Lectures on Dirac Operators and Index Theory

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Chapter 1

Clifford algebra

1.1 Introduction

Historically, Dirac operator was discovered by Dirac (who else!) looking for a square root of the Laplace operator. According to Einstein's (special) relativity, a free particle of mass m in $\mathbb{R}^3$ with momentum vector $p = (p_1, p_2, p_3)$ has energy

$$E = c\sqrt{m^2c^2 + p^2} = c\sqrt{m^2c^2 + p_1^2 + p_2^2 + p_3^2}.$$

For simplicity, we assume that all physical constants are one, including $c = 1$ (the general case is a recaling). Passing to quantum mechanics, one replaces E by the operator $i\frac{\partial}{\partial t}$, and p_j by $-i\frac{\partial}{\partial x_j}$. Therefore the particle now is described by a state function $\Psi(t, x)$ satisfying the equation

$$i\frac{\partial \Psi}{\partial t} = \sqrt{m^2 + \Delta}\Psi.$$

Here the Laplacian

$$\Delta = -\sum_j \frac{\partial^2}{\partial x_j^2}.$$

This motivates Dirac to look for a (Lorentz invariant) square root of Δ. In other words, Dirac looks for a first order differential operator with constant coefficients

$$D = \gamma_j \frac{\partial}{\partial x_j} + m\gamma_0$$

such that $D^2 = m^2 + \Delta$. It follows that

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 0 \quad \text{if } 0 \le i \ne j \le 3; \quad \gamma_0^2 = 1 \text{ and } \gamma_i^2 = -1 \quad \text{for } i = 1, 2, 3.$$

Dirac realized that, to have solutions, the coefficients γ_i will have to be complex matrices. These now come to be known as the Γ-matrices.

Note that the above equations can be written as

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2\eta_{ij}, \tag{1.1}$$

where η_{ij} denotes the canonical Minkowski metric. The mathematical study of this equation is the subject of Clifford algebra and its representations.

1.2 Clifford algebra

Let V be a n-dimensional real vector space, g a non-degenerate quadratic form on V of signature (p,q). Then there are "orthonormal basis" e_i such that

$$\begin{array}{l} g_{ij} = g(e_i, e_j) = 0, \quad i \neq j, \\ g_{ii} = g(e_i, e_i) = \left\{ \begin{array}{ll} +1, & i = 1, \cdots, p \\ -1, & i = p+1, \cdots, p+q = n \end{array} \right. . \end{array} \tag{2.2}$$

Namely g is diagonalized by e_i:

$$g = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2.$$

A concrete construction of Clifford algebra, $Cl(V,g) = Cl_{p,q}$, can be obtained by defining it to be the real algebra generated by $1, e_1, \cdots, e_n$ subject only to the relations

$$e_i e_j + e_j e_i = -2g_{ij}. \tag{2.3}$$

In other words,

$$\begin{array}{l} e_i e_j = -e_j e_i, \quad \text{if } i \neq j; \\ e_i^2 = -1, \quad \text{for} \quad i = 1, \cdots, p, \quad \text{and} \quad e_i^2 = 1, \quad \text{for} \quad i = p+1, \cdots, n. \end{array}$$

This definition is quite explicit. For example, it is clear that

$$1, e_1, \cdots, e_n, e_1 e_2, \cdots, e_{i_1} e_{i_2} \cdots e_{i_k} (i_1 < i_2 < \cdots < i_k), \cdots, e_1 \cdots e_n \tag{2.4}$$

is a vector space basis for $Cl_{p,q}$. So as a vector space $Cl(V,g) \cong \Lambda^* V$, the exterior algebra. Thus we have $\dim Cl_{p,q} = 2^n = 2^{p+q}$.

Denote $Cl_n = Cl_{n,0}$. Then we have

$Cl_1 \equiv \mathbb{C}$, where e_1 corresponds to i.

$Cl_2 \equiv \mathbb{H}$, the quaternions, and the basis vectors $e_1, e_2, e_1 e_2$ correspond to I, J, K.

$Cl_{0,1} \equiv \mathbb{R} \oplus \mathbb{R}$. Here 1 corresponds to $(1,1)$ and e_1 to $(1,-1)$.

$Cl_{0,2} \equiv M_2(\mathbb{R}) = \mathbb{R}(2)$. Here e_1 corresponds to the diagonal matrix with entries $1, -1$, and e_2 to the off-diagonal matrix with entries $1, 1$.

The concrete definition comes with a price: ambiguity. It is not hard to remedy though—we simply translate what the words in the definition means into rigorous mathematics. The result is a basis-free definition.

First of all, the free algebra generated by V should be the tensor algebra of V. To impose the Clifford relation, we look for an ideal in the tensor algebra. Note that the defining relation (2.3) can be rewritten in a basis-free fashion:

$$vw + wv = -2g(v, w) \tag{2.5}$$

for all $v, w \in V$. Equivalently (by polarization),

$$v^2 = -g(v, v). \tag{2.6}$$

Definition: Let $\mathcal{T}(V) = \sum_{r=0}^{\infty} \otimes^r V$ be the full tensor algebra of V and $\mathcal{I}_g(V)$ be the ideal generated by all elements of the form $v \otimes v + g(v)1$, where $g(v) = g(v, v)$. We define the Clifford algebra

$$Cl(V, g) = \mathcal{T}(V)/\mathcal{I}_g(V).$$

In this definition, g is not required to be nondegenerate. Thus, one obtains the exterior algebra $\Lambda^* V$ by taking $g = 0$. It also follows from the definition that the Clifford algebra has universal property in the following sense. For any algebra A with unit and any linear map

$$f : \ V \to A$$

satisfying $f(v)f(v) = -g(v)1$ extends uniquely to an algebra homomorphism

$$\tilde{f} : Cl(V, g) \to A.$$

In particular, given two vector spaces V, V' with quadratic forms g, g' respectively, any linear map $f : \ V \to V'$ preserving the quadratic forms, i.e., $f^* g' = g$, extends to an algebra homomorphism

$$\tilde{f} : Cl(V, g) \to Cl(V', g').$$

One verifies that $\widetilde{f_1 \circ f_2} = \tilde{f}_1 \circ \tilde{f}_2$. It follows then that we have the following representation

$$f \in SO(V, g) \to \tilde{f} \in Aut(Cl(V, g)). \tag{2.7}$$

As a simple but important example, $\alpha : \ V \to V$ sending v to $-v$ extends to an automorphism $\alpha : \ Cl(V, g) \to Cl(V, g)$ satisfying $\alpha^2 = 1$. This gives rise to the decomposition

$$Cl(V, g) = Cl^0(V, g) \oplus Cl^1(V, g), \tag{2.8}$$

where $Cl^0(V, g)$ (respectively $Cl^1(V, g)$) is the $+1(-1)$-eigenspace of α. Clearly,

$$Cl^i(V, g) \cdot Cl^j(V, g) \subset Cl^{i+j}(V, g) \tag{2.9}$$

where the superscripts are taken mod 2.

In terms of a basis e_i, Cl^0 is spanned by all elements of the form $e_{i_1} \cdots e_{i_k}$ with k even, and Cl^1 is spanned by all elements of the form $e_{i_1} \cdots e_{i_k}$ with k odd.

α is an example of an operation on $Cl(V,g)$. Some other important operations are defined as follows. The reversion of an element is defined so that

$$(e_{i_1}\cdots e_{i_k})^R = e_{i_k}\cdots e_{i_1}.$$

The conjugation is then defined by composing α with the reversion: $\varphi^* = \alpha((\varphi)^R)$.

Example: For $Cl_1 \equiv \mathbb{C}$, the conjugation is the usual complex conjugation. Similarly, for $x = x_0+x_1I+x_2J+x_3K \in Cl_2 \equiv \mathbb{H}$, $x^* = x_0-x_1I-x_2J-x_3K = \bar{x}$.

The "norm" of an element $\varphi \in Cl(V,g)$ is then defined to be

$$N(\varphi) = \varphi \cdot \varphi^* \in Cl(V,g). \tag{2.10}$$

This is not really a norm as one in general do not get a scalar, but one verifies easily that

$$N(v) = g(v), \quad N(\varphi_1\varphi_2) = N(\varphi_1)N(\varphi_2),$$

provided that $N(\varphi_2)$ is a scalar.

1.3 The group Pin and Spin

In matrix algebra $M_n(\mathbb{R})$, one considers the multiplicative group of invertible elements, which is $Gl(n,\mathbb{R})$. In $Gl(n,\mathbb{R})$, one obtains the orthogonal group $O(n)$ and special orthogonal group $SO(n)$ by bringing in the inner product and orientation. Similar procedure for Clifford algebra leads us to the groups *Pin* and *Spin*.

For simplicity, we restrict ourselves to $Cl_n = Cl_{n,0}$. We define $Cl_n^\times$ to be the multiplicative group of the invertible elements in Cl_n:

$$Cl_n^\times = \{\varphi \in Cl_n \mid \exists \varphi^{-1} \in Cl_n \text{ such that } \varphi^{-1}\varphi = \varphi\varphi^{-1} = 1\}.$$

We note that $Cl_n^\times$ is an open submanifold of Cl_n and therefore a Lie group with the Lie algebra Cl_n. The Lie bracket is given by the commutator

$$[\varphi,\psi] = \varphi\psi - \psi\varphi.$$

Note also that $V - \{0\} \subset Cl_n^\times$ since $v^{-1} = -\frac{v}{g(v)}$.

Definition: We define the *Pin* group to be the multiplicative subgroup of $Cl_n^\times$ generated by unit vectors of V:

$$Pin(n) = \{v_1\cdots v_k \mid v_i \in V,\ g(v_i) = 1\},$$

and the *Spin* group

$$Spin(n) = Pin(n) \cap Cl_n^0 = \{v_1\cdots v_{2k} \mid v_i \in V,\ g(v_i) = 1\}.$$

Example: First of all, one has an identification $Cl_n^0 \equiv Cl_{n-1}$ which preserves the norm when restricted to the spin group. It follows then $Spin(2) \equiv S^1 \subset \mathbb{C} \equiv Cl_1 \equiv Cl_2^0$.

Similarly $Spin(3) \subset Cl_3^0 \equiv Cl_2 = \mathbb{H}$ and $1 = N(x) = xx^* = x\bar{x} = |x|^2$ for $x \in Spin(3)$. Therefore $Spin(3)$ lies inside the space of unit quaternions, i.e., S^3. We will see that in fact $Spin(3) = S^3$ from the next theorem.

Exercise. What is $Spin(1)$?

To understand these groups we need the following definition.

Definition: The twisted adjoint representation is

$$\begin{array}{cccc} \widetilde{Ad}_\varphi : & Cl_n & \rightarrow & Cl_n \\ & x & \rightarrow & \alpha(\varphi)x\varphi^{-1} \end{array} \tag{3.11}$$

for $\varphi \in Cl_n^\times$.

Clearly, one has $\widetilde{Ad}_{\varphi_1\varphi_2} = \widetilde{Ad}_{\varphi_1} \circ \widetilde{Ad}_{\varphi_2}$. It turns out that the twisted adjoint representation is closely related to reflections.

Lemma 1.3.1 *If $v \in V$ and $v \neq 0$, then $\widetilde{Ad}_v : V \rightarrow V$. In fact,*

$$\widetilde{Ad}_v(w) = w - 2\frac{g(v,w)}{g(v)}v$$

is the reflection across the hyperplane $v^\perp = \{u \in V \mid g(u,v) = 0\}$.

Proof: One computes

$$\widetilde{Ad}_v(w) = \alpha(v)wv^{-1} = -vw\frac{-v}{g(v)} = \frac{v(-vw - 2g(v,w))}{g(v)} = w - 2\frac{g(v,w)}{g(v)}v.$$

$\blacksquare$

Now reflection is an element of $O(n)$. We have

Theorem 1.3.2 *There is an exact sequence*

$$1 \rightarrow \mathbb{Z}_2 \rightarrow Spin(n) \rightarrow SO(n) \rightarrow 1,$$

where the first two arrows are given by natural inclusions and the third given by the twisted adjoint representation. In fact, Spin(n) *is the universal cover of $SO(n)$ for $n \geq 3$, and the nontrivial double cover when $n = 2$.*

Proof: By Lemma 1.3.1, we have a map $\rho : Spin(n) \rightarrow SO(n)$, with $\rho(\varphi) = \widetilde{Ad}_\varphi$. We need to show ρ is onto and $\ker\rho = \mathbb{Z}_2$. That ρ is onto follows from a theorem of Cartan-Diedonne which says that elements of $SO(n)$ are products of even numbers of reflections.

Clearly, $\mathbb{Z}_2 \subset \ker\rho$. Now if $\varphi \in Spin(n) = Pin(n) \cap CL_n^0$ such that $\widetilde{Ad}_\varphi|_V = id$, then

$$\varphi v = v\varphi, \quad \forall v \in V.$$

We claim that only scalars satisfy this equation. In fact, take $\{e_i\}$ an orthonormal basis and write

$$\varphi = a_0 + e_1a_1,$$

where a_0 is even and contains no e_1, a_1 is odd contains no e_1. $\varphi e_1 = e_1\varphi$ implies $a_1 = 0$ so φ has no e_1 in it. Similarly φ has no $e_2, \cdots, e_n$ so $\varphi = t \in \mathbb{R}$. But $\varphi \in Spin(n)$ so $\varphi = v_1 \cdots v_{2k}$ with $g(v_i) = 1$. Hence $t^2 = N(\varphi) = N(v_1)\cdots N(v_{2k}) = g(v_1)\cdots g(v_{2k}) = 1$, and $t = \pm 1$. Hence $\ker\rho \subset \mathbb{Z}_2$.

To see that $Spin(n)$ is the universal cover, we only need to see that the two points $\{-1, 1\} = \mathbb{Z}_2$ in the kernel of ρ can be connected by a continuous path in $Spin(n)$. Such a path is furnished by

$$\gamma(t) = \cos t + e_1e_2 \sin t = (e_1 \cos\frac{t}{2} + e_2 \sin\frac{t}{2}) \cdot (-e_1 \cos\frac{t}{2} + e_2 \sin\frac{t}{2}) \in Spin(n), \tag{3.12}$$

where $0 \le t \le \pi$. This shows that $Spin(n)$ is a nontrivial double cover of $SO(n)$ when $n \ge 2$. Therefore it is the universal cover for $n \ge 3$ since then $\pi_1(SO(n)) = \mathbb{Z}_2$. ■

Remark One also has the short exact sequence

$$1 \to \mathbb{Z}_2 \to Pin(n) \to O(n) \to 1.$$

Remark $SO(n)$ is the rotation group. $Spin(n)$ describes the self-spin of elementary particles. The short exact sequence here is related to the so called "4π periodicity".

Example: Continuing from the last example, we see that nontrivial double covering $Spin(2) = S^1 \to SO(2) = S^1$ is given by the square map $z \in S^1 \to z^2 \in S^1$.

Also from the last example, $Spin(3)$ is a Lie subgroup of S^3. From the theorem we see that they have the same Lie algebra. Therefore $Spin(3) = S^3$ as claimed before.

Similar argument leads to the identification $Spin(4) \equiv S^3 \times S^3 = Spin(3) \times Spin(3) \equiv SU(2) \times SU(2)$.

