1}{2}}$ the character on $\mathbb{R}^\times \times \mathbb{C}^\times$ attached to a positive definite symmetric quadratic form on $\mathbb{R}^n$ by (4.3.1). Let $h : \widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \times \mathbb{H}_n \to \mathbb{C}^\times$ be the automorphy factor for metaplectic group defined as in [Shi00, Theorem A2.4]. We know that for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $h((g,\epsilon),z)^2$ equals $\det(cz+d)$ up to root of unity. We define the canonical section $f_{\widetilde{G},\infty}^{k-\frac{n+1}{2}}(s,\widetilde{\mathrm{sgn}}^{k-\frac{n+1}{2}})$ in $I_{Q_{\widetilde{G}},\infty}(s,\widetilde{\mathrm{sgn}}^{k-\frac{n+1}{2}})$ and the Gaussian function $G_{2\beta',\infty}$ on $M_{n,n+1}(\mathbb{R})$ as

$$f_{\widetilde{G},\infty}^{k-\frac{n+1}{2}}(s,\widetilde{\mathrm{sgn}}^{k-\frac{n+1}{2}})(g,\epsilon) = h\left((g,\epsilon),\sqrt{-1}\mathbf{1}_{n+1}\right)^{-2k+n+1}, \quad G_{2\beta',\infty}(x) = e^{-\pi\mathrm{Tr}x\beta'{}^{\mathrm{t}}x}. \tag{4.5.1}$$

Proposition 4.5.1. *Let $\beta' \in \mathrm{Sym}(n+1,\mathbb{Q})_{>0}$ and k be an integer. Then*

$$FJ_{\beta',\infty}\left(g, g'_{z'}, x, M_{Q_{H'}}(s,\mathrm{sgn}^k) f^k_\infty(s,\mathrm{sgn}^k)\right)\Big|_{s=n+1-k} = \frac{\sqrt{-1}^{(2-n)k}2^{(2n+1)k-\frac{5n^2}{2}-\frac{3n}{2}}\pi^{(n+1)(2n+1+k)}}{\Gamma_{2n+1}(n+1)\Gamma_{n+1}(k)}$$
$$\times (\det 2\beta')^{k-\frac{n+2}{2}}(\det \mathrm{Im} z')^{\frac{k}{2}}\mathbf{e}_\infty(\mathrm{Tr}\beta' z')\cdot S\left(g, x; f^{k-\frac{n+1}{2}}_{\widetilde{G},\infty}(s, \widetilde{\mathrm{sgn}}^{k-\frac{n+1}{2}}), G_{2\beta',\infty}\right).$$

Proof. The right translation by the maximal compact subgroup of $G(\mathbb{R})$ on both sides of the identity is by $j(\cdot,\sqrt{-1})^{-k}$, so it suffices to check the identity for $g = g_z, z \in \mathbb{H}_n$. Let

$$Z' = X' + \sqrt{-1}Y' = \begin{array}{cc} \begin{array}{cc} n & n+1 \end{array} & \\ \begin{pmatrix} z & z_0 \\ {}^{\mathrm{t}}z_0 & z' \end{pmatrix} & \begin{array}{c} n \\ n+1 \end{array} \end{array} \in \mathbb{H}_{2n+1}, \qquad h'_{Z'} = \begin{pmatrix} \sqrt{Y'} & X'{}^{\mathrm{t}}\sqrt{Y'}^{-1} \\ 0 & {}^{\mathrm{t}}\sqrt{Y'}^{-1}\end{pmatrix} \in H'(\mathbb{R}).$$

The evaluation at $z = z_0$ of the integral

(4.5.2)

$$FJ_{\beta',\infty}\left(h'_{Z'}, x, f^k_\infty(s,\mathrm{sgn}^k)\right)$$
$$= \int_{\mathrm{Sym}(n+1,\mathbb{R})} f^k_\infty(s,\mathrm{sgn}^k)\left(\iota_{H'}(1, w_{Q_{G'}})\begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ {}^{\mathrm{t}}x & \mathbf{1}_{n+1} & 0 & 0 \\ 0 & 0 & \mathbf{1}_n & -x \\ 0 & 0 & 0 & \mathbf{1}_{n+1}\end{pmatrix}\iota_{H'}\left(\mathbf{1}_n, \begin{pmatrix}\mathbf{1}_{n+1} & \varsigma \\ 0 & \mathbf{1}_{n+1}\end{pmatrix}\right)h'_{Z'}\right)\mathbf{e}_\infty(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma$$

gives $FJ_{\beta',\infty}\left(g_z, g'_{z'}, x, f^k_\infty(s,\mathrm{sgn}^k)\right)$. Because

$$\iota_{H'}(1, w_{Q_{G'}})\begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ {}^{\mathrm{t}}x & \mathbf{1}_{n+1} & 0 & 0 \\ 0 & 0 & \mathbf{1}_n & -x \\ 0 & 0 & 0 & \mathbf{1}_{n+1}\end{pmatrix}\iota_{H'}\left(\mathbf{1}_n, \begin{pmatrix}\mathbf{1}_{n+1} & \varsigma \\ 0 & \mathbf{1}_{n+1}\end{pmatrix}\right)h'_{Z'} = \left[\begin{array}{c|c} * & * \\ \hline \begin{pmatrix} 0 & 0 \\ {}^{\mathrm{t}}x & \mathbf{1}_{n+1}\end{pmatrix}\sqrt{Y'} & \begin{pmatrix} 0 & 0 \\ {}^{\mathrm{t}}x & \mathbf{1}_{n+1}\end{pmatrix}X'{}^{\mathrm{t}}\sqrt{Y'}^{-1} + \begin{pmatrix}\mathbf{1}_n & -x \\ 0 & \varsigma\end{pmatrix}{}^{\mathrm{t}}\sqrt{Y'}^{-1}\end{array}\right],$$

$$\sqrt{-1}\begin{pmatrix} 0 & 0 \\ {}^{\mathrm{t}}x & \mathbf{1}_{n+1}\end{pmatrix}\sqrt{Y'} + \left(\begin{pmatrix} 0 & 0 \\ {}^{\mathrm{t}}x & \mathbf{1}_{n+1}\end{pmatrix}X'{}^{\mathrm{t}}\sqrt{Y'}^{-1} + \begin{pmatrix}\mathbf{1}_n & -x \\ 0 & \varsigma\end{pmatrix}{}^{\mathrm{t}}\sqrt{Y'}^{-1}\right) = \begin{pmatrix}\mathbf{1}_n & 0 \\ {}^{\mathrm{t}}xz + {}^{\mathrm{t}}z_0 & \mathbf{1}_{n+1}\end{pmatrix}\begin{pmatrix}\mathbf{1}_n & -x \\ 0 & {}^{\mathrm{t}}xzx + {}^{\mathrm{t}}z_0x + {}^{\mathrm{t}}xz_0 + z' + \varsigma\end{pmatrix}\sqrt{Y'}^{-1},$$

the integrand in (4.5.2) equals

$$\det\left({}^{\mathrm{t}}xzx + {}^{\mathrm{t}}z_0x + {}^{\mathrm{t}}xz_0 + z' + \varsigma\right)^{-k}\left|\det\left({}^{\mathrm{t}}xzx + {}^{\mathrm{t}}z_0x + {}^{\mathrm{t}}xz_0 + z' + \varsigma\right)\right|^{k-(s+n+1)}(\det Y')^{\frac{s+n+1}{2}}\mathbf{e}_\infty(-\mathrm{Tr}\beta'\varsigma Z).$$

Writing $z = u + \sqrt{-1}v$, $z' = u' + \sqrt{-1}v'$, $z_0 = u_0 + \sqrt{-1}v_0$ with $u, v \in \mathrm{Sym}(n,\mathbb{R})$, $u', v' \in \mathrm{Sym}(n+1,\mathbb{R})$, $u_0, v_0 \in M_{n,n+1}(\mathbb{R})$, we have

$$\begin{aligned}(4.5.2) = {} & (\det Y')^{\frac{s+n+1}{2}}\int_{\mathrm{Sym}_{n+1}(\mathbb{R})}\det\left(\sqrt{-1}({}^{\mathrm{t}}xvx + {}^{\mathrm{t}}v_0x + {}^{\mathrm{t}}xv_0 + v') + ({}^{\mathrm{t}}xux + {}^{\mathrm{t}}u_0x + {}^{\mathrm{t}}xu_0 + u' + \varsigma)\right)^{-k} \\ & \times\left|\det\left(\sqrt{-1}({}^{\mathrm{t}}xvx + {}^{\mathrm{t}}v_0x + {}^{\mathrm{t}}xv_0 + v') + ({}^{\mathrm{t}}xux + {}^{\mathrm{t}}u_0x + {}^{\mathrm{t}}xu_0 + u' + \varsigma)\right)\right|^{k-(s+n+1)}\mathbf{e}_\infty(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma \\ = {} & (\det Y')^{\frac{s+n+1}{2}}\mathbf{e}_\infty\left(\mathrm{Tr}\beta'({}^{\mathrm{t}}xux + {}^{\mathrm{t}}u_0x + {}^{\mathrm{t}}xu_0 + u')\right)\int_{\mathrm{Sym}(n+1,\mathbb{R})}\det\left(\sqrt{-1}({}^{\mathrm{t}}xvx + {}^{\mathrm{t}}v_0x + {}^{\mathrm{t}}xv_0 + v') + \varsigma\right)^{-k} \\ & \times\left|\det\left(-\sqrt{-1}({}^{\mathrm{t}}xvx + {}^{\mathrm{t}}v_0x + {}^{\mathrm{t}}xv_0 + v') + \varsigma\right)\right|^{k-(s+n+1)}\mathbf{e}_\infty(-\mathrm{Tr}\beta'\varsigma)\,d\varsigma.\end{aligned}$$

By [Shi82, (4.34K)(4.35K)],

$$\begin{aligned}(4.5.2)|_{s=k-n-1} = {} & (\det Y')^{\frac{s+n+1}{2}}\mathbf{e}_\infty\left(\mathrm{Tr}\beta'({}^{\mathrm{t}}xux + {}^{\mathrm{t}}u_0x + {}^{\mathrm{t}}xu_0 + u')\right) \\ & \times\xi_{n+1}\left({}^{\mathrm{t}}xvx + {}^{\mathrm{t}}v_0x + {}^{\mathrm{t}}xv_0 + v', \beta'; \frac{k}{2} + \frac{s+n+1}{2}, -\frac{k}{2} + \frac{s+n+1}{2}\right)\Big|_{k-n-1} \\ = {} & \frac{\sqrt{-1}^{(n+1)k}2^{-\frac{(n+1)n}{2}}\pi^{(n+1)k}}{\Gamma_{n+1}(k)}\cdot(\det 2\beta')^{k-\frac{n+2}{2}}\cdot(\det Y')^{\frac{k}{2}}\mathbf{e}_\infty\left(\mathrm{Tr}\beta'({}^{\mathrm{t}}xzx + {}^{\mathrm{t}}z_0x + {}^{\mathrm{t}}xz_0 + z')\right).\end{aligned}$$

Combining with the formulas (3.5.6)(3.5.7) for the intertwining operator acting on $f^k_\infty(s, \mathrm{sgn}^k)$, we get the desired identity.