The map $\rho : Spin(n) \to SO(n)$ induces isomorphism $\rho_* : Lie(Spin(n)) \to so(n)$. We identify a two form with a skew adjoint matrix by the prescription

$$e_i \wedge e_j \to (e_i \wedge e_j)x = -\langle e_j, x\rangle e_i + \langle e_i, x\rangle e_j.$$

In particular, for an orthonormal basis e_i, the skew adjoint matrix corresponding to $e_i \wedge e_j$ has entries -1 on the (i, j)-slot and 1 on the (j, i)-slot. Thus $A = (A_{ij}) \in so(n)$ is identified with $\frac{1}{2}A_{ij}e_i \wedge e_j$.

Lemma 1.3.3 *$\rho_*^{-1} : so(n) \to Lie(Spin(n)) \subset Cl_n$ is given by*

$$\rho_*^{-1}(A) = \frac{1}{2}\sum_{i<j} A_{ij}e_ie_j = \frac{1}{4}A_{ij}e_ie_j,$$

if $A = (A_{ij}) \in so(n)$. Or, using the identification above,

$$\rho_*^{-1}(\frac{1}{2}A_{ij}e_i \wedge e_j) = \frac{1}{4}A_{ij}e_ie_j.$$

Proof: We first observe that $Lie(Spin(n)) = span\langle e_ie_j | i < j\rangle$. In fact, $\gamma(t) = \cos t + e_ie_j \sin t = (e_i \cos\frac{t}{2} + e_j \sin\frac{t}{2}) \cdot (-e_i \cos\frac{t}{2} + e_j \sin\frac{t}{2}) \in Spin(n)$ with $\gamma(0) = 1, \gamma'(0) = e_ie_j$ implies $Lie(Spin(n)) \supset span\langle e_ie_j | i < j\rangle$. By dimension counting we have equality.

To compute ρ_*, we do it on basis: $\forall x \in \mathbb{R}^n$,

$$\begin{aligned}\rho_*(e_ie_j)x &= \frac{d}{dt}|_{t=0}(\rho(\gamma(t))x) = \frac{d}{dt}|_{t=0}[\gamma(t)x(\gamma(t))^{-1}] \\ &= e_ie_jx - xe_ie_j = -2\langle e_j, x\rangle e_i + 2\langle e_i, x\rangle e_j.\end{aligned}$$

■

1.4 Classification of Clifford algebra

For simplicity, we consider the complexification of the Clifford algebra

$$\mathbb{C}l_n = Cl_n \otimes \mathbb{C} \equiv Cl(\mathbb{C}^n, g_{\mathbb{C}}).$$

The simplification comes from the fact that all nondegenerate quadratic forms are standard over $\mathbb{C}$: $g_{\mathbb{C}} = \sum_{i=1}^n z_i^2$.

Theorem 1.4.1 *One has the mod 2 periodicity*

$$\mathbb{C}l_{n+2} \equiv \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2.$$

Proof: In terms of basis, $\mathbb{C}l_{n+2}$ is generated by $e_1, \cdots, e_{n+2}$ such that

$$e_i^2 = -1, \quad e_ie_j = -e_je_i,\ i \neq j.$$

$\mathbb{C}l_n$ is generated by $e'_1, \cdots, e'_n$ such that

$$(e'_i)^2 = -1, \quad e'_ie'_j = -e'_je'_i,\ i \neq j.$$

$\mathbb{C}l_2$ is generated by e''_1, e''_2 such that

$$(e''_i)^2 = -1, \quad e''_1e''_2 = -e''_2e''_1.$$

Now define

$$f:\ \mathbb{C}^{n+2} \to \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2$$

by sending e_i to $e'_i \otimes \sqrt{-1}e''_1e''_2$ for $1 \leq i \leq n$, and e_{n+1} to $1 \otimes e''_1$, and e_{n+2} to $1 \otimes e''_2$. Clearly,

$$f(e_i)f(e_j) + f(e_j)f(e_i) = -2\delta_{ij}.$$

Thus f extends to an algebra homomorphism from $\mathbb{C}l_{n+2}$ to $\mathbb{C}l_n \otimes \mathbb{C}l_2$. But f is surjective since the image contains a set of generators:

$$-e'_i \otimes \sqrt{-1} = f(e_i)f(e_{n+1})f(e_{n+2}), \quad 1 \otimes e''_1 = f(e_{n+1}), \quad 1 \otimes e''_2 = f(e_{n+2}).$$

Consequnently, f is an isomorphism. ■

Remark This isomorphism tells us how to build all the Γ-matrices from the Pauli matrices.

The mod 2 periodicity can now be seen as follows. First of all, we have

$$\mathbb{C}l_1 = Cl_1 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \equiv \mathbb{C} \oplus \mathbb{C},$$

$$\mathbb{C}l_2 = Cl_2 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \equiv \mathbb{C}(2).$$

Here the first line comes from explicit construction of isomorphism between $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \oplus \mathbb{C}$, and the second isomorphism can be constructed using the Pauli matrices as the basis for the quaternions. The theorem above then implies that

$$\mathbb{C}l_n = \begin{cases} End(\mathbb{C}^{2^{n/2}}) \text{ if } n \text{ is even;} \\ End(\mathbb{C}^{2^{(n-1)/2}}) \oplus End(\mathbb{C}^{2^{(n-1)/2}}) \text{ if } n \text{ is odd.} \end{cases} \tag{4.13}$$

Definition: The vector space of complex n-spinors is defined to be

$$\Delta_n = \mathbb{C}^{2^{[n/2]}},$$

where $[n/2]$ denotes the integer part of $n/2$. Elements of Δ_n are called complex spinors.

So the space of complex spinors are characterized by

$$\mathbb{C}l_n = \begin{cases} End(\Delta_n) \text{ if } n \text{ is even;} \\ End(\Delta_n) \oplus End(\Delta_n) \text{ if } n \text{ is odd.} \end{cases}$$

Remark Without complexification, the periodicity of Cl_n is 8:

$$Cl_{n+8} \equiv Cl_n \otimes_{\mathbb{R}} Cl_8 \equiv Cl_n \otimes_{\mathbb{R}} \mathbb{R}(16).$$

This is because one is then not allowed to use $\sqrt{-1}$ in the isomorphism and $(\sqrt{-1})^4 = 1$.

1.5 Spin representation

As we see, $\mathbb{C}l_n$ is essentially the matrix algebra. This gives rise to natural (algebra) representation of $\mathbb{C}l_n$. We obtain spin representation by restricting Clifford algebra representation to $Spin(n)$.

Note

$$\mathbb{C}l_n = \begin{cases} End(\Delta_n) & n \text{ even} \\ End(\Delta_n) \oplus End(\Delta_n) & n \text{ odd} \end{cases}$$

So $\mathbb{C}l_n \to End(\Delta_n)$ is identity when n is even, and projections when n odd. Any element of $\mathbb{C}l_n$ therefore acts on the complex spinors. This is called Clifford multiplication. In particular, for $v \in V = \mathbb{R}^n$, we have

Definition: The Clifford multiplication by v on the spinors, denoted by $c(v)$, is the endomorphism $c(v) \in End(\Delta_n)$ from $V \subset Cl_n \subset \mathbb{C}l_n \to End(\Delta_n)$. That is

$$c(v): \ \Delta_n \to \Delta_n.$$

Clearly, the Clifford multiplication satisfies the Clifford relation

$$c(v)c(w) + c(w)c(v) = -2\langle v, w\rangle. \tag{5.14}$$

Definition: The composition $\Delta_n : Spin(n) \hookrightarrow Cl_n \hookrightarrow \mathbb{C}l_n \to End(\Delta_n)$ is called the spin representation of $Spin(n)$.
Remark Here we restrict ourselves to complex spin representations.
Remark When n is odd, the two irreducible representations of $\mathbb{C}l_n$ coming from the different projections restrict to the same (i.e. equivalent) spin representation. (Cl_n^0 sits diagonally.)
Remark The fact that these representations comes from algebra representations shows that -1 is represented by $-Id$. Therefore they are not induced from representations of $SO(n)$.
Definition: The complex volume element $\omega_{\mathbb{C}} \in \mathbb{C}l_n$ is

$$\omega_{\mathbb{C}} = i^{[\frac{n+1}{2}]} e_1 \cdots e_n,$$

where $\{e_i\}$ is an orthonormal basis. It is independent of the choices of $\{e_i\}$ if we fix the orientation. And

$$\omega_{\mathbb{C}}^2 = 1.$$

Lemma 1.5.1 *When n is odd, $\omega_{\mathbb{C}}$ commutes with every element of the Clifford algebra. In other words, $\omega_{\mathbb{C}}$ is a central element. When n is even, $\omega_{\mathbb{C}}$ anti-commutes with every $v \in V$. In particular, $\omega_{\mathbb{C}}$ commutes with elements of $\mathbb{C}l_n^0$ and anti-commutes with $\mathbb{C}l_n^1$.*

Proof: Clearly it suffices to look at the commutativity of $\omega_{\mathbb{C}}$ with a unit vector $e \in V$. We extend e into a positively oriented orthonormal basis $e_1 = e, e_2, \cdots, e_n$ of V. In terms of this basis, $\omega_{\mathbb{C}}$ clearly commutes with e when n is odd, and anti-commutes with e when n is even. ■

When n is odd, the fact that $\omega_{\mathbb{C}}$ is central gives rise to an algebra decomposition

$$\mathbb{C}l_n = \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-$$

via the ± 1-eigenspaces of $\omega_{\mathbb{C}}$. This decomposition turns out to coincide with the decomposition (4.13).

We now assume n =even. Then $\omega_{\mathbb{C}}$ anti-commutes with $\forall v \in V$, and so commutes with all elements of $Spin(n)$. Therefore the spin representation Δ_n decomposes as

$$\Delta_n = \Delta_n^+ \oplus \Delta_n^-,$$

where Δ_n^+ is the $+1$-eigenspace of $c(\omega_{\mathbb{C}})$, and Δ_n^- is the -1-eigenspace of $c(\omega_{\mathbb{C}})$.
Definition: [Weyl spinors] Elements of $\Delta_n^\pm$ are called Weyl spinors (chiral spinors) of $\pm$ chirality.

Summarizing, we have

Theorem 1.5.2 *If n =odd, Δ_n is an irreducible representation of $Spin(n)$. If n =even, Δ_n decomposes into $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ two irreducible representations of $Spin(n)$. The Clifford multiplication of any $v \in V$ interchanges $\Delta_n^\pm$.*

Proof: We only need to verify the irreducibility. This follows from the isomorphism $\mathbb{C}l_n^0 \cong \mathbb{C}l_{n-1}$, the fact that $\mathbb{C}l_n$ is a simple algebra when n is even, and the fact that $Spin(n)$ contains a vector space basis of $\mathbb{C}l_n^0$. ■

In fact, for n = even, Δ_n has the following nice explicit description. In this case, $V = \mathbb{R}^n = \mathbb{C}^{n/2}$ comes with a complex structure J. The complex structure J gives rise to the decomposition

$$V \otimes \mathbb{C} = V^{(1,0)} \oplus V^{(0,1)}.$$

We write $v = v^{1,0} + v^{0,1}$ in terms of the decomposition. Then $\Delta_n = \Lambda^*(V^{(0,1)})$, with the Clifford multiplication given by

$$c(v) = \sqrt{2}(v^{(0,1)} \wedge - \, i_{v^{(1,0)}}).$$

Here the interior multiplication

$$i_{v^{(1,0)}} : \ \Lambda^p(V^{(0,1)}) \to \Lambda^{p-1}(V^{(0,1)})$$

is given by contraction using the hermitian metric $(\ ,\ )$.

One also has $\Delta_n^\pm = \Lambda^\pm(V^{(0,1)})$ with $\Lambda^\pm(V^{(0,1)})$ given by exterior algebra elements of even or odd degree, respectively.
Remark 1) The complex spinor space Δ_n is a complex vector space of course. But sometimes it comes with more structures. When $n \equiv 2, 3, 4, 5 \mod 8$, Δ_n turns out to be quarternionic, which is useful in index theory. When $n \equiv 6, 7, 0, 1 \mod 8$, Δ_n has a real structure, which is useful in physics.

2) One version of Bott periodicity states

$$K^{-n}(pt) = K(S^n) = \begin{cases} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

Write $S^n = S^n_+ \cup S^n_-$, $S^n_+ \cap S^n_- = S^{n-1}$. Now any $x \in S^{n-1} \subset \mathbb{R}^n$ defines a Clifford multiplication

$$c(x) = x_i \gamma_i : \ \Delta_n \to \Delta_n,$$

and when n is even, $c(x)$ defines an isomorphism $\Delta_n^+ \xrightarrow{\sim} \Delta_n^-$. Thus

$$\xi = S^n_+ \times \Delta_n^+ \cup_c S^n_- \times \Delta_n^-$$

defines a complex vector bundle on S^n. Here the notation $\cup_c$ indicate that the bundles are glued along $S^n_+ \cap S^n_- = S^{n-1}$ via $c(x)$. The Atiyah-Bott-Shapiro construction asserts that $[\xi]$ generates $K(S^n)$.

Chapter 2

Dirac Operators

2.1 Introduction

In $\mathbb{R}^n$, the Dirac operator is

$$D = \gamma_i \frac{\partial}{\partial x_i}$$

where $\gamma_i = c(e_i)$ are the Γ-matrices ($\{e_i\}$ are the canonical basis of $\mathbb{R}^n$). D acts on the spinor fields

$$f : \mathbb{R}^n \to \Delta_n,$$

which are vector valued functions. We now generalize it to manifold setting.

Let (M^n, g) be a Riemannia manifold. For any $x \in M$, (T_xM, g_x) is a vector space equipped with a quadratic form g_x, and therefore we can form the corresponding Clifford algebra bundle $Cl(T_xM, g_x)$.

Following the construction in $\mathbb{R}^n$, one choose an orthonormal basis $\{e_i\}$ on an open neighborhood U and then set $\gamma_i = c(e_i)$ to be the matrix (spin) representation. Also, one naturally replaces $\frac{\partial}{\partial x_i}$ by ∇_{e_i} on a manifold. Thus, intuitively, the Dirac operator should now be

$$D = \gamma_i \nabla_{e_i} = c(e_i)\nabla_{e_i}$$

acting on spinor fields $\psi : \ U \to \Delta_n$. The tricky part here is that the spinor fields depend on the choice of the basis through the identification

$$Cl(T_xM, g_x) \cong Cl_n = \begin{cases} \mathrm{End}(\Delta_n), & n \text{ even}, \\ \mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n), & n \text{ odd}. \end{cases}$$

Thus, one needs to study the "coordinate (or rather, basis)" dependence in this construction.