$\square$

4.6. **Places dividing N.** Assume $v \mid N$. By (3.5.11), for $\mathfrak{a}_v \in \mathrm{GL}(n+1, \mathbb{Q}_v)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Q}_v)$ and $\beta' \in \mathrm{Sym}(n+1, \mathbb{Q})$, we have

$$FJ_{\beta',v}\left(g, m(\mathfrak{a}_v), x, M_{Q_{H'},v}(s, \eta\chi) f_{\underline{\mathrm{I}},\kappa,v}(s)\right) = FJ_{\beta',v}\left(g, m(\mathfrak{a}_v), x, f^{\alpha_v^{\mathrm{vol}}}(-s, \eta^{-1}\chi^{-1})\right).$$

We compute the right hand side for "big cell" sections with more general Schwartz function α_v.

Before stating the result, we introduce the following operator acting on smooth functions on $G(\mathbb{Q}_v)$,

$$U_{Q_G,v} = \int_{\mathrm{Sym}(n,\mathbb{Z}_v)} R\left(\begin{pmatrix} \mathbf{1}_n & \sigma \\ 0 & \mathbf{1}_n \end{pmatrix}\begin{pmatrix} q_v \mathbf{1}_n & 0 \\ 0 & q_v^{-1}\mathbf{1}_n \end{pmatrix}\right) d\sigma. \tag{4.6.1}$$

This operator resembles the $\mathbb{U}_p$-operator $U_{p,(1,1,\dots,1)}$, but there is no normalization factor here and it is purely local.

For the metaplectic group $\widetilde{G}$, we can also associate to a Schwartz function $\alpha_{n,v}$ on $\mathrm{Sym}(n, \mathbb{Q}_v)$ a "big cell" section $f^{\alpha_{n,v}}_{\widetilde{G},v}(s, \xi\widetilde{\lambda}_{2\beta'})$ inside the degenerate principal series $I_{Q_{\widetilde{G}},v}(s, \xi\widetilde{\lambda}_{2\beta',v})$ as

$$f^{\alpha_{n,v}}_{\widetilde{G},v}\left(s, \xi\widetilde{\lambda}^{-1}_{2\beta'}\right)\left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \epsilon\right) = \xi_v^{-1}(\det c)\widetilde{\lambda}_{2\beta',v}(\det c^{-1}, \epsilon)^{-1} \cdot \alpha_{n,v}(c^{-1}d). \tag{4.6.2}$$

Proposition 4.6.1. *Suppose that the Schwartz function α_v on $\mathrm{Sym}(2n+1, \mathbb{Q}_v)$ can be written as $\alpha_v = \begin{pmatrix} \alpha_{v,\text{up-left}} & \alpha_{v,\text{off-diag}} \\ {}^{\mathrm{t}}\alpha_{v,\text{off-diag}} & \alpha_{v,\text{low-right}} \end{pmatrix}$ (where the notation is as in §2.3.3) and $\alpha_{v,\text{up-left}} = \mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}$. Then for $r \gg 0$,*

$$\begin{aligned} FJ_{\beta',v}\left(g, m(\mathfrak{a}_v), x, f^{\alpha_v}(s, \xi)\right) = & \,\xi_v^{-1}(\det \mathfrak{a}_v)|\det \mathfrak{a}_v|_v^{-s+n+1}\, \widehat{\alpha}_{v,\text{low-right}}({}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v)\, \gamma(\mathbf{e}_v \circ 2\beta')^{-n} \\ & \times \xi_v(q_v)^{rn}|q_v|_v^{rn(s+n+1)}\, U^r_{Q_G,v} \cdot S\left(g, x; f^{\alpha_{v,\text{up-left}}}_{\widetilde{G}}\left(s, \xi_v\widetilde{\lambda}^{-1}_{2\beta'}\right), R(-q_v^{-r} \cdot 2\beta'\mathfrak{a}_v)\, \widehat{\alpha}_{v,\text{off-diag}}\right) \end{aligned}$$

Remark 4.6.2. The formula here shows that in general $FJ_{\beta',v}$ is not a simple product of a section on $\widetilde{G}(\mathbb{Q}_v)$ for Siegel Eisenstein series and a Schwartz function acted on by $\widetilde{G}(\mathbb{Q}_v)$ via Weil representation, but a linear combination of such products.

Proof.

$$\begin{aligned} FJ_{\beta',v}\left(g, m(\mathfrak{a}_v), x, f^{\alpha_v}(s, \xi)\right) &= \int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} f^{\alpha_v}(s, \xi)\left(\begin{pmatrix} a & 0 & b & 0 \\ 0 & 0 & 0 & -{}^{\mathrm{t}}\mathfrak{a}_v^{-1} \\ c & 0 & d & -x{}^{\mathrm{t}}\mathfrak{a}_v^{-1} \\ {}^{\mathrm{t}}xa & \mathfrak{a}_v & {}^{\mathrm{t}}xb & \varsigma{}^{\mathrm{t}}\mathfrak{a}_v^{-1} \end{pmatrix}\right) \mathbf{e}_v\left(-\mathrm{Tr}\beta'\varsigma\right) d\varsigma \\ &= \xi_v^{-1}(\det \mathfrak{a}_v \det c)|\det \mathfrak{a}_v \det c|_v^{-(s+n+1)} \int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)} \alpha_v\left(\begin{pmatrix} c^{-1}d & -c^{-1}x{}^{\mathrm{t}}\mathfrak{a}_v^{-1} \\ -\mathfrak{a}_v^{-1}{}^{\mathrm{t}}x{}^{\mathrm{t}}c^{-1} & \mathfrak{a}_v^{-1}{}^{\mathrm{t}}xac^{-1}x{}^{\mathrm{t}}\mathfrak{a}_v^{-1} + \mathfrak{a}_v^{-1}\varsigma{}^{\mathrm{t}}\mathfrak{a}_v^{-1} \end{pmatrix}\right) \mathbf{e}_v\left(-\mathrm{Tr}\beta'\varsigma\right) d\varsigma \\ &= \xi_v^{-1}(\det \mathfrak{a}_v)|\det \mathfrak{a}_v|_v^{-s+1}\, \widehat{\alpha}_{v,\text{low-right}}({}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v) \cdot \xi_v\widetilde{\lambda}_{2\beta',v}(\det c^{-1}, 1)^{-1}|\det c|_v^{-s-\frac{n+1}{2}}\, \alpha_{v,\text{up-left}}(c^{-1}d) \\ &\quad \times \widetilde{\lambda}_{2\beta',v}(\det c^{-1}, 1)|\det c|_v^{-\frac{n+1}{2}}\, \mathbf{e}_v\left(\mathrm{Tr}\, ac^{-1}x\beta'{}^{\mathrm{t}}x\right) \alpha^{\mathrm{vol}}_{v,\text{off-diag}}(c^{-1}x{}^{\mathrm{t}}\mathfrak{a}_v^{-1}) \end{aligned}$$

Since $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n & -c^{-1}d \\ 0 & \mathbf{1}_n \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} \begin{pmatrix} {}^{\mathrm{t}}c^{-1} & a \\ 0 & c \end{pmatrix}$, we have

$$\begin{aligned} &\widetilde{\lambda}_{2\beta',v}(\det c^{-1}, 1)|\det c|_v^{-\frac{n+1}{2}} \mathbf{e}_v\left(\operatorname{Tr} ac^{-1}x\beta'{}^{\mathrm{t}}x\right) \alpha_{v,\text{off-diag}}(c^{-1}x{}^{\mathrm{t}}\mathfrak{a}_v^{-1}) \\ &= \omega_{2\beta'}\left((g,1)\right) \omega_{2\beta'}\left(\left(\left(\begin{smallmatrix} \mathbf{1}_n & -c^{-1}d \\ 0 & \mathbf{1}_n \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right), 1\right)\right) R({}^{\mathrm{t}}\mathfrak{a}_v^{-1})\, \alpha_{v,\text{off-diag}}(x) \\ &= \gamma(\mathbf{e}_v \circ 2\beta')^{-n} |\det \mathfrak{a}_v|_v^n \cdot \omega_{2\beta'}\left(g\left(\begin{smallmatrix} \mathbf{1}_n & -c^{-1}d \\ 0 & \mathbf{1}_n \end{smallmatrix}\right), 1\right) R(-2\beta'\mathfrak{a}_v)\, \widehat{\alpha}_{v,\text{off-diag}}(x). \end{aligned}$$

Plugging this into the above equation,

(4.6.3)
$$\begin{aligned} FJ_{\beta',v}\left(g, m(\mathfrak{a}_v), x, f^{\alpha_v}(s,\xi)\right) &= \xi_v^{-1}(\det \mathfrak{a}_v)|\det \mathfrak{a}_v|_v^{-s+n+1}\widehat{\alpha}_{v,\text{low-right}}({}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v) \cdot \gamma(\mathbf{e}_v \circ 2\beta')^{-n} \\ &\times f_{\widetilde{G}}^{\alpha_{v,\text{up-left}}}\left(s, \xi_v\widetilde{\lambda}_{2\beta',v}^{-1}\right)(g,1) \cdot \omega_{2\beta'}\left(g\left(\begin{smallmatrix} \mathbf{1}_n & -c^{-1}d \\ 0 & \mathbf{1}_n \end{smallmatrix}\right), 1\right) R(-2\beta'\mathfrak{a}_v)\, \widehat{\alpha}_{v,\text{off-diag}}(x). \end{aligned}$$

If we assume conditions on $\mathfrak{a}_v$ and β' such that ${}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v$, $c^{-1}d$, $x \cdot 2\beta'\mathfrak{a}_v$ belong to the support of $\widehat{\alpha}_{v,\text{low-right}}^{\text{vol}}$, $\alpha_{v,\text{up-left}}^{\text{vol}}$, $\widehat{\alpha}_{v,\text{off-diag}}^{\text{vol}}$ only if $c^{-1}d \cdot x\beta'{}^{\mathrm{t}}x \in \mathrm{Sym}(n,\mathbb{Z}_v)^*$, then we can simply get rid of the term $\begin{pmatrix} \mathbf{1}_n & -c^{-1}d \\ 0 & \mathbf{1}_n \end{pmatrix}$ in the above equation and obtain a nice formula for $FJ_{\beta',v}\left(g, m(\mathfrak{a}_v), x, f^{\alpha_v}(s,\xi)\right)$ without introducing the operator $U_{Q_G,v}$. However, such such conditions on $\mathfrak{a}_v$ and β' can be inconvenient for potential applications, so we need to do a little bit more work.