Thus, let $\{e'_i\}$ be a local orthonormal basis on another open neighborhood V. Then on the overlap, they are related by

$$e'_i(x) = e_j(x)\Omega_{ji}(x), \quad \forall x \in U \cap V,$$

with $\Omega = (\Omega_{ji}) : \ U \cap V \to SO(n)$ is the transition function. Since $\{e_i\}, \{e_i'\}$ (over $U \cap V$) generate the same Clifford algebra, their matrix representations are equivalent. i.e. there exists $S(x)$ such that

$$S\gamma_i S^{-1} = \gamma_i' = \gamma_j \Omega_{ij}.$$

Thus, for Spinor field $\psi(x)$ for γ and spinor field $\psi'(x)$ for γ' to be defined consistently, they should be related by

$$\psi'(x) = S(x)\psi(x).$$

In other words, under coordinate (frame) change $\Omega(x)$, the spinor field should transform by $S(x)$. Note that $S(x)$ are lifts of $\Omega(x)$:

$$\rho(S(x)) = \Omega(x), \quad \rho : Spin(n) \to SO(n).$$

It is now clear how one should proceed. On the Riemannian manifold (M, g), one chooses local orthonormal frames which are related by transition functions. One then lifts the transition functions from $SO(n)$ to $Spin(n)$ and use that to define spinor fields on M. It turns out that there might be some problem with consistent choice of lifts S so that the spinor fields are well defined on the manifold. The obstruction here is the cocycle condition and therefore a topological one. This leads us to the notion of spin structure.
Remark One might wonder why it took so long for such an important concept to be fully recognized. Part of the reason may be that this is not naturally associated with the tangent bundle.

2.2 Spin structures

In general, let $E \to M$ be an oriented real vector bundle of rank k with fiberwise metric $\langle \ , \ \rangle$. Its oriented orthonormal frame bundle $P(E)$ is a principal $SO(k)$-bundle: $SO(k) \to P(E) \to M$. Locally $P(E)$ is described by the transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(k),$$

where $\{U_\alpha\}$ is an open covering of M.

Note that E is then the associated vector bundle via the canonical representation $\rho_0 = id : SO(k) \to SO(k) \subset GL(k, \mathbb{R})$,

$$E = P \times_{\rho_0} \mathbb{R}^k = P \times \mathbb{R}^k / \sim$$

with $(p \cdot g, v) \sim (p, \rho(g)v)$. Locally, transition functions for E are obtained via

$$\varphi_{\alpha\beta} : \begin{tikzcd} U_\alpha \cap U_\beta \arrow[rr] \arrow[dr, "g_{\alpha\beta}"'] & & GL(k, \mathbb{R}) \\ & SO(k) \arrow[ur, "\rho_0"'] & \end{tikzcd}$$

A very important condition that transition function satisfies is the cocycle condition:

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma.$$

Such a collection $\{g_{\alpha\beta}\}$ defines a Cech cocycle $[g_{\alpha\beta}] \in H^1_{Ceck}(M, SO(k))$. In fact, $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ defines the same principal bundle iff $[\{g_{\alpha\beta}\}] = [\{g'_{\alpha\beta}\}]$, i.e. $H^1(M; SO(k))$ classifies all principal $SO(k)$-bundle. Same thing holds when $SO(k)$ is replaced by Lie group G.

Now $\rho : Spin(k) \to SO(k)$ is a homomorphism with $\ker\rho = \{-1, 1\}$, i.e. a double cover. Hence with a good choice of $\{U_\alpha\}$, e.g. a good cover where all intersections are simply connected,

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(k)$$

can always be lifted to $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(k)$. There is only one problem: cocycle condition! Since $\rho(\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}) = g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$, we have

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = \pm 1.$$

Thus, the cocycle condition can fail for the lifts $\{\tilde{g}_{\alpha\beta}\}$.
Definition: A spin structure on $E \to M$ is a collection of lifts $\{\tilde{g}_{\alpha\beta}\}$ such that the cocycle condition holds

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = 1.$$

E is said to be spin of there is a spin structure on E.

To measure the failure of cocycle condition in general, define

$$\omega = \{\omega_{\alpha\beta\gamma}\} : \omega_{\alpha\beta\gamma} = \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z}_2. \tag{2.1}$$

This defines a Čech cocycle: $\delta\omega = 1$ where

$$(\delta\omega)_{\alpha\beta\gamma\delta} = \omega_{\alpha\beta\gamma}[\omega_{\beta\gamma\delta}]^{-1}\omega_{\alpha\gamma\delta}[\omega_{\alpha\beta\delta}]^{-1}.$$

Therefore $[\omega] \in H^2(M, \mathbb{Z}_2)$.
Definition: $w_2(E) = [\omega] \in H^2(M, \mathbb{Z}_2)$ is called the second Stiefel-Whitney class of E.

Theorem 2.2.1 *1) $w_2(E)$ is independent of the choice of the lifts. 2) E has a spin structure iff $w_2(E) = 0$. 3) If $w_2(E) = 0$, then distinct spin structures are parametrized by $H^1(M; \mathbb{Z}_2)$: $\{$distinct spin structures on $E\} \cong H^1(M; \mathbb{Z}_2)$.*

Proof: 1) To show that $w_2(E)$ is independent of choice of lifts, let $\tilde{g}'_{\alpha\beta}$ be a different lift, then $\tilde{g}'_{\alpha\beta} = f_{\alpha\beta}\tilde{g}_{\alpha\beta}$ with $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}_2$. So $\omega' - \omega = \delta\{f_{\alpha\beta}\}$ and $[\omega'] = [\omega] \in H^2(M, \mathbb{Z}_2)$.

For 2), we note that one way is trivial: the existence of a spin structure obviously implies $w_2(E) = 0$. Conversely, if $w_2(E) = 0$, then ω is a coboundary, $\omega = \delta\{f_{\alpha\beta}\}$ with

$$f_{\alpha\beta} :\ U_{\alpha\beta} \to \mathbb{Z}_2.$$

We can then modify the original lifts $\tilde{g}_{\alpha\beta}$ by $f_{\alpha\beta}$: $\tilde{g}'_{\alpha\beta} = f_{\alpha\beta}\tilde{g}_{\alpha\beta}$. Since $\omega = \delta\{f_{\alpha\beta}\}$, the modified lifts $\tilde{g}'_{\alpha\beta}$ clearly satisfies the cocycle condition.

3) Roughly speaking, each $[\{f_{\alpha\beta}\}] \in H^1(M, \mathbb{Z}_2)$ gives rise to a different modification. ■

Remark E is orientable iff $w_1(E) = 0 \in H^1(M, \mathbb{Z}_2)$.

Exercise: define $w_1(E)$ by studying the orientability.

Definition: An orientable manifold M is called spin if TM is spin.

Remark By a theorem of Wu, $w_i(M)$ are all homotopy invariants.

Examples of spin manifolds: clearly, all manifolds which have a good cover with at most two members are spin. Here by good cover we mean a cover by contractible open sets all of whose intersections are simply connected (note that we do not require contractible intersections as is usually the case, since we only need simply connectedness to ensure the local lifts). Examples include the Euclidean space and all contractible manifolds, and the spheres. Also, products of spin manifolds are still spin. Some additional examples are as follows.

1) All orientable surfaces.

2) All oriented 3-manifolds since they are parallelizable.

3) All Lie groups since they are also parallelizable.

4) A complex manifold X is spin iff $c_1(X) \equiv 0 \pmod 2$ since $w_2(X) \equiv c_1(X) \mod 2$.

5) $\mathbb{RP}^n$ spin iff $n \equiv 3 \mod 4$; $\mathbb{CP}^n$ spin iff n odd ($n \equiv 1 \mod 2$); $\mathbb{HP}^n$ always spin (total Stiefel-Whitney class $w = (1+g)^{n+1}$, where g is the generator of cohomology ring in dim 1,2,4 respectively).

6) The two (distinct) spin structures of S^1: cover S^1 by two intervals intersecting at two intervals U_+ and U_-. Since the lifts can always be modified by a global $\mathbb{Z}_2$ function, we can always choose one of the lifts, say $\tilde{g}_+ = 1$. This leaves us with two possible choices for $\tilde{g}_-$: ± 1. The choice of $\tilde{g}_- = 1$ corresponds to the periodic boundary condition on the interval and is called Ramond sector in string theory. The other choice, $\tilde{g}_- = -1$, corresponds to the anti-periodic boundary condition on the interval and is called Neveu-Schwarz sector. Note that the Ramond sector is actually the nontrivial spin structure since it does not extend to the disk. The Neveu-Schwarz sector is the trivial spin structure.

7) $T^2 = S^1 \times S^1$ has 4 spin structures coming from different combinations of the spin structures on each S^1 factor. These are referred as R-R, NS-NS, R-NS, NS-R respectively in string thery. Mathematically, $H^1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. More generally, for Σ_g a compact Riemannian surface of genus g, $c_1(\Sigma_g) = \chi(\Sigma_g) = 2 - 2g \equiv 0 \mod 2$. It has $|H^1(\Sigma_g, \mathbb{Z}_2)| = |Hom(\pi_1(\Sigma_g), \mathbb{Z}_2)| = |\mathbb{Z}_2^{2g}| = 2^{2g}$ spin structures. They can be described as follows:

Let $H^1(\Sigma_g; \theta^*)$ be the Cech cohomology of the sheaf θ^* of nowhere vanishing holomorphic functions on Σ_g, which is the equivalent classes of holomorphic $\mathbb{C}$-line bundles on Σ_g. $H^1(\Sigma_g; \theta^*)$ is an abelian group and the group operation coincides with tensor product of $\mathbb{C}$-line bundle. One has an exact sequence

$$0 \to J \to H^1(\Sigma_g; \theta^*) \xrightarrow{c_1} H^2(\Sigma_g; \mathbb{Z}) \to 0,$$

where $J \cong H^1(\Sigma_g;\mathbb{R})/H^1(\Sigma_g;\mathbb{Z}) \cong T^{2g}$, the Jacobian variety, which describe different holomorphic on a topological line bundle.

Let $\tau_0 = [T\Sigma_g] = [K^{-1}_{\Sigma_g}] \in H^1(\Sigma_g;\theta^*)$.

1) There are exactly $2^{2g}\tau \in H^1(\Sigma_g;\theta^*)$ such that $\tau^2 = \tau_0$.

2) Each such τ determines a distinct spin structure on Σ_g since the natural map $\tau \to \tau^2$ is of the form $z \to z^2$ in the fibres and therefore restricts to $P(\tau) \to P(\tau^2)$ the principal S^1-bundles, and in the fibre, is $z \to z^2$ for $S^1 \to S^1$.

We say the spin structures of Σ_g are parametrized by holomorphic square roots $K^{1/2}_{\Sigma_g}$ of K_{Σ_g}.

More generally, if X is a compact Kahler manifold with $c_1(X) \equiv 0 (\mod 2)$, then the spin structures on X are parametrized by holomorphic square roots $K^{1/2}_X$ of the canonical (line) bundle K_X.

The definition for the spin structure is intuitive but less formal. One can formalize it using the language of principal bundles.

Definition: A spin structure on E is a principal $Spin(k)$ bundle P_{Spin} together with a two-sheeted covering map

$$\xi : \; P_{Spin} \to P_{SO}$$

such that $\xi(p \cdot g) = \xi(p) \cdot \rho(g)$ for any $p \in P_{Spin}$, $g \in Spin(k)$. I.e., on each fiber, ξ restricts to the nontrivial double cover $\rho : \; Spin(k) \to SO(k)$.

2.3 Spin^c-structures

All oriented manifolds of dim ≤ 3 are spin. This is not true anymore in dimension 4, e.g. $\mathbb{C}P^2$ is not spin. It turns out that there is important generalization of spin structure called *spin^c* structure which can be viewed as the complex analog of spin structure and exists on all almost complex manifolds.

To motivate the definition, recall that in the case when the lift $\tilde{g}_{\alpha\beta}$ to the *Spin* group of the transition function $g_{\alpha\beta}$ does not satisfy the cocycle condition, but rather $\omega_{\alpha\beta\gamma} = \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}$ is a coboundary, $\omega_{\alpha\beta\gamma} = \delta\{\tilde{f}_{\alpha\beta}\}$, with

$$\tilde{f}_{\alpha\beta} : \; U_{\alpha\beta} \to \mathbb{Z}_2,$$

one can modify the lifts by $\tilde{f}_{\alpha\beta}$ so that $\tilde{g}'_{\alpha\beta} = \tilde{g}_{\alpha\beta}/\tilde{f}_{\alpha\beta}$ satisfies the cocycle condition. This makes one wonder whether there are more general modifications, even if $w_2 \neq 0$, by suitably valued $\tilde{f}_{\alpha\beta}$ so that the cocycle condition holds. Of course $\tilde{f}_{\alpha\beta}$ can not be $\mathbb{Z}_2$-valued now. It turns out that $S^1 \subset \mathbb{C}$ valued $\tilde{f}_{\alpha\beta}$ can still make it work, being more general than $\mathbb{Z}_2$-valued while still freely commute with other factors. This leads to the so called *Spin^c* structure.

Thus, a *Spin^c* structure consists of lifts $\tilde{g}_{\alpha\beta}$ together with

$$\tilde{f}_{\alpha\beta} : \; U_{\alpha\beta} \to S^1 \subset \mathbb{C}$$

such that

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = \tilde{f}_{\alpha\beta}\tilde{f}_{\beta\gamma}\tilde{f}_{\gamma\alpha}.$$

To see what kind of geometric object such $\tilde{f}_{\alpha\beta}$ corresponds, we note that its square

$$f_{\alpha\beta} = \tilde{f}_{\alpha\beta}^2 : \; U_{\alpha\beta} \to S^1$$

does satisfy the cocycle condition and therefore defines a complex line bundle $L \to M$ with $c_1(L)$ mod 2 precisely the Čech cocycle defined by

$$\tilde{f}_{\alpha\beta}\tilde{f}_{\beta\gamma}\tilde{f}_{\gamma\alpha} : \; U_{\alpha\beta\gamma} \to \mathbb{Z}_2.$$

It follows then that such "modifying line bundle" satisfies

$$w_2(E) \equiv c_1(L) \text{ mod } 2, \tag{3.2}$$

and the corresponding modifications define a so-called *spin*^{c} structure.

To give a precise definition, we again use the formal language of principal bundles.

Definition: [Spin^c group] $\text{Spin}^c(n) \equiv \text{Spin}\,(n) \times U(1)/\{(-1,-1)\} = Spin(n) \times_{\mathbb{Z}_2} U(1) \subset \mathbb{C}l_n = Cl_n \otimes \mathbb{C}$.

Hence we have a short exact sequence

$$1 \to \mathbb{Z}_2 \to Spin^c(n) \to SO(n) \times U(1) \xrightarrow{\rho^c} 1, \tag{3.3}$$

where

$$\rho^c([\varphi, z]) = (\rho(\varphi), z^2). \tag{3.4}$$

Definition: [Spin^c structure] A Spin^c structure for a real oriented vector bundle $E \to M$ consists of a principal Spin$^c(n)$-bundle $P_{Spin^c(n)}$, a principal $U(1)$-bundle $P_{U(1)}$ (or equivalently a complex line bundle), and an equivariant bundle map

$$P_{Spin^c(n)} \to P_{SO(n)} \times P_{U(1)}.$$

The class $c = c_1(P_{U(1)}) \in H^2(M;\mathbb{Z})$ is called the canonical class of the Spin^c-structure.

Theorem 2.3.1 *E has a Spin^c-structure iff $w_2(E) \equiv c \mod 2$ for some $c \in H^2(M,\mathbb{Z})$.*

Remark $H^2(M,\mathbb{Z}) \cong H^1(M,S^1)$ classifies all principal $U(1)$-bundle (or equivalently $\mathbb{C}$-line bundles).

Example: If E is spin, it has canonical Spin^c structure defined by $P_{Spin^c} = P_{spin} \times U(1)/\mathbb{Z}_2$.