Recall that the proposition assumes that $\alpha_{v,\text{up-left}} = \mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}$. When $\beta', \mathfrak{a}_v, \alpha_{v,\text{off-diag}}$ are fixed, for sufficiently large r, the action of

$$\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}(c^{-1}d) \cdot \omega_{2\beta'}\left(g\left(\begin{smallmatrix} \mathbf{1}_n & -c^{-1}d \\ 0 & \mathbf{1}_n \end{smallmatrix}\right), 1\right) = \sum_{\sigma \in \mathrm{Sym}(n,\mathbb{Z}/q_v^{2r})} \mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}\left(\frac{c^{-1}d + \sigma}{q_v^{2r}}\right) \cdot \omega_{2\beta'}\left(g\left(\begin{smallmatrix} \mathbf{1}_n & -c^{-1}d \\ 0 & \mathbf{1}_n \end{smallmatrix}\right), 1\right)$$

on $R(-2\beta'\mathfrak{a}_v)\, \widehat{\alpha}_{v,\text{off-diag}}$ is the same as that of

$$\sum_{\sigma \in \mathrm{Sym}(n,\mathbb{Z}/q_v^{2r})} \mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}\left(\frac{c^{-1}d + \sigma}{q_v^{2r}}\right) \cdot \omega_{2\beta'}\left(g\left(\begin{smallmatrix} \mathbf{1}_n & \sigma \\ 0 & \mathbf{1}_n \end{smallmatrix}\right), 1\right).$$

Therefore,

$$\begin{aligned} (4.6.3) &= \xi_v^{-1}(\det \mathfrak{a}_v)|\det \mathfrak{a}_v|_v^{-s+n+1}\widehat{\alpha}_{v,\text{low-right}}({}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v) \cdot \gamma(\mathbf{e}_v \circ 2\beta')^{-n} \\ &\quad\times \xi_v(q_v)^{rn}|q_v|_v^{rns} \sum_{\sigma \in \mathrm{Sym}(n,\mathbb{Z}/q_v^{2r})} f_{\widetilde{G}}^{\alpha_{v,\text{up-left}}}\left(s, \xi_v\widetilde{\lambda}_{2\beta',v}^{-1}\right)\left(g\left(\begin{smallmatrix} \mathbf{1}_n & \sigma \\ 0 & \mathbf{1}_n \end{smallmatrix}\right)\left(\begin{smallmatrix} q_v^r\mathbf{1}_n & 0 \\ 0 & q_v^{-r}\mathbf{1}_n \end{smallmatrix}\right), 1\right) \\ &\quad\times \omega_{2\beta'}\left(g\left(\begin{smallmatrix} \mathbf{1}_n & \sigma \\ 0 & \mathbf{1}_n \end{smallmatrix}\right)\left(\begin{smallmatrix} q_v^r\mathbf{1}_n & 0 \\ 0 & q_v^{-r}\mathbf{1}_n \end{smallmatrix}\right), 1\right) R(-2q_v^{-r} \cdot \beta'\mathfrak{a}_v)\widehat{\alpha}_{v,\text{off-diag}}(x) \\ &= \xi_v^{-1}(\det \mathfrak{a}_v)|\det \mathfrak{a}_v|_v^{-s+n+1}\widehat{\alpha}_{v,\text{low-right}}({}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v) \cdot \gamma(\mathbf{e}_v \circ 2\beta')^{-n} \\ &\quad\times \xi_v(q_v)^{rn}|q_v|_v^{rn(s+n+1)}\, U_{G,v}^r \cdot S\left(g, x; f_{\widetilde{G}}^{\alpha_{v,\text{up-left}}}\left(s, \xi_v\widetilde{\lambda}_{2\beta',v}^{-1}\right), R(-q_v^{-r} \cdot 2\beta'\mathfrak{a}_v)\widehat{\alpha}_{v,\text{off-diag}}\right). \end{aligned}$$

□

4.7. **The place p.** Like in the previous section, we work with a general "big cell" section $f^{\alpha_v}(s,\xi) \in I_{Q_{H'},v}(s,\xi)$ with $\alpha_{v,\text{up-left}} = \mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}$.

Proposition 4.7.1. *Suppose that $\alpha_{v,\text{up-left}} = \mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}$. Then for $\mathfrak{a}_v \in \mathrm{GL}(n+1,\mathbb{Q}_v)$ and $r \gg 0$,*

$$\begin{aligned} &FJ_{\beta',v}\left(g, m(\mathfrak{a}_v), x, M_{Q_{H'}}(s,\xi)f^{\alpha_v}(s,\xi)\right) \\ &= \xi_v(\det \mathfrak{a}_v)^{-1}|\det \mathfrak{a}_v|_v^{-s+n+1}\widehat{\alpha}_{v,\text{low-right}}({}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v) \cdot c_v\left(s - \frac{n}{2}, \xi, \beta'\right) \cdot \lambda_{2\beta',v}(-1)^{n+1} \\ &\times \xi_v(q_v)^{rn}|q_v|_v^{rn(s+n+1)}\, U_{Q_G,v}^r \cdot S\left(g, x; M_{Q_{\widetilde{G}}}(s, \xi_v\widetilde{\lambda}_{2\beta'})f_{\widetilde{G}}^{\alpha_{v,\text{up-left}}}(s, \xi_v\widetilde{\lambda}_{2\beta',v}), R(-q_v^{-r} \cdot 2\beta'\mathfrak{a}_v)\widehat{\alpha}_{v,\text{off-diag}}\right), \end{aligned}$$

where the factor $c_v\left(s-\frac{n}{2},\xi,\beta'\right)$ *is given as*

(4.7.1)

$$c_v\left(s-\frac{n}{2},\xi,\beta'\right)=\gamma(\det 2\beta',\mathbf{e}_v)^n\begin{cases}h_{\mathbb{Q}_v}(2\beta')\gamma_{\mathbb{Q}_v}(\mathbf{e}_v)^{\frac{n^2+2n}{2}}, & n\ \textit{even}\\ \gamma_{\mathbb{Q}_v}(\mathbf{e}_v)^{\frac{(n+1)^2}{2}}\cdot\gamma_v\left(s-\frac{n-1}{2},\xi\lambda_{2\beta'}\right), & n\ \textit{odd}\end{cases}$$

$$\times\,\xi_v(\det 2\beta')^{-1}|\det 2\beta'|_v^{-s+\frac{n}{2}}\,\gamma_v(s-n,\xi)^{-1}\prod_{j=1}^{\left[\frac{n+1}{2}\right]}\gamma_v\left(2s-2n-1+2j,\xi^2\right)^{-1}.$$

Proof. First it is easy to check that for $\mathfrak{a}_v\in\mathrm{GL}(n+1,\mathbb{Q}_v)$,

$$FJ_{\beta',v}\left(g,m(\mathfrak{a}_v),x,M_{Q_{H'}}(s,\xi)f^{\alpha_v}(s,\xi)\right)=\xi_v(\det\mathfrak{a}_v)|\det\mathfrak{a}_v|_v^{s+1}\,FJ_{{}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v,v}\left(g,\mathbf{1}_{2n+2},x\,{}^{\mathrm{t}}\mathfrak{a}_v^{-1},M_{Q_{H'}}(s,\xi)f^{\alpha_v}(s,\xi)\right),$$

so it suffices to prove the identity for $\mathfrak{a}_v=\mathbf{1}_{n+1}$. Let

$$\begin{pmatrix}\mathfrak{A}(g,x,\tau,\varsigma) & \mathfrak{B}(g,x,\tau,\varsigma)\\ \mathfrak{C}(g,x,\tau,\varsigma) & \mathfrak{D}(g,x,\tau,\varsigma)\end{pmatrix}$$
$$=w_{Q_{H'}}\begin{pmatrix}\mathbf{1}_{2n+1} & \tau\\ 0 & \mathbf{1}_{2n+1}\end{pmatrix}\iota_{H'}(\mathbf{1}_{2n},w_{Q_{G'}})\begin{pmatrix}\mathbf{1}_n & 0 & 0 & 0\\ {}^{\mathrm{t}}x & \mathbf{1}_{n+1} & 0 & 0\\ 0 & 0 & \mathbf{1}_n & -x\\ 0 & 0 & 0 & \mathbf{1}_{n+1}\end{pmatrix}\iota_{H'}\left(g,\begin{pmatrix}\mathbf{1}_{n+1} & \varsigma\\ 0 & \mathbf{1}_{n+1}\end{pmatrix}\right).$$

Then

$$\begin{aligned}&FJ_{\beta',v}\left(g,\mathbf{1}_{2n+2},x,M_{Q_{H'}}(s,\xi)f^{\alpha_v}(s,\xi)\right)\\ &=\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)}\int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)}f^{\alpha_v}(s,\xi)\begin{pmatrix}A(g,x,\tau,\varsigma) & B(g,x,\tau,\varsigma)\\ C(g,x,\tau,\varsigma) & D(g,x,\tau,\varsigma)\end{pmatrix}\mathbf{e}_v\left(-\mathrm{Tr}\beta'\varsigma\right)\,d\tau\,d\varsigma\\ &=\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)}\int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)}\xi_v\left(C(g,x,\tau,\varsigma)\right)^{-1}|C(g,x,\tau,\varsigma)|^{-(s+n+1)}\\ &\qquad\qquad\times\alpha_v\left(C(g,x,\tau,\varsigma)^{-1}D(g,x,\tau,\varsigma)\right)\mathbf{e}_v\left(-\mathrm{Tr}\beta'\varsigma\right)\,d\tau\,d\varsigma.\end{aligned}$$

Write $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $\tau=\begin{array}{cc}\begin{array}{cc} n & n+1\end{array} & \\ \begin{pmatrix}\tau_1 & \tau_2\\ {}^{\mathrm{t}}\tau_2 & \tau_4\end{pmatrix} & \begin{array}{c} n\\ n+1\end{array}\end{array}$. We have

$$C(g,x,\tau,\varsigma)=\begin{pmatrix}a+\tau_1c+\tau_2{}^{\mathrm{t}}xa & \tau_2\\ {}^{\mathrm{t}}\tau_2c+\tau_4{}^{\mathrm{t}}xa & \tau_4\end{pmatrix}=\begin{pmatrix}\mathbf{1}_n & \tau_2\tau_4^{-1}\\ 0 & \mathbf{1}_{n+1}\end{pmatrix}\begin{pmatrix}a+\tau_1'c & 0\\ {}^{\mathrm{t}}\tau_2c & \tau_4\end{pmatrix}\begin{pmatrix}\mathbf{1}_n & 0\\ {}^{\mathrm{t}}xa & \mathbf{1}_{n+1}\end{pmatrix},$$
$$D(g,x,\tau,\varsigma)=\begin{pmatrix}b+\tau_1d+\tau_2{}^{\mathrm{t}}xb & -\tau_1x+\tau_2\varsigma\\ {}^{\mathrm{t}}\tau_2d+\tau_4{}^{\mathrm{t}}xb & -{}^{\mathrm{t}}\tau_2x+\tau_4\varsigma-\mathbf{1}_{n+1}\end{pmatrix}=\begin{pmatrix}\mathbf{1}_n & \tau_2\tau_4^{-1}\\ 0 & \mathbf{1}_{n+1}\end{pmatrix}\begin{pmatrix}b+\tau_1'd & -\tau_1'x+\tau_2\tau_4^{-1}\\ {}^{\mathrm{t}}\tau_2d+\tau_4{}^{\mathrm{t}}xb & -{}^{\mathrm{t}}\tau_2x+\tau_4\varsigma-\mathbf{1}_{n+1}\end{pmatrix},$$

with $\tau_1'=\tau_1-\tau_2\tau_4^{-1}{}^{\mathrm{t}}\tau_2$. Direct computation shows

$$C(g,x,\tau,\varsigma)^{-1}D(g,x,\tau,\varsigma)=\left(\begin{array}{c:c}(a+\tau_1'c)^{-1}(b+\tau_1'd) & (a+\tau_1'c)^{-1}(-\tau_1'x+\tau_2\tau_4^{-1})\\ \hdashline {}^{\mathrm{t}}(-\tau_1'x+\tau_2\tau_4^{-1}){}^{\mathrm{t}}(a+\tau_1'c)^{-1} & \begin{array}{c}-\tau_4^{-1}{}^{\mathrm{t}}\tau_2c(a+\tau_1'c)^{-1}(-\tau_1'x+\tau_2\tau_4^{-1})-\tau_4^{-1}\tau_2x\\ +\varsigma-\tau_4^{-1}-{}^{\mathrm{t}}xa(a+\tau_1'c)^{-1}(-\tau_1'x+\tau_2\tau_4^{-1})\end{array}\end{array}\right)$$
$$=\left(\begin{array}{c:c}(a+\tau_1'c)^{-1}(b+\tau_1'd) & \tau_2'\\ \hdashline {}^{\mathrm{t}}\tau_2' & \begin{array}{c}\varsigma-\tau_4^{-1}-{}^{\mathrm{t}}x\tau_1'x-{}^{\mathrm{t}}\tau_2'{}^{\mathrm{t}}(a+\tau_1'c)c\tau_2'\\ -{}^{\mathrm{t}}\tau_2'{}^{\mathrm{t}}(a+\tau_1'c)x-{}^{\mathrm{t}}x(a+\tau_1'c)\tau_2'\end{array}\end{array}\right)$$

with $\tau_2' = (a+\tau_1'c)^{-1}(-\tau_1'x+\tau_2\tau_4^{-1})$. By change of variable, we get
(4.7.2)

$$
\begin{aligned}
&FJ_{\beta',v}\left(g,\mathbf{1}_{2n+2},x,M_{Q_{H'},v}(s,\xi)f^{\alpha_v}(s,\xi)\right)\\
&=\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)}\int_{\mathrm{Sym}(2n+1,\mathbb{Q}_v)}\xi_v^{-1}(\det(\tau_4(a+\tau_1c)))|\det\tau_4|_v^{-s-1}|\det(a+\tau_1c))|_v^{-s}\alpha_v\begin{pmatrix}(a+\tau_1c)^{-1}(b+\tau_1d) & \tau_2\\ {}^{\mathrm{t}}\tau_2 & \varsigma\end{pmatrix}\\
&\quad\times\mathbf{e}_v\left(-\mathrm{Tr}\beta'(\varsigma+\tau_4^{-1}+{}^{\mathrm{t}}x\tau_1x+{}^{\mathrm{t}}\tau_2{}^{\mathrm{t}}(a+\tau_1c)c\tau_2+{}^{\mathrm{t}}\tau_2{}^{\mathrm{t}}(a+\tau_1c)x+{}^{\mathrm{t}}x(a+\tau_1c)\tau_2)\right)\,d\tau\,d\varsigma
\end{aligned}
$$

Suppose $\alpha_v=\begin{pmatrix}\alpha_{v,\text{up-left}} & \alpha_{v,\text{off-diag}}\\ {}^{\mathrm{t}}\alpha_{v,\text{off-diag}} & \alpha_{v,\text{low-right}}\end{pmatrix}$. Then

$$FJ_{\beta',v}\left(g,\mathbf{1}_{2n+2},x,M_{Q_{H'},v}(s,\xi)f^{\alpha_v}(s,\xi)\right)=\widehat{\alpha}_{v,\text{low-right}}(\beta')\cdot I_1(s,\xi)\cdot I_2(s,\xi_v,\alpha_v). \tag{4.7.3}$$

with

$$
\begin{aligned}
I_1(s,\xi_v,\beta')&=\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)}\xi_v^{-1}(\det\tau_4)|\det\tau_4|_v^{-s-1}\mathbf{e}_v(-\mathrm{Tr}\beta'\tau_4^{-1})\,d\tau_4,\\
I_2(s,\xi_v,\alpha_v,\beta')&=\int_{\mathrm{Sym}(n,\mathbb{Q}_v)}\xi_v^{-1}(\det(a+\tau_1c))|\det(a+\tau_1c)|_v^{-s}\alpha_{v,\text{up-left}}\left((a+\tau_1c)^{-1}(b+\tau_1d)\right)\mathbf{e}_v(-\mathrm{Tr}\beta'{}^{\mathrm{t}}x\tau_1x)\\
&\qquad\times\int_{M_{n,n+1}(\mathbb{Q}_v)}\alpha_{v,\text{off-diag}}(\tau_2)\mathbf{e}_v\left(-\mathrm{Tr}\beta'({}^{\mathrm{t}}\tau_2{}^{\mathrm{t}}(a+\tau_1c)c\tau_2+{}^{\mathrm{t}}\tau_2{}^{\mathrm{t}}(a+\tau_1c)x+{}^{\mathrm{t}}x(a+\tau_1c)\tau_2)\right)\,d\tau_2\,d\tau_1
\end{aligned}
$$

The proof of Proposition 4.7.1 reduces to computing $I_1(s,\xi_v,\beta')$ and $I_2(s,\xi_v,\alpha_v)$.

Proposition 4.7.2.

$$\widehat{\alpha}_{v,\text{low-right}}(\beta')\cdot I_1(s,\xi_v,\beta')=c_v\left(s-\frac{n}{2},\xi,\beta'\right)\cdot\widehat{\alpha}_{v,\text{low-right}}(\beta'),$$

with the factor $c_v\left(s-\frac{n}{2},\xi,\beta'\right)$ given by (4.7.1).

Proof. First observe that

$$
\begin{aligned}
&W_{\beta',v}\left(\mathbf{1}_{2n+2},M_{Q_{G'},v}\left(s-\frac{n}{2},\xi\right)f^{\alpha_{v,\text{low-right}}}_{G',v}\left(s-\frac{n}{2},\xi\right)\right)\\
&=\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)}\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)}f^{\alpha_{v,\text{low-right}}}_{G',v}\left(s-\frac{n}{2},\xi\right)\left(w_{G'}\begin{pmatrix}\mathbf{1}_{n+1}&\tau_4\\0&\mathbf{1}_{n+1}\end{pmatrix}w_{G'}\begin{pmatrix}\mathbf{1}_{n+1}&\sigma\\0&\mathbf{1}_{n+1}\end{pmatrix}\right)\\
&\qquad\qquad\mathbf{e}_v\left(-\mathrm{Tr}\beta'\sigma\right)\,d\tau_4\,d\sigma\\
&=\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)}\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)}\xi_v^{-1}(\det\tau_4)|\det\tau_4|_v^{-s-1}\alpha_{v,\text{low-right}}(\sigma-\tau_4^{-1})\mathbf{e}_v(-\mathrm{Tr}\beta'\sigma)\,d\tau_4\,d\sigma\\
&=\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)}\xi_v^{-1}(\det\tau_4)|\det\tau_4|_v^{-s-1}\mathbf{e}_v\left(-\mathrm{Tr}\beta'\tau_4^{-1}\right)\,d\tau_4\\
&\qquad\int_{\mathrm{Sym}(n+1,\mathbb{Q}_v)}\alpha_{v,\text{low-right}}(\sigma-\tau_4^{-1})\mathbf{e}_v\left(-\mathrm{Tr}\beta'(\sigma-\tau_4^{-1})\right)\,d\sigma\\
&=\widehat{\alpha}_{v,\text{low-right}}(\beta')\cdot I_1(s,\xi_v,\beta').
\end{aligned}
$$

On the other hand, by the functional equation for $W_{\beta',v}$ [LR05, (14)], we have

$$
\begin{aligned}
&W_{\beta',v}\left(\mathbf{1}_{2n+2},M_{Q_{G'},v}\left(s-\frac{n}{2},\xi\right)f^{\alpha_{v,\text{low-right}}}_{G',v}\left(s-\frac{n}{2},\xi\right)\right)\\
&=c_v\left(s-\frac{n}{2},\xi,\beta'\right)\cdot W_{\beta',v}\left(\mathbf{1}_{2n+2},f^{\alpha_{v,\text{low-right}}}_{G',v}\left(s-\frac{n}{2},\xi\right)\right)=c_v\left(s-\frac{n}{2},\xi,\beta'\right)\cdot\widehat{\alpha}_{v,\text{low-right}}(\beta').
\end{aligned}
$$

The formulas for the $c_v(s,\xi,\beta')$ are given in [Swe95, Proposition 4.8]. $\square$

Proposition 4.7.3. *For $r \gg 0$,*

$$I_2(s,\xi_v,\alpha_v,\beta') = \lambda_{2\beta',v}(-1)^{n+1}\cdot \xi_v(q_v)^{rn}|q_v|_v^{rn(s+n+1)}$$
$$\times U^r_{Q_G,v}\cdot S\left(g,x;M_{Q_{\widetilde{G}}}(s,\xi_v\widetilde{\lambda}_{2\beta',v})f_{\widetilde{G}}^{\alpha_{v,\text{up-left}}}(s,\xi_v\widetilde{\lambda}_{2\beta',v}),R(-q_v^{-r}\cdot 2\beta')\,\widehat{\alpha}_{v,\text{off-diag}}\right)$$

Proof. We can write the interior integral of $I_2(s,\xi_v,\alpha_v,\beta')$ as

$$\int_{M_{n,n+1}(\mathbb{Q}_v)}\alpha_{v,\text{off-diag}}(\tau_2)\,\mathbf{e}_v\left(-\mathrm{Tr}\beta'({}^{\mathrm{t}}\tau_2{}^{\mathrm{t}}(a+\tau_1c)c\tau_2+{}^{\mathrm{t}}\tau_2{}^{\mathrm{t}}(a+\tau_1c)x+{}^{\mathrm{t}}x(a+\tau_1c)\tau_2)\right)\,d\tau_2$$
$$=\widetilde{\lambda}_{2\beta',v}(\det(-a-\tau_1c),1)^{-1}|\det(a+\tau_1c)|^{-\frac{n+1}{2}}\gamma(\mathbf{e}_v\circ 2\beta')^{n+1}\omega_{2\beta''}\left(\begin{pmatrix}0 & -(a+\tau_1c)\\ {}^{\mathrm{t}}(a+\tau_1c)^{-1} & -c\end{pmatrix},1\right)\alpha_{v,\text{off-diag}}(x).$$