Remark A $Spin^c(n)$-equivariant bundle map $P_{Spin^c(n)} \to P_{SO(n)} \times P_{U(1)}$ is equivalent to the existence of local lifts

$$\begin{array}{rccc} [\tilde{g}_{\alpha\beta}, \tilde{f}_{\alpha\beta}] : & U_{\alpha\beta} & \longrightarrow & Spin^c(n) \\ & || & & \downarrow \rho^c \\ & U_{\alpha\beta} & \longrightarrow & SO(n) \times U(1) \end{array}$$

satisfying the cocylce condition. In other words,

$$\rho(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta}, \quad \tilde{f}_{\alpha\beta}^2 = f_{\alpha\beta},$$

and

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = \tilde{f}_{\alpha\beta}\tilde{f}_{\beta\gamma}\tilde{f}_{\gamma\alpha}.$$

Definition: M is Spin^c if TM has Spin^c structure.
Example: 1) Almost $\mathbb{C}$-manifolds are Spin^c: $w_2(X) \equiv c_1(X) \mod 2$.

2) All orientable 4-manifolds are Spin^c. (This uses Wu's formula $x^2 = w_2 \cup x,\ \forall x \in H^2(M^4;\mathbb{Z}_2)$ and Poincaré duality and universal coefficients theorem.)

3) The canonical homomorphism

$$\begin{array}{rccc} j: & U(n) & \to & Spin^c(2n) \\ & || & & \downarrow \rho^c \\ & U(n) & \to & SO(2n)\times U(1) \\ & g & \to & (g^{\mathbb{R}}, \det g) \end{array} \tag{3.5}$$

defines a canonical spin^c structure on an almost complex manifold. Here j is defined as follows. For $g \in U(n)$, choose orthonormal basis e_i of $\mathbb{C}^n$ such that $g = diag(e^{i\theta_1}, \cdots, e^{i\theta_n})$. Then e_i, Je_i is a basis for the real vector space $\mathbb{R}^{2n}$ (J is the complex structure). Now

$$j(g) = \Pi_{k=1}^n(\cos\frac{\theta_k}{2} + \sin\frac{\theta_k}{2}e_kJe_k) \times e^{\frac{i}{2}\sum\theta_k}. \tag{3.6}$$

Remark The question of more general type of "modifications" leads us to the Clifford module.

2.4 Spinors

Let M^n be spin, $P_{Spin(n)}$ the principal Spin (n)-bundle, $\rho_n : Spin(n) \subset Cl_n \subset \mathbb{C}l_n \to End\Delta_n$ the spin representation.
Definition: [$\mathbb{C}$-spinor bundles] The complex spinor bundle $S \to M$ is the associated vector bundle

$$S = P_{spin} \times_{\rho_n} \Delta_n.$$

Remark If $P_G \to M$ is a principal G-bundle, $\rho : G \to GL(W)$ is a representation, then $P_G \times_\rho W = P_G \times W/\sim$, where $(p\cdot g, s) \sim (p, \rho(g)s)$.

In terms of transition functions $\{g_{\alpha\beta}\}$ of P_G: $g_{\alpha\beta} : U_{\alpha\beta} \to G$, the transition functions $\{f_{\alpha\beta}\}$ of the associated vector bundle is $f_{\alpha\beta} = \rho(g_{\alpha\beta}) : U_{\alpha\beta} \to GL(W)$.

For example, if $P_{SO(n)}$ is the oriented orthonormal frames of M, $\rho : SO(n) \to SO(n) \subset GL(\mathbb{R}^n)$ is the standard representation, then $TM = P_{SO(n)} \times_\rho \mathbb{R}^n$. Similarly for exterior bundle, tensor bundle.
Definition: [Clifford algebra bundle] The Clifford algebra bundle $Cl(M) = \cup_{x\in M} Cl(T_xM, g(x))$ is a bundle of Clifford algebras. Note: This is a bundle associated with $P_{SO(n)}$!

Fiberwise Clifford multiplication gives

$$\begin{array}{rcl} C^\infty(T^*M \otimes S) & \to & C^\infty(S) \\ v \otimes s & \to & c(v)s. \end{array}$$

Definition: [Spinor fields] Sections of S are called spinors (or spinor fields) on M.

When n is even, we have the $\frac{1}{2}$-spin representations $\rho_n^{\pm} : Spin(n) \to GL(\Delta_n^{\pm})$ with $\Delta_n = \Delta^+ \oplus \Delta^-$. Therefore we have the chiral spinor bundles $S^{\pm}$ and the decomposition $S = S^+ \oplus S^-$.

We can also define the complex volume element $\omega_{\mathbb{C}}$ which is a smooth section of the Clifford algebra bundle. In fact, $\omega_{\mathbb{C}} = (\sqrt{-1})^{[\frac{n+1}{2}]} e_1 \cdots e_n$ for any positively oriented (local) orthonormal frame e_i of TM. Clearly, $S^{\pm}$ is the ± 1-eigenbundle of $c(\omega_{\mathbb{C}}) = (\sqrt{-1})^{[\frac{n+1}{2}]} c(e_1) \cdots c(e_n)$:

$$S^{\pm} = \{ s \in S \mid c(\omega_{\mathbb{C}}) s = \pm s \}. \tag{4.7}$$

Similarly, for M (or more generally, E) spin^c, we define the spin^c spinor bundle S^c via $S^c = P_{Spin^c} \times_{\rho_n^c} \Delta_n$, where $\rho_n^c : Spin^c(n) \subset Cl_n \otimes \mathbb{C} = \mathbb{C}l_n \to \mathrm{End}\Delta_n$.

Remark Note that $\rho_n^c : Spin^c(n) \to GL(\Delta_n)$ is induced by

$$\begin{aligned} \mathbb{C}l_n = \quad & Cl_n \otimes \mathbb{C} \to \quad \mathrm{End}\Delta_n \\ & e_i \otimes \lambda \to \quad \lambda c(e_i). \end{aligned}$$

Therefore, the transition functions for the $Spin^c(n)$ spinor bundle are given by $\rho_n^c[\tilde{g}_{\alpha\beta}, \tilde{f}_{\alpha\beta}] = \rho_n(\tilde{g}_{\alpha\beta})\tilde{f}_{\alpha\beta}$. It follows that if E is also spin, then the spin^c spinor bundle and the spin spinor bundle are related by

$$S^c = S \otimes L^{1/2}, \tag{4.8}$$

where L is the the complex line bundle associated to the given spin^c structure and $L^{1/2}$ is its square root (which exists as a genuine line bundle by the assumptions).

We also note the following.

1). If M is spin, then M is spin^c with the trivial canonical class. In this case, $S^c = S$ (L is trivial).

2). As mentioned above, an almost complex manifold M has a canonical spin^c structure associated with the almost complex structure. I.e., $P_{Spin^c} = P_{U(n)} \times_j Spin^c(2n)$ with j given in (3.5). Now recall that the complex vector space of spinors Δ_{2n} can be taken to be $\Delta_{2n} = \Lambda^* V^{0,1}$ ($V = \mathbb{R}^{2n} \otimes \mathbb{C} = \mathbb{C}^{2n}$). Moreover

$$\rho_{2n} : \ \mathbb{C}l_{2n} \to End(\Delta_{2n})$$

is given by the Clifford multiplication

$$c(v) = \sqrt{2}(v^{0,1} \wedge - \, i_{v^{1,0}}),$$

if $v = v^{1,0} + v^{0,1}$. It follows then that for the canonical spin^c structure of an almost complex manifold M,

$$S^c = \Lambda^* T^{0,1} M = \Lambda^{0,*} M. \tag{4.9}$$

3). If M is both spin and almost complex, then by the above discussion,

$$S^c = S \otimes K_M^{-1/2}. \tag{4.10}$$

Or,

$$S = \Lambda^{0,*}M \otimes K_M^{1/2}.$$

2.5 Spin connections

Let (M^n, g) be an oriented Riemannian manifold, ∇ its Levi-Civita connection. As we know, this means two things: 1). ∇ is compatible with the metric g; 2). ∇ is torsion free. In a local coordinate x^i, ∇ is given by the Christoffel symbol

$$\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma^k_{ij} \partial_{x^k}.$$

Define 1-forms (in the coordinate neighborhood) $\tilde{\omega}^k_j = \Gamma^k_{ij} dx^i$, then $\nabla \partial_{x^j} = \tilde{\omega}^k_j \partial_{x^k}$. In other words, one can also write

$$\nabla : \ C^\infty(TM) \to C^\infty(T^*M \otimes TM). \tag{5.11}$$

Now, choose a (local) orthonormal basis $e_1, \cdots, e_n$ of TM, then $\nabla e_i = \tilde{\omega}_{ij} e_j$, i.e., $\nabla_X e_i = \tilde{\omega}_{ij}(X) e_j$ (we will not distinguish upper and lower indices in an orthonormal basis and we supress the $\otimes$ sign here). Then the fact that the connection is compatible with the metrics means precisely that $\tilde{\omega}_{ij}$ is skew-symmetric: $\tilde{\omega}_{ij} = -\tilde{\omega}_{ji}$. Thus we can package them into a skew-symmetric matrix with entries differential 1-forms: $\tilde{\omega} = (\tilde{\omega}_{ij}) \in so(n)$. The $so(n)$-valued 1-form $\tilde{\omega}$ still depends on the choice of orthonormal frame e_i (which accounts for the “tilde” here). One obtains a globally defined 1-form by passing to the orthonormal frame bundle $P_{SO(n)}$.

Definition: An affine (spin) connection is a $so(n)$-valued 1-form ω on $P_{so(n)}$, such that

1) $\omega(X_P) = X, \ \forall X \in so(n)$.

2) $g^*\omega = Ad_{g^{-1}}(\omega), \ \ \forall g \in SO(n)$.

Here X_P denotes the vector field on $P_{so(n)}$ induced by the right action of $\exp tX \in SO(n)$. The curvature of ω is the $so(n)$-valued 2-form $\Omega = d\omega + \frac{1}{2}[\omega, \ \omega]$.

An orthonormal frame on $U \subset M$ defines a section $s : U \to P_{so(n)}$. Then $\tilde{\omega} = s^*\omega$ is the 1-form discussed at the beginning.

More generally, this definition works for any principal bundle $P \to M$ with structure group G. One just replaces $SO(n)$ by G and $so(n)$ by the Lie algebra $\mathbf{g}$ of G. So a connection is now a $\mathbf{g}$-valued 1-form on P, with its curvature $\mathbf{g}$-valued 2-form.

A connection on the principal bundle naturally induces connections on its associated vector bundles. Thus if $E = P \times_\rho W$ where $\rho : \ G \to GL(W)$ is the representation, then $\rho_*(\omega)$, where $\rho_* : \ \mathbf{g} \to gl(W)$, defines a connection on the frame bundle $P_{GL(E)}$ (here of course we allow all frames, not just orthonormal ones. Strictly speaking, one needs to use the bundle map $P \to P_{GL(E)} = P \times_\rho GL(W)$, Cf. [KN]). Moreover the induced connection has curvature $\rho_*(\Omega)$.

This has the following concrete description in a local frame. A local section $s: U \to P$ gives a trivialization of P, which in turn induces a local frame $v_1, \cdots, v_m$ for E (via $P_{GL(E)}$). Then we have $\nabla v_i = s^*(\rho_* \omega) v_i$.

Proposition 2.5.1 *If M is spin, any connection on $P_{SO(n)}$ (or equivalently, any connection on M compatible with the metric) naturally induces a connection on $P_{Spin(n)}$, which in turn gives a covariant derivative on the spinor bundle*

$$\nabla^S : C^\infty(S) \to C^\infty(T^*M \otimes S).$$

In a local orthonormal frame e_i and local coordinates x,

$$\nabla^S_{\partial_{x^k}} \psi = \partial_{x^k} \psi + \frac{1}{4} \tilde{\omega}_{ij}(\partial_{x^k}) c(e_i) c(e_j) \psi. \tag{5.12}$$

The induced connection is compatible with the Clifford multiplication

$$\nabla^S_X (c(v)\psi) = c(\nabla_X v)\psi + c(v) \nabla^S_X \psi. \tag{5.13}$$

Remark Note that

$$\frac{1}{4} \tilde{\omega}_{ij} c(e_i) c(e_j) = \frac{1}{8} \tilde{\omega}_{ij} (c(e_i)c(e_j) - c(e_j)c(e_i)) = \frac{1}{8} \tilde{\omega}_{ij} [c(e_i),\ c(e_j)].$$

For physicists, $c(e_i) = \gamma_i$ are the Γ-matrices, and so the connection is usually written as $\frac{1}{8}\tilde{\omega}_{ij}[\gamma_i,\ \gamma_j]$.

Remark If we define as before

$$e_i \wedge e_j : TM \to TM$$

via $(e_i \wedge e_j)(v) = \langle e_i,\ v \rangle e_j - \langle e_j,\ v \rangle e_i$, then the connection 1-form of a P_{SO} connection can be written as $\omega = \frac{1}{2} \omega_{ij} e_i \wedge e_j$. And the covariant derivative by the connection has the local form

$$\nabla_{\partial_{x^k}} X = \partial_{x^k} X + \frac{1}{2} \tilde{\omega}_{ij}(\partial_{x^k})(e_i \wedge e_j)(X).$$

Proof: Since M is spin, we have a principal $Spin(n)$ bundle P_{Spin} together with the double covering map $\xi : P_{Spin(n)} \to P_{SO(n)}$. Thus if ω is the connection 1-form on $P_{SO(n)}$, then $\rho_*^{-1}(\xi^*\omega)$ defines a connection 1-form on $P_{Spin(n)}$, where ρ_* is the isomorphism from $so(n)$ to the Lie algebra of $Spin(n)$ whose inverse is computed in Lemma 1.3.3. (5.12) follows from Lemma 1.3.3. To verify (5.13), we do it locally via (5.12):

$$\begin{aligned}
\nabla^S_X (c(v)\psi) &= X(c(v)\psi) + \frac{1}{4}\tilde{\omega}_{ij}(X) c(e_i) c(e_j) c(v) \psi \\
&= c(v)\nabla^S_X(\psi) + c(Xv)\psi + \frac{1}{4}\tilde{\omega}_{ij}(X)(-2c(e_i)\langle e_j,\ v\rangle + 2\langle e_i,\ v\rangle c(e_j))\psi \\
&= c(v)\nabla^S_X(\psi) + c(Xv + \frac{1}{2}\tilde{\omega}_{ij}(X)(e_i \wedge e_j)(v))\psi \\
&= c(v)\nabla^S_X(\psi) + c(\nabla_X v)\psi.
\end{aligned}$$

■

Recall that $\omega_{\mathbb{C}} = (\sqrt{-1})^{[\frac{n+1}{2}]} e_1 \cdots e_n$ is the complex volume element. The following lemma says that $\omega_{\mathbb{C}}$ is parallel with respect to ∇^S.

Lemma 2.5.2 *We have* $\nabla^S(c(\omega_{\mathbb{C}})\psi) = c(\omega_{\mathbb{C}})\nabla^S\psi$.