Plugging it into the expression of $I_2(s,\xi,\alpha_v,\beta')$, we get

$$I_2(s,\xi,\alpha_v,\beta')=\gamma(\mathbf{e}_v\circ 2\beta')^{n+1}\int_{\mathrm{Sym}(n,\mathbb{Q}_v)}\widetilde{\lambda}_{2\beta',v}(\det(-a-\tau_1c,1))^{-1}\xi_v^{-1}(\det(a+\tau_1c))|\det(a+\tau_1c)|^{-(s+\frac{n+1}{2})}$$
$$\times\alpha_{v,\text{up-left}}\left((a+\tau_1c)^{-1}(b+\tau_1d)\right)\cdot\omega_{2\beta'}\left(\begin{pmatrix}\mathbf{1}_n & -\tau_1\\ 0 & \mathbf{1}_n\end{pmatrix}\begin{pmatrix}0 & -(a+\tau_1c)\\ {}^{\mathrm{t}}(a+\tau_1c)^{-1} & -c\end{pmatrix},1\right)\alpha_{v,\text{off-diag}}(x)\,d\tau_1.$$

Then since

$$\begin{pmatrix}\mathbf{1}_n & -\tau_1\\ 0 & \mathbf{1}_n\end{pmatrix}\begin{pmatrix}0 & -(a+\tau_1c)\\ {}^{\mathrm{t}}(a+\tau_1c)^{-1} & -c\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\mathbf{1}_n & -(a+\tau_1c)^{-1}(b+\tau_1d)\\ 0 & \mathbf{1}_n\end{pmatrix}\begin{pmatrix}0 & -\mathbf{1}_n\\ \mathbf{1}_n & 0\end{pmatrix},$$

we get

$$I_2(s,\xi,\alpha_v,\beta')=\lambda_{2\beta',v}(-1)^{n+1}\int_{\mathrm{Sym}(n,\mathbb{Q}_v)}\widetilde{\lambda}_{2\beta',v}(\det(a+\tau_1c),1)^{-1}\xi_v^{-1}(\det(a+\tau_1c))|\det(a+\tau_1c)|^{-(s+\frac{n+1}{2})}$$
$$\times\alpha_{v,\text{up-left}}\left((a+\tau_1c)^{-1}(b+\tau_1d)\right)\cdot\omega_{2\beta'}\left(g\begin{pmatrix}\mathbf{1}_n & -(a+\tau_1c)^{-1}(b+\tau_1d)\\ 0 & \mathbf{1}_n\end{pmatrix},1\right)R(2\beta')\,\widehat{\alpha}_{v,\text{off-diag}}(x)\,d\tau_1.$$

By a similar manipulation as in Proposition 4.6.1 (recall that $\alpha_{v,\text{up-left}}=\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}$ is still assumed), for $r\gg 0$ we get

$$I_2(s,\xi,\alpha_v,\beta')$$
$$=\lambda_{2\beta',v}(-1)^{n+1}\xi_v(q_v)^{-rn}|q_v|_v^{-rns}\sum_{\sigma\in\mathrm{Sym}(n,\mathbb{Z}/q_v^{2r})}\int_{\mathrm{Sym}(n,\mathbb{Q}_v)}f_{\widetilde{G}}^{\alpha_{v,\text{up-left}}}\left(s,\xi_v\widetilde{\lambda}_{2\beta',v}\right)\left(\begin{pmatrix}0 & -\mathbf{1}_n\\ \mathbf{1}_n & \tau_1\end{pmatrix}g\begin{pmatrix}q_v^r\mathbf{1}_n & q_v^{-r}\mathbf{1}_n\sigma\\ 0 & q_v^{-r}\end{pmatrix}\right)$$
$$\times\,\omega_{2\beta'}\left(g\begin{pmatrix}q_v^r\mathbf{1}_n & q_v^{-r}\sigma\\ 0 & q_v^{-r}\mathbf{1}_n\end{pmatrix},1\right)R(-q_v^{-r}\cdot 2\beta')\,\widehat{\alpha}_{v,\text{off-diag}}(x)\,d\tau_1,$$

and the proposition follows. □

Finally, combining (4.7.3) with Propositions 4.7.2, 4.7.3 proves Proposition (4.7.1). □

4.8. **Summary.** We summarize the computation in the last few sections on the non-degenerate Fourier coefficients of the specializations of the family $\mathcal{E}_{ord}$ on $G\times G'$ constructed in Proposition 2.6.1.

Theorem 4.8.1. *Assume that* ${}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v \in \mathrm{Sym}(n+1,\mathbb{Z}_v)^*$ *and* $2{}^{\mathrm{t}}\mathfrak{a}\beta'\mathfrak{a} \in \mathrm{GL}(n+1,\mathbb{Z}_v)$ *for all* $v \nmid Np\infty$.

$$\mathcal{E}_{\mathrm{ord}}(\underline{\tau},\kappa)_{\beta'}\left(\cdot, m(\mathfrak{a}_f)g'_{z'}\right) = \frac{\sqrt{-1}^{-(3n-1)k}2^{-\frac{n(n+1)}{2}}\pi^{(n+1)k}}{\Gamma_{n+1}(k)}\eta\chi(\det\mathfrak{a}_f)^{-1}|\det\mathfrak{a}_f|_{\mathbb{A}_f}^{k}\prod_{v|N}|\det\mathfrak{a}_v|_v^{n+1}$$
$$\times\prod_{v\nmid p\infty}\eta_v\chi_v(\det 2{}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v)|\det 2{}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v|_v^{-k+\frac{n+2}{2}}\prod_{v|Np}\widehat{\alpha}_{\underline{\tau},\kappa,v,\text{low-right}}({}^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v)\cdot\left(\mathrm{Im}z'\right)^{\frac{k}{2}}\mathbf{e}_\infty\left(\mathrm{Tr}\beta'z'\right)$$
$$\times\left(c_{k,\eta\chi}\prod_{v|Np}U_{Q_G,v}\right)^r\cdot\left(\mathfrak{A}(s,\eta\chi,2\beta')\,E^{\mathrm{Si}}\left(\cdot,f_{\widetilde{G},\underline{\tau},\kappa}(-s)\right)\theta_{2\beta'}\left(\cdot,\phi^r_{2\beta',\mathfrak{a},\underline{\tau},\kappa}\right)\right)\Big|_{s=n+1-k}.$$

Here

$$c_{k,\eta\chi}=\eta_p\chi_p(p)^n\,p^{n(k-2n-2)}\prod_{v|N}\eta_v\chi_v(q_v)^{-n}|q_v|_v^{nk},$$

$$\mathfrak{A}(s,\eta\chi,2\beta')=\prod_{v|N\infty}\left(\gamma_v(s-n,\eta\chi)\prod_{j=1}^{\left[\frac{n+1}{2}\right]}\gamma_v(2s-2n-1+2j,\eta^2\chi^2)\right)\prod_{j=\left[\frac{n+1}{2}\right]}^{n}L^{Np\infty}(2s-2n-1+2j,\eta^2\chi^2)$$
$$\times\begin{cases}h_{\mathbb{Q}_p}(2\beta') & n\ \textit{even}\\ L^{Np\infty}\left(-s+\frac{n+1}{2},\eta^{-1}\chi^{-1}\lambda_{2\beta'}\right)\gamma_p\left(s-\frac{n-1}{2},\eta\chi\lambda_{2\beta'}\right) & n\ \textit{odd}\end{cases}.$$

The section $f_{\widetilde{G},\underline{\tau},\kappa}(-s)\in I_{Q_G}\left(-s,\eta^{-1}\chi^{-1}\widetilde{\lambda}_{2\beta'}^{-1}\right)$ *is the factorizable with local factors given as*

$$f_{\widetilde{G},\underline{\tau},\kappa,v}(-s)=\begin{cases}f^{\mathrm{ur}}_{\widetilde{G},v}(-s,\eta^{-1}\chi^{-1}\widetilde{\lambda}_{2\beta'}^{-1}), & \textit{if } v\nmid Np\infty,\\ f^{k-\frac{n+1}{2}}_{\widetilde{G},\infty}(-s,\widetilde{\mathrm{sgn}}^{k-\frac{n+1}{2}}), & \textit{if } v=\infty,\\ f^{\alpha_{\underline{\tau},\kappa,v,\text{up-left}}}_{\widetilde{G},v}(-s,\eta^{-1}\chi^{-1}\widetilde{\lambda}_{2\beta'}^{-1}), & \textit{if } v|N,\\ M_{Q_{\widetilde{G}}}(s,\eta\chi\widetilde{\lambda}_{2\beta'})\,f^{\alpha_{\underline{\tau},\kappa,v,\text{up-left}}}_{\widetilde{G},v}(s,\eta\chi\widetilde{\lambda}_{2\beta'}), & \textit{if } v=p,\end{cases},$$

and the Schwartz function $\phi^r_{2\beta',\mathfrak{a}_v,\underline{\tau},\kappa}\in\mathcal{S}(M_{n,n+1}(\mathbb{A},\mathbb{C})$ *for the theta series is the product of the local ones given as*

$$\phi^r_{2\beta',\mathfrak{a},\underline{\tau},\kappa,v}=\begin{cases}R({}^{\mathrm{t}}\mathfrak{a}_v^{-1})\mathbb{1}_{M_{n,n+1}(\mathbb{Z}_v)}, & \textit{if } v\nmid Np\infty\\ G_{2\beta',\infty}, & \textit{if } v=\infty\\ R(-q_v^r\cdot 2\beta'\mathfrak{a}_v)\,\widehat{\alpha}_{\underline{\tau},\kappa,v,\text{off-diag}}, & \textit{if } v|N\\ R(-q_v^r\cdot 2\beta'\mathfrak{a}_v)\,\widehat{\alpha}_{\underline{\tau},\kappa,v,\text{off-diag}}, & \textit{if } v=p.\end{cases}$$

See (4.5.1) *(resp.* (4.6.2)*) for the definition of the canonical archimedean section* $f^{k-\frac{n+1}{2}}_{\widetilde{G},\infty}(s,\widetilde{\mathrm{sgn}}^{k-\frac{n+1}{2}})$ *and the Gaussian function* $G_{2\beta',\infty}$ *(resp. the "big cell" sections on* $\widetilde{G}$*). The local operator* $U_{Q_G,v}$ *is defined in* (4.6.1).

4.9. **Expressing $FJ_{\beta',v}$ by using Siegel–Weil sections.** Suppose that η is the quadratic character $\eta_v=(\cdot,d_\eta)_v$ for some $d_\eta\in\mathbb{Q}^\times$, and that the finite part χ (resp. the algebraic part k) of the arithmetic κ is trivial (resp. has the parity $(-1)^k=\mathrm{sgn}\,d_\eta$). Then inside the degenerate principal series $I_{Q_{\widetilde{G}},v}\left(k-n-1,\eta^{-1}\widetilde{\lambda}_{2\beta'}^{-1}\right)$ there are a special type of sections called the Siegel–Weil sections, which come from the theta lift of the trivial representation on $\mathrm{O}(2\beta_*)(\mathbb{Q}_v)$ for a quadratic form $2\beta_*$ of dimension $2k-n-1$ and determinant $(-1)^k(d_\eta\det 2\beta')^{-1}$. They are sections inside the image of the map

$$\begin{aligned} \mathcal{S}(M_{n,2k-n-1}(\mathbb{Q}_v),\mathbb{C}) &\longrightarrow I_{Q_{\widetilde{G}},v}(k-n-1,\widetilde{\lambda}_{2\beta_*}) \\ \phi &\longmapsto \big((g,\epsilon)\mapsto \omega_{2\beta_*}(g,\epsilon)\cdot\phi(0)\big). \end{aligned} \tag{4.9.1}$$

We denote the image as $R_{\widetilde{G},v}(2\beta_*)$.

In this section, we discuss expressing $FJ_{\beta',v}(g,m(\mathfrak{a}_v),x,f_{\underline{\tau},\kappa,v})$ as

$$\omega_{\begin{pmatrix} 2\beta' & 0 \\ 0 & 2\beta_* \end{pmatrix}}(g)\cdot\phi_{\beta',\beta_*,\underline{\tau},\kappa,v}(x,0) \tag{4.9.2}$$

with suitable β_* and $\phi_{\beta',\beta_*,\underline{\tau},\kappa,v}\in\mathcal{S}(M_{n,2k}(\mathbb{Q}_v),\mathbb{C})$. We make a few remarks here:

(1) It is not always possible to express $FJ_{\beta',v}(g,m(\mathfrak{a}_v),x,f_{\underline{\tau},\kappa,v})$ in this way because the Siegel–Weil sections attached to a single $2\beta_*$ do not always span the degenerate principal series.

(2) The map (4.9.1) is not injective. In general the trivial representation is a quotient but not necessarily a natural sub-representation of the Weil representation.

(3) One motivation for trying to express the $FJ_{\beta',v}(g,m(\mathfrak{a}_v),x,f_{\underline{\tau},\kappa,v})$ as (4.9.2) is because it will be useful if one studies the Fourier coefficient via the seesaw diagram

$$\begin{tikzcd}
\text{theta series} & & \text{Siegel Eis series} & \theta(\pi) \\
\widetilde{\mathrm{Sp}}(2n) & \times \arrow[d, dash] & \widetilde{\mathrm{Sp}}(2n) \arrow[dr, dash] & \mathrm{O}\begin{pmatrix} 2\beta' & 0 \\ 0 & 2\beta_* \end{pmatrix} \arrow[d, dash] \arrow[dll, dash] \\
& \mathrm{Sp}(2n) & & \mathrm{O}(2\beta')\times\mathrm{O}(2\beta_*) \\
& \pi & & \text{triv}
\end{tikzcd} . \tag{4.9.3}$$

(4) Another motivation is that it helps deal with the cases excluded from Theorem 4.8.1, *i.e.* vector weights at the archimedean place and the place $v=2$ when n is even.

4.9.1. *Assumptions.* From now on, we only consider β''s satisfying the following conditions:

- At $v\nmid 2Np\infty$, there exists $\mathfrak{a}_v\in\mathrm{GL}(n+1,\mathbb{Q}_v)$ such that
$$^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v\in\mathrm{Sym}(n+1,\mathbb{Z}_v)^*\cap\mathrm{GL}(n+1,\mathbb{Z}_v).$$
- At the place $v=2$, there exists, $\mathfrak{a}_v\in\mathrm{GL}(n+1,\mathbb{Q}_v)$ such that
$$2\,^{\mathrm{t}}\mathfrak{a}_v\beta'\mathfrak{a}_v=\begin{cases} \begin{pmatrix} 0 & \mathbf{1}_{\frac{n+1}{2}} \\ \mathbf{1}_{\frac{n+1}{2}} & 0 \end{pmatrix}, & \text{if } n \text{ is odd,} \\ \begin{pmatrix} 0 & \mathbf{1}_{\frac{n}{2}} & 0 \\ \mathbf{1}_{\frac{n}{2}} & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, & \text{if } n \text{ is even.} \end{cases}$$
- At $v=\infty$, β' is positive definite.

We also fix an $\mathfrak{a}_v$ at each $v\nmid Np\infty$ appearing in the above conditions. (For almost all places we can choose $\mathfrak{a}_v=\mathbf{1}_{n+1}$.)

We assume that the arithmetic character κ, the quadratic character $\eta=(\cdot,d_\eta)$, $d_\eta\in\mathbb{Q}^\times$ and the auxiliary quadratic form $\beta_*\in\mathrm{Sym}(2k-n-1,\mathbb{Q})$ satisfy

$$\det 2\beta_*=(-1)^k(d_\eta\det 2\beta')^{-1}, \qquad 2\beta_*>0,$$
$$d_\eta\in\mathbb{Z}_v^\times,\quad \varepsilon_v(2\beta_*)=1,\quad \text{for all } v\nmid 2Np\infty,\quad (-1)^k d_\eta\in(\mathbb{Q}_2^\times)^2,\quad \varepsilon_2(2\beta_*)=(-1)^{\frac{[k-\frac{n}{2}]([k-\frac{n}{2}]-1)}{2}},$$

where $\varepsilon_v(\cdot)$ denotes the Hasse invariant of a quadratic form over $\mathbb{Q}_v$.

4.9.2. *The unramified places not dividing* 2. From our assumption on β' and β_*, the Weil representations of $\widetilde{G}(\mathbb{Q}_v)$ corresponding to $2\beta'$ and $2\beta_*$ both contain $G(\mathbb{Z}_v)$-fixed vectors, and it is easy to see that we have

Proposition 4.9.1. *For $v \nmid 2Np\infty$, suppose that $\beta', \beta_*, \mathfrak{a}_v, \eta, \kappa$ are given as in §4.9.1. Pick $\mathfrak{s}_v \in \mathrm{GL}(2k-n-1, \mathbb{Q}_v)$ such that ${}^{\mathrm{t}}\mathfrak{s}_v\beta_*\mathfrak{s}_v \in \mathrm{Sym}(2k-n-1,\mathbb{Z}_v)^* \cap \mathrm{GL}(2k-n-1,\mathbb{Z}_v)$. Then*

$$b_{H',v}(s,\eta)\cdot FJ_{\beta',v}\left(g, m(\mathfrak{a}_v), x, M_{Q_{H'},v}(s,\eta) f_{\underline{t},\kappa,v}(s)\right)\big|_{s=n+1-k}$$

is given by Proposition 4.4.1 with $\xi=\eta$, $s=n+1-k$ and

$$S\left(g, x; f^{\mathrm{ur}}_{\widetilde{G},v}(s,\eta\widetilde{\lambda}^{-1}_{2\beta'}), R({}^{\mathrm{t}}\mathfrak{a}_v^{-1})\mathbb{1}_{M_{n,n+1}(\mathbb{Z}_v)}\right)\Big|_{s=n+1-k} = \omega_{\left(\begin{smallmatrix} 2\beta' & 0\\ 0 & 2\beta_*\end{smallmatrix}\right)}(g) R\left(\begin{smallmatrix} {}^{\mathrm{t}}\mathfrak{a}_v^{-1} & 0 \\ 0 & {}^{\mathrm{t}}\mathfrak{s}_v^{-1}\end{smallmatrix}\right)\cdot \mathbb{1}_{M_{n,2k}(\mathbb{Z}_v)}(x,0).$$

4.9.3. *The place* 2. The case when n is odd is the same as the previous subsection.

Proposition 4.9.2. *Suppose that n is odd, $v=2$ and $\beta',\beta_*,\mathfrak{a}_v,\eta,\kappa$ are given as in §4.9.1. Pick $\mathfrak{s}_2 \in \mathrm{GL}(2k-n-1,\mathbb{Q}_v)$ such that $2{}^{\mathrm{t}}\mathfrak{s}_2\beta_*\mathfrak{s}_2 = \begin{pmatrix} 0 & \mathbf{1}_{k-\frac{n-1}{2}} \\ \mathbf{1}_{k-\frac{n-1}{2}} & 0\end{pmatrix}$. Then the same formula in Proposition 4.9.1 holds.*

Now we look at the case when n is even. In this case, the Weil representation of $\widetilde{G}(\mathbb{Q}_v)$ associated to $2\beta'$ or $2\beta_*$ does not contain unramified vector, but the one associated to $\begin{pmatrix} 2\beta' & 0 \\ 0 & 2\beta_*\end{pmatrix}$ does.

Proposition 4.9.3. *Suppose n that is even, $v=2$ and $\beta',\beta_*,\mathfrak{a}_v,\eta,\kappa$ are given as in §4.9.1. Pick $\mathfrak{s}_2 \in \mathrm{GL}(2k-n-1,\mathbb{Q}_v)$ such that $2{}^{\mathrm{t}}\mathfrak{s}_2\beta_*\mathfrak{s}_2 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{k-\frac{n}{2}} \\ 0 & \mathbf{1}_{k-\frac{n}{2}} & 0\end{pmatrix}$. Put $\Upsilon_2 = \begin{pmatrix} \mathbf{1}_n & & \\ & \begin{smallmatrix} 1 & 1 \\ 1 & -1\end{smallmatrix} & \\ & & \mathbf{1}_{2k-n-2}\end{pmatrix}$. Then*

$$\begin{aligned}
&b_{H',2}(s,\eta)\cdot FJ_{\beta',2}\left(g, m(\mathfrak{a}_v), x, M_{Q_{H'},2}(s,\eta) f_{\underline{t},k,2}(s)\right)\big|_{s=n+1-k}\\
=\ &\eta_2(\det\mathfrak{a}_2)|\det\mathfrak{a}_2|^{n+2-k}\cdot\gamma_v(s-n,\eta)^{-1}\prod_{j=1}^{\frac{n}{2}}\gamma_v\left(2s-2n-1+2j,\mathrm{triv}\right)^{-1}\prod_{j=\frac{n}{2}+1}^{n} L_v(2s-2n-1+2j,\mathrm{triv})\\
&\times\omega_{\left(\begin{smallmatrix} 2\beta' & 0\\ 0 & 2\beta_*\end{smallmatrix}\right),2}(g) R\left(\Upsilon_2\left(\begin{smallmatrix} {}^{\mathrm{t}}\mathfrak{a}_2^{-1} & 0 \\ 0 & {}^{\mathrm{t}}\mathfrak{s}_2^{-1}\end{smallmatrix}\right)\right)\cdot\mathbb{1}_{M_{n,2k}(\mathbb{Z}_2)}(x,0)
\end{aligned}$$

Note that when n is even, $FJ_{\beta',2}$ is not a simple product of local sections for Siegel Eisenstein series and theta series, and the Schwartz function $R\left(\Upsilon_2\left(\begin{smallmatrix} {}^{\mathrm{t}}\mathfrak{a}_2^{-1} & 0 \\ 0 & {}^{\mathrm{t}}\mathfrak{s}_2^{-1}\end{smallmatrix}\right)\right)\cdot\mathbb{1}_{M_{n,2k}(\mathbb{Z}_2)}$ is not a simple product of Schwartz functions on $M_{n,n+1}(\mathbb{Z}_2)$ and $M_{n,2k-n-1}(\mathbb{Z}_2)$.

4.9.4. *Pluri-harmonic polynomials and holomorphic differential operators.* Before discussing $FJ_{\beta',\infty}$ evaluated at $s=n+1-k$ for general weight $\underline{t}$, we first introduce the notion of pluri-harmonic polynomials and define a pluri-harmonic polynomial $P_{\underline{t},k}$ on $M_{2n+1,2k}$.