Proof: For any $p \in M$, we once again work in the orthonormal frame e_i such that $\nabla e_i = 0$ at p. Then, using (5.13), we compute at p,

$$\nabla^S(c(\omega_{\mathbb{C}})\psi) = (\sqrt{-1})^{[\frac{n+1}{2}]} c(e_1)\nabla^S(c(e_2)\cdots c(e_n)\psi) = \cdots = c(\omega_{\mathbb{C}})\nabla^S\psi.$$

■

Corollary 2.5.3 *When n is even,* ∇^S *is compatible with the splitting* $S = S^+ \oplus S^-$. *In other words,* ∇^S *is diagonal in this decomposition.*

Remark The whole discussion extends without change to a vector bundle $E \to M$ which is spin.
Remark If E (or M) is spin^c, then we have $\xi^c : P_{Spin^c} \to P_{SO} \times P_{U(1)}$. In this case, any connection on P_{SO} together with a connection on $P_{U(1)}$ determines a connection on S^c.

2.6 Dirac operators

Let (M^n, g) be a spin/spin^c manifold with spinor bundle S.
Definition: [Dirac operator] The Dirac operator D

$$D : C^\infty(S) \to C^\infty(S)$$

is the composition

$$C^\infty(S) \xrightarrow{\nabla^S} C^\infty(T^*M \otimes S) \xrightarrow{c} C^\infty(S),$$

where c is the Clifford multiplication. In local orthonormal basis $\{e_i\}$,

$$D = c(e_i)\nabla^S_{e_i}. \tag{6.14}$$

Here we identify e_i^* with e_i using Riemannian metric.

For example, in $\mathbb{R}^n$, $D = \pm i\frac{\partial}{\partial x}$ when $n = 1$ and $D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ when $n = 2$.

In local coordinates

$$D = c(dx_i)\nabla^S_{\partial_i} = c(dx^i)(\partial_i + \frac{1}{4}\tilde{\omega}_{ijk}(\partial_i)c(e_j)c(e_k)).$$

Therefore it is a first order differential operator.

In general, $P : C^\infty(E) \to C^\infty(F)$, where E, F are vector bundles, is called a differential operator of order m, if, in local coordinates,

$$P = \sum_{|\alpha|\le m} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \sum_{|\alpha|\le m} A_\alpha(x)\partial_x^\alpha, \tag{6.15}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and $A_\alpha(x) : E_x \to F_x$ is a linear map.

The principal symbol of P is defined as follows. For any $\xi \in T_x^*M$, $\xi = \xi_k dx^k$,

$$\sigma_\xi(P) = (\sqrt{-1})^m \sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha : \ E_x \to F_x, \tag{6.16}$$

is a linear map. Here $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. In other words, $\sigma_\xi(P)$ is obtained from the leading order terms of P by replacing ∂_{x^i} by $\sqrt{-1}\xi_i$, which is really a Fourier transform. (The choice of cotangent vector is so that $\sigma_\xi(P)$ will be independent of the local coordinates.) The differential operator P is said to be elliptic if $\sigma_\xi(P)$ is an isomorphism for all $\xi \ne 0 \in T_x^*M$ and all $x \in M$.

Lemma 2.6.1 *The Dirac operator D is elliptic. In fact, $\sigma_\xi(D) = \sqrt{-1}c(\xi)$ is (up to a multiple of $\sqrt{-1}$) the Clifford multiplication.*

Proof: This is clear from the local description of D. ■

We also need to put a metric on S. In order to get a metric which are compatible with the existing structures, we seek to make the representation

$$\rho_n : \ Spin(n) \to GL(\Delta_n)$$

unitary.

Lemma 2.6.2 *There exists an hermitian inner product (,) on Δ_n such that for any unit vector $e \in \mathbb{R}^n$, and $v, w \in \Delta_n$,*

$$(c(e)v, \ c(e)w) = (v, \ w).$$

Equivalently

$$(c(e)v, \ w) = -(v, \ c(e)w).$$

Proof: Let e_i be the standard orthonormal basis of $\mathbb{R}^n$. Take any hermitian inner product $(\, , \,)_0$ on Δ_n, we use the trick of averaging to get an hermitian inner product (,) such that

$$(c(e_i)v, \ c(e_i)w) = (v, \ w) \tag{6.17}$$

for $i = 1, \cdots, n$ and all $v, w \in \Delta_n$. To see this, construct a finite subgroup $\Gamma = \{\pm 1, \pm e_1, \cdots, \pm e_n, \cdots, \pm e_{i_1}e_{i_2} \cdots e_{i_k} \ (i_1 < \cdots < i_k), \cdots, \pm e_1 \cdots e_n\} \subset Cl_n^\times$ and set

$$(v, \ w) = \sum_{g\in\Gamma} (c(g)v, \ c(g)w)_0.$$

Clearly $(c(g)v,\ c(g)w) = (v,\ w)$ for any $g \in \Gamma$, since the insertion of $c(g)$ in the right hand side of the definition above simply results in a rearrangement of the group elements. Thus (6.17) holds. It follows then $(c(e)v,\ c(e)w) = (v,\ w)$ for any unit vector e.
Remark The last step uses the fact the metric on $\mathbb{R}^n$ is Euclidean which fails for indefinite metrics.

Proposition 2.6.3 *The hermitian inner product on Δ_n as above induces an hermitian metric $(\ ,\ )$ on the spinor bundle S such that for any unit vector field e,*

$$(c(e)\psi,\ \psi') = -(\psi,\ c(e)\psi'). \tag{6.18}$$

Moreover, it is compatible with the connection

$$X(\psi,\ \psi') = (\nabla^S_X\psi,\ \psi') + (\psi,\ \nabla^S_X\psi'). \tag{6.19}$$

Proof: The first part is clear from the previous lemma. The second part follows from the fact that a connection is compatible with the metric iff its connection form is $u(k)$ valued ($k = 2^{[n/2]}$ is the rank of S). By the previous lemma, the representation is now in the unitary group $U(k)$. Hence its derivative takes value in $u(k)$. ■

Proposition 2.6.4 *With respect to $(\ ,\ )$, D is formally self adjoint, i.e.,*

$$\int_M (D\psi,\ \psi')dvol = \int_M (\psi,\ D\psi')dvol$$

for all $\psi, \psi' \in C_0^\infty(S)$.

Proof: Once again, we fix a point $p \in M$ and work with a local orthonormal frame e_i such that $\nabla e_i = 0$ at p. Then, at p,

$$\begin{aligned}
(D\psi,\ \psi') &= (c(e_i)\nabla^S_{e_i}\psi,\ \psi') \\
&= -(\nabla^S_{e_i}\psi,\ (c(e_i)\psi') \\
&= -e_i(\psi,\ c(e_i)\psi') + (\psi,\ \nabla^S_{e_i}(c(e_i)\psi')) \\
&= (\psi,\ D\psi') - e_i(\psi,\ c(e_i)\psi').
\end{aligned}$$

The second term looks like a divergence term, and it is. Define a vector field X on M by

$$\langle X,\ Y\rangle = -(\psi,\ c(Y)\psi')$$

for all vector fields Y. Then (again computing at a fixed arbitrary point p)

$$\begin{aligned}
\mathrm{div} X &= \langle\nabla_{e_i}X,\ e_i\rangle \\
&= e_i\langle X,\ e_i\rangle = -e_i(\psi,\ c(e_i)\psi').
\end{aligned}$$

Now the proposition follows from integration by parts. ■

Remark If M is compact but with nonempty boundary ∂M, then the proof above gives the Green's formula

$$\int_M (D\psi,\ \psi')dvol = \int_M (\psi,\ D\psi')dvol + \int_{\partial M} (c(n)\psi,\ \psi')dvol. \tag{6.20}$$

Here n denotes the unit outer normal of ∂M.
Remark Similarly we have a spin^c Dirac operator D_c enjoying all the previous properties, provided one has a spin^c structure.
Remark There is also twisted Dirac operator: if M is spin, $E \to M$ is an hermitian vector bundle with a unitary connection ∇^E, then

$$D_E = c(e_i)(\nabla^S_{e_i} \otimes 1 + 1 \otimes \nabla^E_{e_i}) :\ C^\infty(S \otimes E) \to C^\infty(S \otimes E). \tag{6.21}$$

All properties extend as usual.
Remark As we discussed before, when M is spin, all spin^c structures are determined by the square root $L^{1/2}$ of some line bundle L (the square root exists under the assumption). We have $S^c = S \otimes L^{1/2}$ and $D_c = D_{L^{1/2}}$.
Remark Finally, when M is almost complex, it carries a canonical spin^c structure associated with the almost complex structure, and $S^c = \Lambda^{0,*}M$. When M is Kähler, it can be shown that $D_c = \sqrt{2}(\bar\partial + \bar\partial^*)$.

2.7 Clifford module

From the construction of spinor bundle and Dirac operator, one can extract out the useful idea of Clifford module, which we now discuss.

A complex vector bundle $\mathcal{E} \to M$ is a Clifford module if there is a Clifford multiplication

$$\begin{array}{cccc} c: & C^\infty(T^*M \otimes \mathcal{E}) & \to & C^\infty(\mathcal{E}) \\ & v^* \otimes s & \to c(v^*)s & \end{array} \tag{7.22}$$

satisfying the Clifford relation, i.e.,

$$c(v^*)c(w^*) + c(w^*)c(v^*) = -2g(v^*,\ w^*).$$

A Clifford connection on $\mathcal{E}$ is a connection $\nabla^{\mathcal{E}}$ compatible with the Clifford multiplication:

$$\nabla^{\mathcal{E}}_X(c(v^*)s) = c(\nabla_X v^*)s + c(v^*)\nabla^{\mathcal{E}}_X s.$$

Such a connection always exists (since it exists locally).

Given these, a Dirac operator on $\mathcal{E}$ can be constructed as before, via the composition:

$$\mathcal{D} :\ C^\infty(\mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} C^\infty(T^*M \otimes \mathcal{E}) \xrightarrow{c} C^\infty(\mathcal{E}). \tag{7.23}$$

As before, locally, $\mathcal{D} = c(e_i)\nabla^{\mathcal{E}}_{e_i}$.

If M is spin, it turns out that all Clifford modules are of the form $S \otimes E$ for some vector bundle E ([BGV]), and the corresponding Dirac operator is simply the twisted Dirac operator. But, as the case of spin^c shows, Clifford modules exist even when M is not spin. In fact, a very interesting example here is the exterior algebra bundle.

1) de Rham operator—starting with the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0,$$

we roll up the complex to get $\mathcal{E} = \Lambda^* M = \oplus_{p=0}^n \Lambda^p M$. This can be made into a Clifford module by the Clifford multiplication

$$c(v^*)\alpha = v^* \wedge \alpha - i_{v^*}\alpha,$$

where the contraction $i_{v^*}: \ \Lambda^p M \to \Lambda^{p-1} M$ via the metric. It can be characterized by the following two properties:

(i). If $\alpha \in \Lambda^1 M = T^* M$, then

$$i_{v^*}\alpha = g(v^*, \ \alpha).$$

(ii). It acts as a graded derivation

$$i_{v^*}(\alpha \wedge \beta) = (i_{v^*}\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge (i_{v^*}\beta).$$

Exercise: Show the Clifford relation holds.

The Levi-Civita connection, extended to $\mathcal{E}$, is a Clifford connection (in fact, that is exactly how the connection is extended).

Proposition 2.7.1 *The Dirac operator associated to the Clifford module* $\mathcal{E} = \Lambda^* M$ *and the Levi-Civita connection is*

$$\mathcal{D} = d + d^* : \ \Omega^*(M) \to \Omega^*(M).$$

Proof: The connection being torsion free implies that $d = e^i \wedge \nabla_{e_i}$. This, together with the fact that the connection is also compatible with the metric, implies that $d^* = -i_{e_i}\nabla_{e_i}$ (the adjoint of $e^i \wedge$ is i_{e_i}, and the metricity implies that the adjoint of ∇_{e_i} is $-\nabla_{e_i}$). ■

We note that $\mathcal{D}^2 = dd^* + d^*d = \Delta$ is the Hodge Laplacian. We also note that, while $d + d^*$ does not preserve the degree, Δ does: $\Delta: \ \Omega^p(M) \to \Omega^p(M)$. We will denote by a subscript p the restriction to $\Omega^p(M)$.

Remark The de Rham theorem states that for closed manifold, $H^p(M, \mathbb{R}) \cong \ker d_p / \mathrm{Im} d_{p-1}$. The Hodge theorem does even better: the p-th cohomology is isomorphic to the space of harmonic p-forms, $H^p(M, \mathbb{R}) \cong \ker \Delta_p$. In other word, $\dim \ker \Delta_p$ is the p-Betti number, a topological invariant. (Note $\ker \Delta_p = \ker d_p \cap \ker d^*_{p-1} = \ker d_p \cap (\mathrm{Im} d_{p-1})^\perp$.)

2). The $\bar\partial$-operator—We first recall the definition of an almost complex manifold.

Definition: An almost complex manifold M is a manifold with an almost complex structure. That is, an endomorphism

$$J: \ TM \to TM$$

such that

$$J^2 = -1.$$

It follows that the (real) dimension of M must be even, $\dim M = 2n$.

With the almost complex structure, the complexified tangent bundle decomposes into the $(\pm i)$-eigenbundles of J:

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M, \tag{7.24}$$

which induces

$$T^*M \otimes \mathbb{C} = (T^{1,0}M)^* \oplus (T^{0,1}M)^*.$$

Hence,

$$\Lambda^k M \otimes \mathbb{C} = \oplus_{p+q=k} \Lambda^{p,q} M,$$

where

$$\Lambda^{p,q}M = \Lambda^p (T^{1,0}M)^* \otimes \Lambda^p (T^{0,1}M)^*.$$

Then $\mathcal{E} = \Lambda^{0,*}M$ is a Clifford module with the Clifford multiplication

$$c(v) = \sqrt{2}(v^{0,1} \wedge - \, i_{v^{1,0}}).$$

A complex manifold is an almost complex manifold, of course. Conversely, by the Newlander-Nirenberg theorem, an almost complex manifold with an integral almost complex structure is a complex manifold. That is, there are complex local coordinates $z_1, \cdots, z_n$ such that the transition functions are holomorphic. In this case, $\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}$ is a basis of $T^{1,0}$, which is then called the holomorphic tangent bundle. Similarly, $\frac{\partial}{\partial \bar{z}_1}, \cdots, \frac{\partial}{\partial \bar{z}_n}$ is a basis of the anti-holomorphic tangent bundle $T^{0,1}$.

For a complex manifold M, we have

$$d = \partial + \bar{\partial},$$

where

$$\begin{array}{rccc} \partial : & C^\infty(M, \Lambda^{p,q}M) & \to & C^\infty(M, \Lambda^{p+1,q}), \\ & f dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} & \to & \partial f \wedge dz_{i_1} \wedge \cdots \wedge d\bar{z}_{j_q}, \end{array}$$

and

$$\begin{array}{rccc} \bar{\partial} : & C^\infty(M, \Lambda^{p,q}M) & \to & C^\infty(M, \Lambda^{p,q+1}), \\ & f dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} & \to & \bar{\partial} f \wedge dz_{i_1} \wedge \cdots \wedge d\bar{z}_{j_q}, \end{array}$$

with

$$\partial f = \frac{\partial f}{\partial z_i} dz_i, \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i.$$

More generally, if $W \to M$ is a holomorphic complex vector bundle (i.e. transition functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to GL(n, \mathbb{C})$ are holomorphic), the $\bar{\partial}$-operator extends to

$$\bar{\partial} : C^\infty(M, \Lambda^{0,q} \otimes W) \to C^\infty(M, \Lambda^{0,q+1} \otimes W)$$

by acting only on the form part. (By the holomorphy of W, this is well defined.) By taking $W = \Lambda^{p,0} M = \Lambda^p (T^{1,0}M)^*$ we recover the previous discussion.