Let $\mathbb{C}[M_{m,l}]$ be the space of polynomials on $m\times l$ matrices. For $1\le i,j\le m$, define the operator Δ_{ij} as

$$\Delta_{ij}P(x) = \sum_{r=0}^{l}\frac{\partial^2}{\partial x_{ir}\partial x_{jr}}P(x), \qquad\qquad P(x)\in\mathbb{C}[M_{m,l}].$$

As in [KV78], a polynomial in $\mathbb{C}[M_{m,l}]$ is called pluri-harmonic if it is annihilated by Δ_{ij} for all $1\le i,j\le m$. The subspace of pluri-harmonic polynomials in $\mathbb{C}[M_{m,l}]$ is denoted as $\mathfrak{H}_{m,l}$. Pluri-harmonic polynomials are introduced *loc. cit* to study the Weil representation of $\mathrm{Sp}(2m,\mathbb{R})\times\mathrm{O}(l,\mathbb{R})$

when l is even. (In this section, by writing $\mathrm{O}(l,\mathbb{R})$ we mean the definite orthogonal group for the quadratic form given by $\mathbf{1}_l$.)

There is a natural embedding of $\mathbb{C}[M_{m,l}]$ into the space of Schwartz functions on $M_{m,l}(\mathbb{R})$,

$$\begin{aligned}\mathscr{I}_{m,l}: \mathbb{C}[M_{m,l}] &\hookrightarrow \mathcal{S}\left(M_{m,l}(\mathbb{R}),\mathbb{C}\right)\\ P &\longmapsto \left(x\mapsto P(x)\cdot \mathbf{e}_\infty\left(\frac{\sqrt{-1}}{2}\mathrm{Tr}x{}^{\mathrm{t}}x\right)\right).\end{aligned}$$

This embedding relates the action of Δ_{ij} on the left hand side to the Lie algebra action of $\mathfrak{u}^-_{\mathrm{Sp}(2m)}\subset \mathrm{Lie}\,\mathrm{Sp}(2m)$ on the right hand side. More precisely,

$$\mathscr{I}_{m,l}\left(\Delta_{ij}P\right) = -4\pi\sqrt{-1}L^-_{ij}\,\mathscr{I}_{m,l}(P), \qquad \text{for all } 1\leq i,j\leq m \text{ and } P\in\mathbb{C}[M_{m,l}],$$

where

$$L^-_{ij} = J\begin{pmatrix}0 & 0\\ E_{ij}+E_{ji} & 0\end{pmatrix}J^{-1}\in\mathfrak{u}^-_{\mathrm{Sp}(2m)}, \qquad J=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{1}_m & i\mathbf{1}_m\\ i\mathbf{1}_m & \mathbf{1}_m\end{pmatrix}.$$

Thus, pluri-harmonic polynomials correspond to vectors in the Weil representation killed by $\mathfrak{u}^-_{\mathrm{Sp}(2m)}$, *i.e.* elements inside the lowest K_∞-types of holomorphic discrete series of $\mathrm{Sp}(2m)$ appearing in the Weil representation.

Specializing to our situation, we define the following two conditions on polynomials on $M_{2n+1,2k}$.

((n, n+1)-ph) A polynomial $P\in\mathbb{C}[M_{2n+1,2k}]$ is called pluri-harmonic with respect to $\mathrm{Sp}(2n)\times\mathrm{Sp}(2n+1)\hookrightarrow\mathrm{Sp}(4n+2)$ if $\Delta_{ij}P=0$ for all $1\leq i,j\leq n$ or $n+1\leq i,j\leq 2n+1$.

(O-inv) A polynomial $P\in\mathbb{C}[M_{2n+1,2k}]$ is called O-invariant if there exists a polynomial $Q_P\in\mathbb{C}[\mathrm{Sym}(2n+1)]$ such that $P(x)=Q_P(x{}^{\mathrm{t}}x)$.

Proposition 4.9.4. *For $k\geq n$ and a dominant weight $\underline{t}$ with $t_1\geq\cdots\geq t_n\geq k$, there exists a unique polynomial $P_{\underline{t},k}\in\mathbb{C}[M_{2n+1,2k}]$ satisfying the conditions ((n, n+1)-ph), (O-inv) and*

(4.9.4)
$$P_{\underline{t},k}(x)\equiv\prod_{j=1}^{n-1}\det{}_j(2x_1{}^{\mathrm{t}}x_2)^{t_j-t_{j+1}}\det{}_n(2x_1{}^{\mathrm{t}}x_2)^{t_n-k}\mod (x_1{}^{\mathrm{t}}x_1, x_2{}^{\mathrm{t}}x_2)\mathbb{C}[x],\quad \text{for } x=\begin{array}{cc} \scriptstyle 2k & \\ \begin{pmatrix}x_1\\ x_2\end{pmatrix} & \begin{array}{c}\scriptstyle n\\ \scriptstyle n+1\end{array}\end{array},$$

where $(x_1{}^{\mathrm{t}}x_1, x_2{}^{\mathrm{t}}x_2)\mathbb{C}[x]$ denotes the ideal in $\mathbb{C}[M_{2n+1,2k}]$ generated by entries of $x_1{}^{\mathrm{t}}x_1, x_2{}^{\mathrm{t}}x_2$.

We will see from the proof that the Schwartz function $\mathscr{I}_{2n+1,k}(P_{\underline{t},k})$ is the highest weight vector inside the lowest K_∞-type of the summand isomorphic to $\mathcal{D}_{\underline{t}}\boxtimes\mathcal{D}_{(\underline{t},k)}$ of the Weil representation of $\mathrm{Lie}\left(\mathrm{Sp}(2n,\mathbb{R})\times\mathrm{Sp}(2n+2,\mathbb{R})\right)$ on $\mathcal{S}(M_{2n+1,2k}(\mathbb{R}))^{\mathrm{O}(2k,\mathbb{R})}$.

Proof. Let $\mu_{Q_{H'}}$ be the $(2n+1)\times(2n+1)$ matrix with the (i,j)-th entry being $\left(\begin{smallmatrix}0 & E_{ij}+E_{ji}\\ 0 & 0\end{smallmatrix}\right)$. Its entries constitute a basis of $\mathfrak{u}_{Q_{H'}}$, the Lie algebra of the unipotent radical of $Q_{H'}$. Then

$$Q\left(\frac{\mu_{Q_{H'}}}{2\pi\sqrt{-1}}\right)\cdot\mathbf{e}_\infty\left(\frac{\sqrt{-1}}{2}\mathrm{Tr}x{}^{\mathrm{t}}x\right) = Q(x{}^{\mathrm{t}}x)\,\mathbf{e}_\infty\left(\frac{\sqrt{-1}}{2}\mathrm{Tr}x{}^{\mathrm{t}}x\right),\quad Q\in\mathbb{C}[\mathrm{Sym}(2n+1)].$$

Inside the Weil representation,

$$U(\mathrm{Lie}\,H')(\mathbb{R})\cdot\mathbf{e}_\infty\left(\frac{\sqrt{-1}}{2}\mathrm{Tr}x{}^{\mathrm{t}}x\right) = U(\mathfrak{u}_{Q_{H'}})(\mathbb{R})\cdot\mathbf{e}_\infty\left(\frac{\sqrt{-1}}{2}\mathrm{Tr}x{}^{\mathrm{t}}x\right).\tag{4.9.5}$$

Hence, $\mathscr{I}_{2n+1,2k}$ induces a bijection between (4.9.5) and $\mathbb{C}[\mathrm{Sym}(2n+1)]$. By [JV79, Proposition 2.2, Corollary 2.3], there is a multiplicity free decomposition

$$(4.9.6)\qquad (4.9.5)|_{(\mathrm{Lie}\,G\times\mathrm{Lie}\,G')(\mathbb{R})}\cong\bigoplus_{a_1\geq\cdots\geq a_n\geq 0}\mathcal{D}_{a_1+k,\dots,a_n+k}\boxtimes\mathcal{D}_{a_1+k,\dots,a_n+k,k}.$$

Therefore, there exists a polynomial $Q_{\underline{t},k}\in\mathbb{C}[\mathrm{Sym}(2n+1)]$ such that $Q_{\underline{t},k}(\mu_{Q_{H'}})\cdot\mathbf{e}_\infty\left(\frac{\sqrt{-1}}{2}\mathrm{Tr}x{}^{\mathrm{t}}x\right)$ spans the highest weight space inside the lowest K_∞-type of $\mathcal{D}_{\underline{t}}\boxtimes\mathcal{D}_{\underline{t},k}$. Moreover, from the discussion in [Liu16, §3.4], after rescaling we have

$$Q_{\underline{t},k}(\mu_{Q_{H'}})\equiv\prod_{j=1}^{n-1}\det{}_j(\widehat{\mu}_0^+)^{t_j-t_{j-1}}\det{}_n(\widehat{\mu}_0^+)^{t_n-k}\equiv\prod_{j=1}^{n-1}\det{}_j(\mu_{Q_{H'},0})^{t_j-t_{j-1}}\det{}_n(\mu_{Q_{H'},0})^{t_n-k}$$

Define $P_{\underline{t},k}\in\mathbb{C}[M_{2n+1,k}]$ by $P_{\underline{t},k}(x)=Q_{\underline{t},k}(x{}^{\mathrm{t}}x)$. Then $P_{\underline{t},k}$ satisfies the conditions ($(n,n+1)$-ph), (O-inv) and (4.9.4). If there exists another $P'_{\underline{t},k}$ satisfying ($(n,n+1)$-ph), (O-inv) and (4.9.4), then the first two conditions imply that $Q_{P'_{\underline{t},k}}(\mu_{H'})$ sends the Gaussian function to the space spanned by lowest K_∞-types (with respect to $G(\mathbb{R})\times G'(\mathbb{R})$), and the condition (4.9.4) further implies that the K_∞-type is $\underline{t}\boxtimes(\underline{t},k)$ and the weight is $\underline{t},(\underline{t},k)$. By the multiplicity freeness of the decomposition (4.9.6), we know that $P'_{\underline{t},k}=P_{\underline{t},k}$. □

4.9.5. *The archimedean place.* With the polynomials $P_{\underline{t},k}\in\mathbb{C}[M_{2n+1,2k}]$ and $Q_{\underline{t},k}\in\mathbb{C}[\mathrm{Sym}(2n+1)]$ as defined in Proposition 4.9.4, we express $FJ_{\beta',\infty}$ evaluated at $s=n+1-k$ for general weight $\underline{t}$ as follows.

Proposition 4.9.5. *Keep the setting of Proposition 4.5.1. For $t_1\geq\cdots\geq t_n\geq k\geq n$, let $D_{\underline{t},k}=Q_{\underline{t},k}\left(\frac{\mu_{Q_{H'}}}{4\pi\sqrt{-1}}\right)\in U(\mathrm{Lie}\,H')$. Then*

$$\begin{aligned}FJ_{\beta',\infty}\left(g,g'_{z'},x,M_{Q_{H'}}(s,\mathrm{sgn}^k)D_{\underline{t},k}f^k_\infty(s,\mathrm{sgn}^k)\right)\Big|_{s=n+1-k}&=\frac{\sqrt{-1}^{(2-n)k}2^{(2n+1)k-\frac{5n^2}{2}-\frac{3n}{2}}\pi^{(n+1)(2n+1+k)}}{\Gamma_{2n+1}(n+1)\Gamma_{n+1}(2k)}\\&\times(\det 2\beta')^{k-\frac{n+2}{2}}(\det\mathrm{Im}z')^{\frac{k}{2}}\mathbf{e}_\infty(\mathrm{Tr}\beta'z')\cdot\omega_{\begin{pmatrix}2\beta'&0\\0&2\cdot\mathbf{1}_{2k-n-1}\end{pmatrix}}(g)\phi_{\underline{t},k,2\beta',\infty}(z';(x,0)),\end{aligned}$$

where the Schwartz function $\phi_{\underline{t},k,2\beta',\infty}(z;\cdot)$ on $M_{n,n+1}(\mathbb{R})\times M_{n,2k-n-1}(\mathbb{R})$ is defined as

$$\phi_{\underline{t},k,2\beta',\infty}(z';(x,y))=P_{\underline{t},k}\begin{pmatrix}x\sqrt{\beta'}&y\\\sqrt{\mathrm{Im}z'}\sqrt{\beta'}&0\end{pmatrix}\mathbf{e}_\infty\left(\sqrt{-1}\mathrm{Tr}(x\beta'{}^{\mathrm{t}}x+y{}^{\mathrm{t}}y)\right).$$

Remark 4.9.6. The archimedean section $D_{\underline{t},k}f^k_\infty(s,\mathrm{sgn}^k)$ gives rise to holomorphic forms when evaluated at $s=n+1-k$, and differs from the section $f_{\underline{\tau},\kappa,\infty}(s)$ we choose in §2.3.2. However, after ordinary projection on $G\times G'$, the corresponding Siegel modular forms are the same.

Proof. Let $A_k=\frac{\sqrt{-1}^{(2-n)k}2^{(2n+1)k-\frac{5n^2}{2}-\frac{3n}{2}}\pi^{(n+1)(2n+1+k)}}{\Gamma_{2n+1}(n+1)\Gamma_{n+1}(2k)}$. By the computation in §4.5,

$$FJ_{\beta',\infty}\left(h',x,f^k_\infty(s,\mathrm{sgn}^k)\right)=A_k\,\det(2\beta')^{k-\frac{n+2}{2}}\cdot\omega_{\begin{pmatrix}2\beta'&0\\0&2\mathbf{1}_{2k-n-1}\end{pmatrix}}(h')\,G_{\begin{pmatrix}2\beta'&0\\0&2\cot\mathbf{1}_{2k-n-1}\end{pmatrix}}\begin{pmatrix}x&0\\\mathbf{1}_{n+1}&0\end{pmatrix}.$$

The Gaussian function here is defined as

$$G_{\begin{pmatrix}2\beta'&0\\0&2\cdot\mathbf{1}_{2k-n-1}\end{pmatrix}}(X)=\mathbf{e}_\infty\left(\mathrm{Tr}\sqrt{-1}X\begin{pmatrix}\beta'&0\\0&\mathbf{1}_{2k-n-1}\end{pmatrix}{}^{\mathrm{t}}X\right),\qquad X\in M_{2n+1,2k}(\mathbb{R}).$$

Hence

$$FJ_{\beta',\infty}\left(h',x,D_{\underline{t},k}f^k_\infty(s,\mathrm{sgn}^k)\right)=A_k\,\det(2\beta')^{k-\frac{n+2}{2}}\cdot\omega_{\begin{pmatrix}2\beta' & 0\\ 0 & 2\cdot\mathbf{1}_{2k-n-1}\end{pmatrix}}(h')\,\phi_{\underline{t},k,2\beta',\infty}\begin{pmatrix}x & 0\\ \mathbf{1}_{n+1} & 0\end{pmatrix},$$

$$\phi_{\underline{t},k,2\beta',\infty}(X)=Q_{\underline{t},k}\left(X\begin{pmatrix}\beta' & 0\\ 0 & \mathbf{1}_{2k-n-1}\end{pmatrix}{}^{\mathrm{t}}X\right)G_{\begin{pmatrix}2\beta' & 0\\ 0 & 2\cdot\mathbf{1}_{2k-n-1}\end{pmatrix}}(X),$$

and the proposition follows. □

4.9.6. *The places with "big cell" sections.* The discussion in this subsection and the next is mainly based on results in [KR92, Swe]. We consider $FJ_{\beta',v}\left(g,m(\mathfrak{a}_v),x,f^{\alpha_v}(-s,\eta^{-1})\right)\big|_{s=n+1-k}$ for a Schwartz function $\alpha_v=\begin{pmatrix}\alpha_{v,\text{up-left}} & \alpha_{v,\text{off-diag}}\\ {}^{\mathrm{t}}\alpha_{v,\text{off-diag}} & \alpha_{v,\text{low-right}}\end{pmatrix}$ on $\mathrm{Sym}(2n+1,\mathbb{Q}_v)$ with $\alpha_{v,\text{up-left}}=\mathbb{1}_{\mathrm{Sym}(n+1,\mathbb{Z}_v)}$.

By Proposition 4.6.1, we need to examine when the "big cell" section $f^{\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}}_{\widetilde{G},v}\left(k-n-1,\widetilde{\lambda}_{2\beta_*,v}\right)$ belongs to the image of the map (4.9.1). First, when $2\beta_*$ admits sufficiently large isotropic subspace, we have the easy proposition below.

Proposition 4.9.7. *Suppose $v\neq 2$, $\beta',\beta_*,\mathfrak{a}_v,\eta,\kappa$ are given as in §4.9.1, and $\alpha_{v,\text{up-left}}=\mathbb{1}_{\mathrm{Sym}(n+1,\mathbb{Z}_v)}$. If there exists $\mathfrak{s}_v\in\mathrm{GL}(2k-n-1,\mathbb{Q}_v)$ such that*

$$2{}^{\mathrm{t}}\mathfrak{s}_v\beta_*\mathfrak{s}_v=\begin{array}{c}\begin{array}{ccc}n & n & 2k-3n-1\end{array}\\ \begin{pmatrix}0 & \mathbf{1}_n & 0\\ \mathbf{1}_n & 0 & 0\\ 0 & 0 & *\end{pmatrix}\end{array}\begin{array}{c}\\ n\\ n\\ 2k-3n-1\end{array}\in\mathrm{Sym}(2k-n-1,\mathbb{Z}_v),$$

then

$$FJ_{\beta',v}\left(g,m(\mathfrak{a}_v),x,f^{\alpha_v}(-s,\eta^{-1})\right)\big|_{s=n+1-k}$$

is given by the formula in Proposition 4.6.1 with $\xi=\eta^{-1}$, $s=k-n-1$ and

$$\begin{aligned}&S\left(g,x;f^{\alpha_{v,\text{up-left}}}_{\widetilde{G}}\left(s,\eta^{-1}\widetilde{\lambda}^{-1}_{2\beta'}\right),R(-q_v^{-r}\cdot 2\beta'\mathfrak{a}_v)\widehat{\alpha}_{v,\text{off-diag}}\right)\Big|_{s=k-n-1}\\ &=\omega_{\begin{pmatrix}2\beta' & 0\\ 0 & 2\beta_*\end{pmatrix}}(g)R\begin{pmatrix}-q_v^r\cdot 2\beta'\mathfrak{a}_v & 0\\ 0 & {}^{\mathrm{t}}\mathfrak{s}_v^{-1}\end{pmatrix}\cdot\left(\widehat{\alpha}_{v,\text{off-diag}},\mathbb{1}_{\mathrm{GL}(n,\mathbb{Z}_v)},\mathbb{1}_{M_{n,2k-n-1}(\mathbb{Z}_v)}\right)(x,0)\end{aligned}$$

The condition on $2\beta_*$ in the above proposition is always satisfied if $k\geq\frac{3n+5}{2}$, *i.e.* the size of $2\beta_*$ is larger or equal to $2n+4$, which is exactly the range where the map (4.9.1) is surjective for all $2\beta_*$ according to [KR92, Swe]. Also, it is obvious that the size of $2\beta_*$ must be at least $2n$ in order for the condition to be satisfied. Thus, when the condition is satisfied, the corresponding critical point for $L(s,\pi\times\eta)$ is far from the center (unless $n=1$).

When the condition on $2\beta_*$ in Proposition 4.9.7 is not satisfied, the image of the map (4.9.1) for a single $2\beta_*$ is a proper subspace of the degenerate principal series, and in general does not contain the "big cell" sections. Let $2\beta_*^{\pm}\in\mathrm{Sym}(2k-n-1,\mathbb{Q})$ be a quadratic form of dimension $2k-n-1$ and determinant $(-1)^k(d_\eta\det 2\beta')^{-1}$ with Hasse invariant (over $\mathbb{Q}_v$) equal to ±1. Denote by $R(2\beta_*^{\pm})$ the image of the map (4.9.1) associated to $2\beta_*^{\pm}$ (if $2\beta_*^-$ does not exist then we set $R(2\beta_*^-)=0$).

Theorem 4.9.8 ([KR92,Swe]). *$R(2\beta_*^+)+R(2\beta_*^-)=I_{Q_{\widetilde{G}},v}(k-n-1,\widetilde{\lambda}_{2\beta_*^+})$ if and only if $k-n-1\geq 0$.*

It follows that when $k\geq n+1$ but the condition in Proposition 4.9.7 is not satisfied, one can still express $FJ_{\beta',v}\left(g,m(\mathfrak{a}_v),x,f^{\alpha_v}(-s,\eta^{-1})\right)\big|_{s=n+1-k}$ by using Siegel–Weil sections but one needs to take a linear combination over $2\beta_*^+$ and $2\beta_*^-$.

If $k < n+1$, then according to Theorem 4.9.8 one does not expect the "big cell" section to be a Siegel–Weil section. (There are some reasons to think of "big cell" sections as unlikely to lie inside a proper sub-representation.)

4.9.7. *The places with intertwining operator applied to "big cell" sections.*

Theorem 4.9.9 ([KR92, Swe]). *Let $2\beta_*^+, 2\beta_*^-$ be as in Theorem 4.9.8. Then*

$$R(2\beta_*^+) + R(2\beta_*^-) = M_{Q_{\widetilde{G}}}\left(n+1-k, (\widetilde{\lambda}_{2\beta_*^+})^{-1}\right) I_{Q_{\widetilde{G}},v}\left(n+1-k, (\widetilde{\lambda}_{2\beta_*^+,v})^{-1}\right)$$

if and only if $0 \le k \le n+1$.

Therefore, when $k < n+1$, the "big cell" section $f_{\widetilde{G},v}^{\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}}\left(k-n-1, \widetilde{\lambda}_{2\beta_*^+}\right)$ is not a Siegel–Weil section, but the section $M_{Q_{\widetilde{G}}}\left(n+1-k, \widetilde{\lambda}_{2\beta_*^+}^{-1}\right) f_{\widetilde{G},v}^{\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_v)}}\left(n+1-k, \widetilde{\lambda}_{2\beta_*^+}^{-1}\right)$ can be written as sum of Siegel–Weil sections.

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