A covariant derivative ∇ on W is called holomorphic if

$$\nabla^{0,1} = \bar{\partial},$$

where

$$\nabla^{0,1} : C^\infty(M, W) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes W) \xrightarrow{proj.} C^\infty(M, \Lambda^{0,1} M \otimes W).$$

Proposition 2.7.2 (Chern connection) *If $W \to M$ is a holomorphic vector bundle with an hermitian metric, then there is a unique holomorphic covariant derivative ∇^W on W which preserves the metric.*

Proof: In a local coordinate basis, the hermitian metric is given by an hermitian matrix valued function $h(z)$. Then

$$\nabla^W = d + h^{-1} \partial h$$

is the desired connection. ■

In particular, the curvature of the Chern connection is given by

$$F_h = h^{-1} \bar{\partial} \partial h - (h^{-1} \bar{\partial} h) \wedge (h^{-1} \partial h).$$

If $W = L$ is a holomorphic line bundle, this simplifies to

$$F_h = \bar{\partial} \partial \log h = \bar{\partial} \partial \log |s|^2$$

if s is a local nonvanishing holomorphic section.

We now assume that M is Kähler, i.e., there are Riemannian metric g and almost complex structure J such that J is parallel with respect to the Levi-Civita connection, $\nabla J = 0$. Clearly, this implies that ∇ preserves the splitting (7.24).

Proposition 2.7.3 *Let W be a holomorphic vector bundle with hermitian metric on a Kähler manifold M, and ∇, ∇^W be the Levi-Civita and Chern connection respectively. Then the tensor product connection $\nabla^{\mathcal{E}} = \nabla \otimes 1 + 1 \otimes \nabla^W$ is a Clifford connection on the Clifford module $\mathcal{E} = \Lambda^{0,*} M \otimes W$. The associated Dirac operator is*

$$\mathcal{D} = \sqrt{2}(\bar{\partial} + \bar{\partial}^*).$$

Proof: Let Z_i be a local frame of $T^{1,0}M$ and Z^i the dual frame for $T^{0,1}M$. Then, $Z_i, \bar{Z}_i$ is a local frame of $TM \otimes \mathbb{C}$. Hence,

$$d = Z^i \wedge \nabla_{Z_i} + \bar{Z}^i \wedge \nabla_{\bar{Z}_i}.$$

Since $\nabla^{\mathcal{E}}$ preserves the splitting (7.24), we conclude

$$\bar{\partial} = \bar{Z}^i \wedge \nabla_{\bar{Z}_i}.$$

One can then show that
$$\bar{\partial}^* = -i_{Z^i} \wedge \nabla_{\bar{Z}_i}.$$
■

Finally we remark that $\bar{\partial}^2 = 0$ and the Delbeault theorem says that
$$H^q(M, \mathcal{O}(W)) = \ker \bar{\partial}_q / \mathrm{Im}\bar{\partial}_{q-1}.$$

2.8 Bochner-Lichnerowicz-Weitzenbock formula

Let ∇ be a connection (say, the Levi-Civita connection) on (M, g) and $E \to M$ be a vector bundle with a connection ∇^E.
Definition: [Connection Laplacian] The connection Laplacian (or flat Laplacian) $\nabla^*\nabla : \ C^\infty(E) \to C^\infty(E)$ is defined by
$$\nabla^*\nabla s = -\sum_{i=1}^{n} (\nabla^E_{e_i}\nabla^E_{e_i} - \nabla^E_{\nabla_{e_i} e_i}) s,$$
where e_i is a local orthonormal frame for TM.

Clearly, $\nabla^*\nabla$ is independent of the choice of local orthonormal frame. It is a second order differential operator with principal symbol $\sigma_\xi(\nabla^*\nabla) = |\xi|^2$, hence also elliptic.

Now let ∇ be the Levi-Civita connection and ∇^E a unitary connection with respect to an hermitian metric $(\ , \ )$ on E.

Proposition 2.8.1 *The connection Laplacian is a nonnegative operator and is formally self adjoint. In fact,*
$$\int_M (\nabla^*\nabla s, s') d\mathrm{vol} = \int_M (\nabla^E s, \nabla^E s') d\mathrm{vol}$$
for all sections s, s' with compact support.

Proof: As before, we fix an arbitrary point $p \in M$ and choose e_i such that $\nabla e_i = 0$ at p. Then, at p,
$$\begin{aligned} (\nabla^*\nabla s, s') &= -(\nabla^E_{e_i}\nabla^E_{e_i} s, s') \\ &= -e_i(\nabla^E_{e_i} s, s') + (\nabla^E_{e_i} s, \nabla^E_{e_i} s') \\ &= (\nabla^E s, \nabla^E s') + \mathrm{div} V \end{aligned}$$
where the vector field V is defined by $\langle V, W \rangle = (\nabla^E_W s, s')$ for all vector fields W. Now, the desired formula follows by integration by parts. ■

We now assume that M is spin. Then the spinor bundle S is an hermitian vector bundle on M with an unitary connection induced from the Levi-Civita connection. The Dirac operator D acts on sections of S. The following theorem reveals the intimate connection between the Dirac operator and the scalar curvature.

Theorem 2.8.2 (Lichnerowicz formula) *We have*

$$D^2 = \nabla^*\nabla + \frac{1}{4}R$$

where R is the scalar curvature of (M, g).

Proof: Fix p and e_i as before. Then, at p,

$$\begin{aligned}
D^2 &= c(e_i)\nabla^S_{e_i}c(e_j)\nabla^S_{e_j} = c(e_i)c(e_j)\nabla^S_{e_i}\nabla^S_{e_j} \\
&= -\nabla^S_{e_i}\nabla^S_{e_i} + \sum_{i\neq j} c(e_i)c(e_j)\nabla^S_{e_i}\nabla^S_{e_j} \\
&= -\nabla^S_{e_i}\nabla^S_{e_i} + \frac{1}{2}\sum_{i\neq j} c(e_i)c(e_j)(\nabla^S_{e_i}\nabla^S_{e_j} - \nabla^S_{e_j}\nabla^S_{e_i}) \\
&= \nabla^*\nabla + \frac{1}{2}c(e_i)c(e_j)R^S(e_i, e_j)
\end{aligned}$$

So far, we have only used the Clifford relation and the compatibility of the connection with the hermitian metric. Now we have $R^S(e_i, e_j) = \frac{1}{4}R_{ijkl}c(e_k)c(e_l)$ where R_{ijkl} is the Riemann curvature tensor. Thus

$$\begin{aligned}
\frac{1}{2}c(e_i)c(e_j)R^S(e_i, e_j) &= \frac{1}{8}R_{ijkl}c(e_i)c(e_j)c(e_k)c(e_l) \\
&= \frac{1}{8}\sum_l [\frac{1}{3}\sum_{i,j,k\ distinct}(R_{ijkl} + R_{jkil} + R_{kijl})c(e_i)c(e_j)c(e_k) \\
&\quad + \sum_{i,j} R_{ijil}c(e_i)c(e_j)c(e_i) + \sum_{i,j} R_{ijjl}c(e_i)c(e_j)c(e_j)]c(e_l) \\
&= \frac{1}{4}R_{ijil}c(e_j)c(e_l) = -\frac{1}{4}\mathrm{Ric}(e_j, e_l)c(e_j)c(e_l) = \frac{1}{4}R.
\end{aligned}$$

Here we have used the symmetries of Riemann curvature tensor, including the first Bianchi identity. ∎

Corollary 2.8.3 *If, in addition, M is closed and $R > 0$, then $\ker D = 0$. In other words, if g has positive scalar curvature, then there is no harmonic spinors on (M, g).*

Proof: If s is a harmonic spinor, $Ds = 0$, then, using Lichnerowicz formula,

$$0 = D^2 s = \nabla^*\nabla s + \frac{1}{4}Rs.$$

Taking inner product of the above identity with s and integrating over M yields

$$0 = \int_M |\nabla^S s|^2 d\mathrm{vol} + \frac{1}{4}\int_M R|s|^2 d\mathrm{vol}.$$

Since $R > 0$, we must have $s = 0$. ∎

As we point out in the proof of Lichnerowicz formula, one actually have the following generalization. If $\mathcal{E} \to M$ is a Clifford module with Clifford connection $\nabla^{\mathcal{E}}$, and $\mathcal{D}$ is the associated Dirac operator, then we have the following Bochner-Lichnerowicz-Weitzenbock formula

$$\mathcal{D}^2 = \nabla^* \nabla + \mathcal{R}, \tag{8.25}$$

where

$$\mathcal{R} = \frac{1}{2} c(e_i) c(e_j) R^{\mathcal{E}}(e_i, e_j).$$

As an application, we deduce the following Bochner formula.

Corollary 2.8.4 *If ω is a 1-form on a Riemannian manifold M, then*

$$\Delta \omega = \nabla^* \nabla \omega + \mathrm{Ric}(\omega),$$

where $\mathrm{Ric}(\omega)$ is the Ricci transformation defined by

$$\mathrm{Ric}(\omega) = \mathrm{Ric}(\omega^{\#}, e_i) e^i$$

in terms of local basis e_i and its dual basis e^i; $\omega^{\#}$ denoting the metric dual of ω.

Proof: We apply the general Bochner-Lichnerowicz-Weitzenbock formula (8.25) to the Clifford module $\mathcal{E} = \Lambda^* M$ and $\mathcal{D} = d + d^*$. Since both $\mathcal{D}^2 = \Delta$ and $\nabla^* \nabla$ preserve the degree, so does $\mathcal{R}$. We now compute

$$\begin{aligned} \mathcal{R}(\omega) &= \frac{1}{2} c(e_i) c(e_j) R(e_i, e_j) \omega \\ &= -\frac{1}{2} c(e_i) c(e_j) \langle R(e_i, e_j) \omega^{\#}, e_k \rangle e^k \\ &= -\frac{1}{2} c(e_i) c(e_j) \langle R(e_i, e_j) \omega^{\#}, e_k \rangle c(e_k) 1. \end{aligned}$$

We are now in the exact form to apply the first Bianchi identity as in the proof of Lichnerowicz formula. Hence

$$\begin{aligned} \mathcal{R}(\omega) &= \langle R(e_i, e_j) \omega^{\#}, e_j \rangle c(e_i) 1 \\ &= \mathrm{Ric}(e_i, \omega^{\#}) e^i. \end{aligned}$$

■

Noting that $\langle \mathrm{Ric}(\omega), \omega \rangle = \mathrm{Ric}(\omega^{\#}, \omega^{\#})$, one deduce the classic Bochner vanishing theorem.

Corollary 2.8.5 *If a closed Riemannian manifold M has positive Ricci curvature, then its first Betti number vanishes.*

Proof: Proceeding exactly as before, one obtains that $\ker \Delta_1 = 0$. By Hodge theory, $b_1(M) = \dim \ker \Delta_1 = 0$. ∎

Remark A variation of the Bochner formula is

$$\Delta |\omega|^2 = 2|\nabla\omega|^2 - 2\langle \omega, \Delta\omega\rangle + 2\mathrm{Ric}(\omega^\#, \omega^\#).$$

In particular, for $\omega = df$, one has

$$\Delta |\nabla f|^2 = 2|\mathrm{Hess}(f)|^2 - 2\langle \nabla f, \nabla \Delta f\rangle + 2\mathrm{Ric}(\nabla f, \nabla f).$$

To see this, one notes

$$\Delta|\omega|^2 = -2\langle \omega, \nabla^*\nabla\omega\rangle + 2|\nabla\omega|^2$$

and plugs in the Bochner formula for $\nabla^*\nabla\omega$.

The Bochner vanishing theorem shows that the topological invariant $b_1(M)$ is an obstruction to positive Ricci curved metrics. As we learned earlier, on a spin manifold, positive scalar curvature implies $\ker D = 0$. Unfortunately, $\dim\ker D$ is not a topological invariant. It turns out that the topological obstruction here is coming from something more subtle—the index of the Dirac operator.

Chapter 3

Index of Dirac Operators

If E and F are complex vector bundles on closed manifold M and $P: C^\infty(M,E) \to C^\infty(M,F)$ is an elliptic differential operator, then the standard elliptic theory implies that P is Fredholm, i.e., $\dim\ker P < \infty$ and $\dim \mathit{coker} P < \infty$. This leads to the definition of index:

$$\mathrm{ind} P = \dim_{\mathbb{C}} \ker P - \dim_{\mathbb{C}} \mathit{coker} P \in \mathbb{Z} \tag{0.1}$$

The most important property about index is the homotopy invariance. That is, for a continuous family of elliptic differential operators $P_t: C^\infty(M,E) \to C^\infty(M,F)$, $\mathrm{ind}P_t$ is constant in t. This leads Gelfand to ask

Question: How to express $\mathrm{ind}P$ in terms of topological quantities?

The Atiyah-Singer index theorem is a full and far-reaching answer to this question.

3.1 Index of Fredholm operators

Let H^0, H^1 be Hilbert spaces, $P: H^0 \to H^1$ a closed linear operator with dense domain.

Definition: [Fredholm index] P is called a Fredholm operator if $\dim\ker P < \infty, \dim \mathrm{coker} P < \infty$, where $\mathit{coker} P = H^1/ImP$. And we define its index

$$\mathrm{ind} P = \dim\ker P - \dim \mathrm{coker} P \in \mathbb{Z}. \tag{1.2}$$

Recall that the formal adjoint $P^*: H^1 \to H^0$ is defined via the equation

$$(Pu, v)_{H^1} = (u, P^* v)_{H^0}.$$

We have some simple properties of Fredholm operators.

Proposition 3.1.1 *a) If P is Fredholm, then its adjoint $P^*: H^1 \to H^0$ is also Fredholm.*

b)If P is Fredholm, we have the orthogonal decompositions

$$H^0 = \ker P \oplus (\ker P)^\perp = \ker P \oplus \mathrm{Im} P^*, \quad H^1 = \ker P^* \oplus (\ker P^*)^\perp = \ker P^* \oplus \mathrm{Im} P.$$

In particular, $\mathrm{Im}P$ *and* $\mathrm{Im}P^*$ *are closed subspaces. And*

$$\mathrm{ind} P = \dim \ker P - \dim \ker P^* = -\mathrm{ind} P^*. \tag{1.3}$$

c) There exists a bounded operator $G_0 : H^1 \to H^0$ such that $1 - G_0 P$ and $1 - PG_0$ are orthogonal projections onto $\ker P$ *and* $\ker P^*$ *respectively.*

Proof: We sketch the proof of the second property. Since $\dim \mathrm{coker} P = d < \infty$, there are d vectors $v_1, \cdots, v_d \in H^1$ such that any $v \in H^1$ can be written as

$$v = Pu + c_1 v_1 + \cdots + c_d v_d$$

with $u \perp \ker P$. Moreover, this representation is unique. It follows then that

$$\bar{P} : \ (\ker P)^\perp \oplus \mathbb{C}^d \to H^1$$

defined by $\bar{P}(u, c_1, \cdots, c_d) = Pu + c_1 v_1 + \cdots + c_d v_d$ which is a closed operator with dense domain, has its inverse $\bar{P}^{-1}$ everywhere defined. By the Banach theorem, $\bar{P}^{-1}$ is bounded. That is,

$$\|u\| \leq C \|Pu\|, \quad \forall u \in Dom(P) \cap (\ker P)^\perp$$

for some constant $C > 0$. This implies that $\mathrm{Im}P$ is closed and hence $H^1 = \ker P^* \oplus (\ker P^*)^\perp = \ker P^* \oplus \mathrm{Im} P$. ■

The operator G_0, which is essentially the inverse of P restricted to $(\ker P)^\perp$, is called the Green's operator of P. It follows from c) that

$$\mathrm{ind} P = \mathrm{Tr}(1 - G_0 P) - \mathrm{Tr}(1 - PG_0).$$

We now review about trace class operators.

If $T : H^0 \to H^1$ is a compact operator, then T^*T is a compact self-adjoint operator and hence its spectrum consists of eigenvalues of finite multiplicity except the zero eigenvalue. Let $s_i > 0$ be the nonzero eigenvalues of T^*T repeated according to multiplicity.

Definition: 1) The trace norm is

$$\|T\|_1 = \sum_{i=1}^{\infty} s_i^{1/2}.$$

2) T is of trace class if $\|T\|_1 < \infty$.

Note that if T is trace class and A, B are bounded, then BTA is also of trace class and $\|BTA\|_1 \leq \|A\| \|B\| \|T\|_1$. In particular, the space of trace class operators is a two-sided ideal in the space of bounded operators.

Definition: [Trace] If H is a Hilbert space and $T : H \to H$ is of trace class, then

$$\mathrm{Tr}(T) = \sum_{i=1}^{\infty} (Te_i, e_i),$$

with $\{e_i\}$ an orthonormal basis, is independent of the choices of the orthonormal basis, and is called the trace of T.

The following is an important property of the trace: for B bounded and T trace class,

$$\mathrm{Tr}(BT) = \mathrm{Tr}(TB), \quad \text{or equivalently} \quad \mathrm{Tr}([B,T]) = 0.$$

More generally, $\mathrm{Tr}(B_1 B_2 \cdots B_n) = \mathrm{Tr}(B_n B_1 \cdots B_{n-1})$ or with any cyclic permutations, as long as one of B_i is of trace class and the rest are bounded. In fact

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

as long as both AB, BA are of trace class (even though A, B may not be individually bounded).

Also,

$$\mathrm{Tr}(T^*) = \overline{\mathrm{Tr}(T)}.$$

The Lidskii Theorem gives the trace in terms of the eigenvalues and kernel representation.

Theorem 3.1.2 (Lidskii Theorem) *Let λ_k denote the eigenvalues of T counted with multiplicity. Then*

$$\mathrm{Tr}(T) = \sum_k \lambda_k.$$

If $T: L^2 \to L^2$ is defined by a continuous kernel function $T(x,y)$, i.e.,

$$(Tf)(x) = \int T(x,y) f(y) dy,$$

then

$$\mathrm{Tr}(T) = \int T(x,x) dx.$$

Remark Note that the trace norm $\|T\|_1 = \mathrm{Tr}(T^*T)^{\frac{1}{2}}$. Similarly, one can define $\|T\|_p = [\mathrm{Tr}(T^*T)^{\frac{p}{2}}]^{\frac{1}{p}}$.

Coming back to the index, we note the several equivalent formula for the index.

$$\begin{aligned}
\mathrm{ind} P &= \dim\ker P - \dim \mathrm{coker} P \\
&= \dim\ker P - \dim\ker P^* \\
&= \dim\ker P^*P - \dim\ker PP^* \\
&= \mathrm{Tr}(1 - G_0 P) - \mathrm{Tr}(1 - PG_0).
\end{aligned}$$

Theorem 3.1.3 (Heat equation method) *Assume that P is Fredholm, and P^*P and PP^* have discrete spectrum; further e^{-tP^*P} and e^{-tPP^*} are trace class for all $t > 0$. Then*

$$\mathrm{ind}P = \mathrm{Tr}(e^{-tP^*P}) - \mathrm{Tr}(e^{-tPP^*})$$

for all $t > 0$.

Proof: We present two proofs here.
1). We compute

$$\begin{aligned}
\frac{d}{dt}[\mathrm{Tr}(e^{-tP^*P}) - \mathrm{Tr}(e^{-tPP^*})] &= -\mathrm{Tr}(P^*Pe^{-tP^*P}) + \mathrm{Tr}(PP^*e^{-tPP^*}) \\
&= -\mathrm{Tr}(P^*e^{-tPP^*}P) + \mathrm{Tr}(PP^*e^{-tPP^*}) \\
&= \mathrm{Tr}[P, P^*e^{-tPP^*}] = 0.
\end{aligned}$$

Thus $\mathrm{Tr}(e^{-tP^*P}) - \mathrm{Tr}(e^{-tPP^*})$ is a constant. Now

$$\lim_{t\to\infty}[\mathrm{Tr}(e^{-tP^*P}) - \mathrm{Tr}(e^{-tPP^*})] = \dim\ker P^*P - \dim\ker PP^* = \mathrm{ind}P,$$

which finishes the proof.
2). Denote the eigenspace for P^*P with eigenvalue λ by $E_{P^*P}(\lambda)$:

$$E_{P^*P}(\lambda) = \{\ u \in H^0 \mid P^*Pu = \lambda u\ \}.$$

Similarly,

$$E_{PP^*}(\lambda) = \{\ v \in H^1 \mid PP^*v = \lambda v\ \}.$$

Clearly, $P:\ E_{P^*P}(\lambda) \to E_{PP^*}(\lambda)$. Moreover, for $\lambda \neq 0$, P defines an isomorphism $P:\ E_{P^*P}(\lambda) \xrightarrow{\cong} E_{PP^*}(\lambda)$ since its inverse is given by $\lambda^{-1}P^*$. It follows then that

$$\begin{aligned}
\mathrm{Tr}(e^{-tP^*P}) - \mathrm{Tr}(e^{-tPP^*}) &= \sum_\lambda e^{-t\lambda}\dim E_{P^*P}(\lambda) - \sum_\lambda e^{-t\lambda}\dim E_{PP^*}(\lambda) \\
&= \dim\ker P^*P - \dim\ker PP^* = \mathrm{ind}P.
\end{aligned}$$

■

The following formula, used very often in noncommutative geometry, already hints at the invariant property of index.

Theorem 3.1.4 *The operator $P:\ H^0 \to H^1$ is Fredholm if and only if there is a bounded linear operator $G:\ H^1 \to H^0$ such that $1 - GP$ and $1 - PG$ are both trace class. Moreover,*

$$\mathrm{ind}\,P = \mathrm{Tr}(1 - GP) - \mathrm{Tr}(1 - PG).$$

Proof: The proof of the first part can be found in any standard functional analysis book. For the formula, we compute, using the Green's operator G_0:

$$\begin{aligned}\operatorname{ind} P &= \operatorname{Tr}(1-G_0P) - \operatorname{Tr}(1-PG_0) \\ &= \operatorname{Tr}(1-GP) - \operatorname{Tr}(1-PG) + \operatorname{Tr}((G-G_0)P) - \operatorname{Tr}(P(G-G_0)) \\ &= \operatorname{Tr}(1-GP) - \operatorname{Tr}(1-PG),\end{aligned}$$

where we have used the property of trace and the fact that both $(G-G_0)P$ and $P(G-G_0)$ are trace class since (say) $(G-G_0)P = (1-G_0P)-(1-GP)$. ■

Such G is called a parametrix of P.

We now continue with some more properties of index.

1). Trivial properties: $\operatorname{ind} P = 0$ if P is self adjoint. Also, $\operatorname{ind} P = \dim H^0 - \dim H^1$ if H^0, H^1 are finite dimensional.

2). Stability: Let $P: H^0 \to H^1$ be Fredholm and G a parametrix. If $B: H^0 \to H^1$ is a small perturbation in the sense that $\|BG\| < 1$ and $\|GB\| < 1$, then $P-B$ is also Fredholm, and $\operatorname{ind}(P-B) = \operatorname{ind} P$.

Proof: A parametrix G' of $P-B$ can be obtained by the Neumann series

$$G' = G\sum_{n=0}^{\infty}(BG)^n = (\sum_{n=0}^{\infty}(GB)^n)G.$$

It follows that

$$\begin{aligned}1-G'(P-B) &= (\sum_{n=0}^{\infty}(GB)^n)(1-GP) \\ 1-(P-B)G' &= (1-PG)(\sum_{n=0}^{\infty}(BG)^n)\end{aligned}$$

are both trace class. Furthermore, the traces of $(GB)^n(1-GP)$ and $(1-PG)(BG)^n$ coincide for $n \geq 1$, since the factors under trace sign can be interchanges cyclically. ■

Of course, the stability implies the homotopy invariance for elliptic differential operators.

3). Additivity: If $P: H^0 \to H^1$ and $Q: H^1 \to H^2$ are Fredholm, then so is $QP: H^0 \to H^2$ and $\operatorname{ind} QP = \operatorname{ind} Q + \operatorname{ind} P$.

Proof: Let G_P and G_Q be parametrixes of P and Q, respectively. It is easy to see that G_PG_Q is a parametrix of QP. Further

$$\begin{aligned}\operatorname{Tr}(1-G_PG_QQP) &= \operatorname{Tr}(1-G_PP) + \operatorname{Tr}[G_P(1-G_QQ)P] \\ &= \operatorname{Tr}(1-G_PP) + \operatorname{Tr}[PG_P(1-G_QQ)] \\ &= \operatorname{Tr}(1-G_QQ) + \operatorname{Tr}(1-G_PP) - \operatorname{Tr}[(1-PG_P)(1-G_QQ)].\end{aligned}$$

Similarly

$$\operatorname{Tr}(1-QPG_PG_Q) = \operatorname{Tr}(1-QG_Q) + \operatorname{Tr}(1-G_PP) - \operatorname{Tr}[(1-G_QQ)(1-PG_P)].$$

The formula follows. ■

4). Multiplicativity:

Remark Index theory and SUSY: In any QFT, the Hilbert space H splits $H = H^+ \oplus H^-$ with H^+ the space of Bosonic states and H^- that of Fermionic states. Define

$$(-1)^F = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix}.$$

(Here F stands for Fermion counting operator.) A SUSY theory is by definition a QFT where there are hermitian operators (supersymmetries) Q_i, $i = 1, \cdots, N$ such that

a). Q_i interchanges $H^{\pm}$. Or in terms of the anti-commutators $\{Q_i, (-1)^F\} = Q_i(-1)^F + (-1)^F Q_i = 0$.

b). Q_i are symmetries of the theory. In other words, if H is the Hamiltonian, then $[Q_i, H] = 0$.

c). One further has the following anti-commutating relations:

$$\{Q_i, Q_j\} = 0, \quad \text{if } i \neq j; \quad \{Q_i, Q_i\} = 2H \quad \text{for all} \quad i.$$

Strictly speaking c) is only satisfied by non-relativistic theory. For relativistic theory, the Lorentz invariance dictates that, for example, when $N = 2$,

$$Q_1^2 = H + P, \quad Q_2^2 = H - P,$$

and

$$Q_1Q_2 + Q_2Q_1 = 0.$$

Here P is the (linear) momentum operator. Thus $H = \frac{1}{2}(Q_1^2 + Q_2^2)$. But this reduces to c) when things can be considered in $H_0 = \ker P$.

One of the most important questions for a SUSY theory is, whether there is a state $|\Omega\rangle$ such that

$$Q_i|\Omega\rangle = 0, \quad \forall\, i? \tag{1.4}$$

Consequences of (1.4) includes the existence of minimum energy (or vacuum) state. Also, the theory will contain bosons and fermions of equal mass.

Alas, nature does not behave that way! Hence in a realistic SUSY theory, one would want no solution of (1.4) and it is said that then SUSY is "spontaneously broken". Index theory can help rule out theories in which SUSY are not "spontaneously broken". In fact, from a), any $Q = Q_i$ will have the form

$$Q = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix}.$$

If $\operatorname{ind} Q_+ \neq 0$, then (1.4) has solutions!

3.2 Chern-Weil theory for characteristic classes

Roughly speaking, the theory of characteristic classes associates cohomology classes to a vector bundle which measures the global twisting of the vector bundle. Euler class is one such example which gives the obstruction to nowhere vanishing sections. Chern-Weil theory uses curvature to construct de Rham cohomology classes, and is thus a geometric approach to characteristic classes (one loses the torsion information however).

Let $E \to M$ be a vector bundle on M, either complex or real.

Definition: A connection on E is a linear map

$$\nabla^E : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$$

satisfying the Leibniz rule

$$\nabla^E(fs) = df \otimes s + f\nabla^E s,$$

for all $f \in C^\infty(M)$ and $s \in C^\infty(M, E)$.

Thus, for trivial bundle, the exterior differentiation d gives us the trivial connection. More generally, $d + A$, for any $A \in C^\infty(M, T^*M \otimes E)$, is a connection. This, and partition of unity, gives us constructions of connections on any vector bundle. Conversely, any connection on the trivial bundle is of the form $d + A$. In fact, if ∇^E, $\tilde{\nabla}^E$ are two connection on a vector bundle E, then, by the Leibniz rule, their difference $\nabla^E - \tilde{\nabla}^E$ is tensorial; i.e., $\nabla^E - \tilde{\nabla}^E = A \in C^\infty(M, T^*M \otimes E)$.

Denote by $\Omega^*(M, E) = C^\infty(M, \Lambda^*M \otimes E)$ the space of E-valued differential forms. The connection ∇^E extends to

$$\nabla^E : \Omega^p(M, E) \to \Omega^{p+1}(M, E)$$

by the Leibniz rule

$$\nabla^E(\omega \otimes s) = df \otimes s + (-1)^{\deg \omega}\omega \wedge \nabla^E s.$$

The curvature of ∇^E is then defined by

$$R^E = (\nabla^E)^2 = \nabla^E \circ \nabla^E : \Omega^0(M, E) \to \Omega^2(M, E).$$

Proposition 3.2.1 *The curvature R^E is tensorial:*

$$R^E(fs) = fR^E s$$

for all $f \in C^\infty(M)$ and $s \in C^\infty(M, E)$.

Proof: Apply Leibniz rule. ∎

Thus $R^E \in \Omega^2(M, \mathrm{End}(E))$ is a 2-form with values in $\mathrm{End}(E)$, where $\mathrm{End}(E) = E \otimes E^*$ is the vector bundle of fiberwise endomorphisms of E. In fact, for any vector fields X, Y,

$$R^E(X, Y) = \nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X - \nabla^E_{[X,Y]}$$

is the usual curvature expression. Similarly, $(R^E)^k \in \Omega^{2k}(M, \mathrm{End}(E))$.

To extract differential forms out of it, we recall the trace functional. Taking fiberwise, it gives a linear map

$$\mathrm{tr}: \ C^\infty(M, \mathrm{End}(E)) \to C^\infty(M).$$

Once again, it extends to

$$\mathrm{tr}: \ \Omega^*(M, \mathrm{End}(E)) \to \Omega^*(M) \tag{2.5}$$

by tracing over the endomorphism part:

$$\mathrm{tr}(\omega A) = \omega \mathrm{tr}(A),$$

where $\omega \in \Omega^*(M)$ and $A \in C^\infty(M, \mathrm{End}(E))$ (and we now suppress the tensor sign). Similarly, the commutator extends to (a super-commutator on) $\Omega^*(M, \mathrm{End}(E))$ via

$$[\omega A, \ \eta B] = (\omega A)(\eta B) - (-1)^{\deg \omega \deg \eta}(\eta B)(\omega A).$$

Here the wedge product on the forms is implied. Clearly, the trace vanishes on the super-commutators:

$$\mathrm{tr}[A, \ B] = 0 \tag{2.6}$$

for all $A, B \in \Omega^*(M, \mathrm{End}(E))$.

Now, for any $A \in \Omega^*(M, \mathrm{End}(E))$, we define

$$[\nabla^E, A] = \nabla^E \circ A - (-1)^{\deg A} A \circ \nabla^E \in \Omega^*(M, \mathrm{End}(E))$$

(One verifies that indeed $[\nabla^E, A]$ is tensorial.) Note second Bianchi identity

$$[\nabla^E, R^E] = 0. \tag{2.7}$$

Lemma 3.2.2 *For any connection ∇^E on E, and $A \in \Omega^*(M, \mathrm{End}(E))$,*

$$d\mathrm{tr}(A) = \mathrm{tr}([\nabla^E, A]).$$

Proof: . First of all, by (2.6) and the tensorial property for difference of connections, the right hand side is independent of ∇^E. The equation is also local. Thus we can verify it locally, where the vector bundle becomes trivial. For the trivial bundle, we can take $\nabla^E = d$ to be the trivial connection. Then the equation follows trivially. ∎

Now let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \in \mathbb{C}[[x]]$$

be a formal power series. Define

$$f(R^E) = a_0 \mathrm{I} + a_1 R^E + a_2 (R^E)^2 + \cdots \in \Omega^*(M, \mathrm{End}(E)). \tag{2.8}$$

Note that this is in fact a finite sum.

Theorem 3.2.3 (Chern-Weil) *1). The differential form* $\mathrm{tr} f(R^E) \in \Omega^*(M)$ *is closed.*
2). The cohomology class $[\mathrm{tr} f(R^E)] \in H^*(M)$ *is independent of the connection* ∇^E*.*

Proof: We have

$$d\mathrm{tr} f(R^E) = \mathrm{tr}[\nabla^E, f(R^E)] = 0$$

by the second Bianchi identity.

To see 2), let ∇^E and $\tilde{\nabla}^E$ be two connections. Consider the path $\nabla^E_t = (1-t)\nabla^E + t\tilde{\nabla}^E$ of connections. First of all

$$\frac{d}{dt}\nabla^E_t = \tilde{\nabla}^E - \nabla^E \in \Omega^1(M, \mathrm{End}(E)). \tag{2.9}$$

Denote its curvature by $R^E_t = (\nabla^E_t)^2$. Then

$$\begin{aligned}
\frac{d}{dt}\mathrm{tr} f(R^E_t) &= \mathrm{tr}(\frac{dR^E_t}{dt} f'(R^E_t)) = \mathrm{tr}(\frac{d(\nabla^E_t)^2}{dt} f'(R^E_t)) \\
&= \mathrm{tr}([\nabla^E_t, \frac{d\nabla^E_t}{dt}] f'(R^E_t)) \\
&= \mathrm{tr}([\nabla^E_t, \frac{d\nabla^E_t}{dt} f'(R^E_t)]) \\
&= d\mathrm{tr}(\frac{d\nabla^E_t}{dt} f'(R^E_t)).
\end{aligned}$$

Integrating, we obtain

$$\mathrm{tr} f(\tilde{R}^E) - \mathrm{tr} f(R^E) = d\omega,$$

with

$$\omega = \int_0^1 \mathrm{tr}(\frac{d\nabla^E_t}{dt} f'(R^E_t)) dt.$$

■

Definition: A characteristic class for E is an element of the ring in $H^*(M)$ generated by all cohomology classes $[\mathrm{tr} f(R^E)]$, for all $f \in \mathbb{C}[[x]]$.

Thus defined, all characteristic classes will be trivial for the trivial bundles.

We now assume that E is a complex vector bundle.

Example: 1) The Chern character is defined by

$$ch(E) = ch(\frac{\sqrt{-1}}{2\pi} R^E) = \mathrm{tr}(e^{\frac{\sqrt{-1}}{2\pi} R^E}). \tag{2.10}$$

Clearly, it satisfies

$$ch(E \oplus F) = ch(E) + ch(F).$$

In fact this holds true for any characteristic class $[\mathrm{tr} f(R^E)]$ defined by a single trace. The Chern character also satisfies

$$ch(E \otimes F) = ch(E)ch(F)$$

which, combined with the additivity, explains the name character. The normalization factor here is so that $ch(\frac{\sqrt{-1}}{2\pi}R^E) \in H^*(M, \mathbb{Q})$ is in fact a rational class.

2). The total Chern class is

$$c(E) = c(\frac{\sqrt{-1}}{2\pi}R^E) = \det(\mathrm{I} + \frac{\sqrt{-1}}{2\pi}R^E) = \exp(\mathrm{tr}[\log(\mathrm{I} + \frac{\sqrt{-1}}{2\pi}R^E)]). \quad (2.11)$$

Since it involves products of traces, the total Chern class is no longer additive. Rather, it satisfies the Whitney product formula

$$c(E \oplus F) = c(E)c(F). \quad (2.12)$$

Here the normalization is so that $c(E) \in H^*(M, \mathbb{Z})$ is integral. Writing

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots$$

into its homogeneous components, we have the i-th Chern class $c_i(E) \in H^{2i}(M, \mathbb{Z})$.

It follows from the Whitney product formula that if E admits a nowhere vanishing section, then the top Chern class $c_k(E)$ ($k = \mathrm{rank}E$) of E vanishes.

Note the following properties:

i). $c_i(E) = 0$ for $i > \mathrm{rank}\, E$. Also, $c_i(E) = 0$ if $2i > n = \dim M$.

ii). (Naturality) $c(f^*(E)) = f^*c(E)$ for any smooth map f from another manifold N to M.

In fact, the Chern classes can be defined axiomatically by i), ii), the Whitney product formula, and the following normalization

$$c(L) = 1 + x$$

for the canonical line bundle $L \to \mathbb{CP}^1$. Here $x \in H^2(\mathbb{CP}^1, \mathbb{Z})$ is the generator.

Remark In our definition, $c(E)$ depends on the complex structure of E and the smooth structure of M.

In practice, characteristic classes are manipulated via the so called Chern roots. By choosing an hermitian metric on E and thus a unitary connection ∇^E, its curvature R^E is skew-hermitian. Hence we can (formally) diagonalize

$$\frac{\sqrt{-1}}{2\pi}R^E = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_k \end{pmatrix}, \quad (2.13)$$

where x_i's are 2-forms. Therefore

$$\begin{aligned} \det(\mathrm{I} + \frac{\sqrt{-1}}{2\pi}R^E) &= \prod_{i=1}^{k}(1 + x_i) \\ &= 1 + \sum_i x_i + \sum_{i \neq j} x_i x_j + \cdots + \prod_i x_i. \end{aligned}$$

Thus, we have

$$\begin{aligned}
c_1(E) &= \sum_i x_i = \mathrm{Tr}(\frac{\sqrt{-1}}{2\pi}R^E), \\
c_2(E) &= \sum_{i\neq j} x_i x_j = (\sum_i x_i)^2 - \sum_i x_i^2 \\
&= (\mathrm{Tr}(\frac{\sqrt{-1}}{2\pi}R^E))^2 - \mathrm{Tr}[(\frac{\sqrt{-1}}{2\pi}R^E)^2], \\
c_i(E) &= \sigma_i(x_1, \cdots, x_k), \quad \text{the i-th elementary symmetric function}, \\
c_k(E) &= \prod_i x_i = \det(\frac{\sqrt{-1}}{2\pi}R^E).
\end{aligned}$$

It follows that all other characteristic classes of E are polynomials of the Chern classes. This illustrates the fundamental importance of Chern classes.

In particular, if $L \to M$ is a (complex) line bundle, we have

$$c_1(L) = \frac{\sqrt{-1}}{2\pi}R^L. \tag{2.14}$$

Remark Dirac magnetic monopole corresponds to complex line bundles over S^2 (or equivalently, principal $U(1)$-bundles over S^2. Then c_1 is the monopole charge. The fact that c_1 is an integer is interpreted as the Dirac quantization condition.

As another example from gauge theory, consider $E \to M$ a vector bundle with structure (gauge) group $SU(N)$. Then the curvature R^E is a $su(N)$-valued 2-form. It follows that $\mathrm{Tr}(R^E) = 0$. Hence $c_1(E) = 0$. Therefore

$$c_2(E) = \frac{1}{4\pi^2}\mathrm{Tr}(R^E)^2 = \frac{1}{4\pi^2}\mathrm{Tr}(R^E \wedge R^E). \tag{2.15}$$

For $\dim M = 4$, this is the so called instanton number. When M is oriented, R^E decomposes into its self-dual and anti-self-dual parts $R^E = R^E_+ + R^E_-$ using the Hodge $*$-operator. Thus

$$\begin{aligned}
c_2(E) &= \frac{1}{4\pi^2}\mathrm{Tr}(R^E \wedge R^E) \\
&= \frac{1}{4\pi^2}[\mathrm{Tr}(R^E_+ \wedge R^E_+) + \mathrm{Tr}(R^E_- \wedge R^E_-) + 2\mathrm{Tr}(R^E_+ \wedge R^E_-)] \\
&= \frac{1}{4\pi^2}[\mathrm{Tr}(R^E_+ \wedge *R^E_+) - \mathrm{Tr}(R^E_- \wedge *R^E_-)] \\
&= -\frac{1}{4\pi^2}(|R^E_+|^2 - |R^E_-|^2)d\mathrm{vol}
\end{aligned}$$

3). The Todd genus for a complex vector bundle $E \to M$ is defined by

$$Td(E) = Td(\frac{\sqrt{-1}}{2\pi}R^E) = \det\left(\frac{\frac{\sqrt{-1}}{2\pi}R^E}{e^{\frac{\sqrt{-1}}{2\pi}R^E} - 1}\right). \tag{2.16}$$

We now turn to real vector bundles.

Example: 1). Pontryagin classes. By putting a metric on a real vector bundle E, we can take an orthogonal connection ∇^E, whose curvature R^E will then be skew symmetric. The total Pontryagin class is defined by

$$\begin{aligned} p(E) &= p(\frac{R^E}{2\pi}) = \det(\mathrm{I} + \frac{R^E}{2\pi}) = \det(\mathrm{I} - \frac{R^E}{2\pi}) \\ &= 1 + p_1(E) + p_2(E) + \cdots . \end{aligned} \tag{2.17}$$

Thus, $p_i \in H^{4i}(M)$.

Here R^E can be diagonalized into block form

$$\frac{1}{2\pi} R^E = \begin{pmatrix} \begin{pmatrix} 0 & y_1 \\ -y_1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & y_l \\ -y_l & 0 \end{pmatrix} \end{pmatrix}.$$

Here y_i are two forms and $2l = k = \operatorname{rank} E$. In the case when $k = 2l + 1$, there is an additional zero row and zero column.

Thus,

$$p(E) = \prod_{i=1}^{l} (1 + y_i^2),$$

and thus $p_i(E) = \sigma_i(y_1^2, \cdots, y_l^2)$ is the i-th elementary symmetric function of $y_1^2, \cdots, y_l^2$. It follows then, for example,

$$p_1(E) = \sum y_i^2 = -\frac{1}{2}\mathrm{Tr}(\frac{R^E}{2\pi})^2 = -\frac{1}{2}\mathrm{Tr}(\frac{R^E}{2\pi} \wedge \frac{R^E}{2\pi}). \tag{2.18}$$

It also follows that all other characteristic classes are polynomials of the Pontryagin classes.

Remark For the complexification $E^{\mathbb{C}} = E \otimes \mathbb{C}$, we have

$$\frac{\sqrt{-1}}{2\pi} R^{E^{\mathbb{C}}} = \begin{pmatrix} y_1 & & & & \\ & -y_1 & & & \\ & & \ddots & & \\ & & & y_l & \\ & & & & -y_l \end{pmatrix}.$$

Therefore $c_i(E^{\mathbb{C}}) = 0$ for i odd, and $p_i(E) = (-1)^i c_{2i}(E^{\mathbb{C}}) \in H^{4i}(M, \mathbb{Z})$.

Definition: We define $p_i(M) = p_i(TM)$. Similarly for other characteristic classes, i.e., the characteristic classes are a real manifold are defined to be those of its tangent bundle.

Remark A priori, $p_i(M)$ depends on the smooth structure of M. But a theorem of Novikov say that $p_i(M)$ is actually a homeomorphism invariant (note that our $p_i(M)$ has no torsion information).

Definition: [$\hat{A}$-genus] Using Riemannian curvature R, we define the $\hat{A}$-class

$$\hat{A}(\frac{R}{2\pi}) = \det{}^{1/2}(\frac{R/4\pi}{\sinh R/4\pi}),$$

and the $\hat{A}$-genus

$$\hat{A}(M) = \int_M \hat{A}(\frac{R}{2\pi}).$$

In terms of the Chern roots,

$$\hat{A}(\frac{R}{2\pi}) = \prod_{i=1}^{[n/2]} \frac{y_i/2}{\sinh y_i/2}.$$

Now (note that $\frac{x/2}{\sinh x/2}$ is an even function.)

$$\hat{A}(x) = \frac{x/2}{\sinh x/2} = 1 - \frac{1}{24}x^2 + \frac{7}{2^7 3^2 5}x^4 + \cdots .$$

We deduce

$$\hat{A} = 1 - \frac{1}{24}p_1 + \cdots . \tag{2.19}$$

Remark Since Pontryagin classes are integral, we see that $\hat{A}(M) \in \mathbb{Q}$. We will see later from Atiyah-Singer index theorem that for spin manifolds, $\hat{A}(M) \in \mathbb{Z}$. This is the so called integrality theorem.

Definition: [L-genus] Similarly, the Hirzebruch L-genus is defined via

$$L(x) = \frac{x}{\tanh x}.$$

That is,

$$L(\frac{R}{2\pi}) = \det{}^{1/2}(\frac{R/2\pi}{\tanh R/2\pi}).$$

and

$$L(M) = \int_M L(\frac{R}{2\pi}).$$

Using

$$L(x) = 1 + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{2k}}{(2k)!}B_k x^{2k},$$

(B_k denotes the Bernoulli numbers) and Chern roots, we find

$$L = 1 + \frac{1}{3}p_1 + \cdots . \tag{2.20}$$

In particular, in dimension 4, (2.19) and (2.20) yield

$$L(M) = -8\hat{A}(M). \tag{2.21}$$

We have defined our characteristic classes using traces and determinants. These are example of invariant polynomials. In general, given a Lie group G with Lie algebra $\mathbf{g}$, an invariant polynomial P of G is a polynomial function on its Lie algebra $\mathbf{g}$ (it depends polynomially on the coefficients in any basis) such that it is invariant under the adjoint action of G on $\mathbf{g}$:

$$P(\mathrm{Ad}_g A) = P(A), \quad \forall\, A \in \mathbf{g}, \quad \forall, g \in G.$$

Then, given any vector bundle $E \to M$ with structure group G, $P(\frac{\sqrt{-1}}{2\pi} R^E)$ (for complex vector bundles) defines a cohomology class. And the Chern-Weil homomorphism is

$$\begin{array}{rcl} I^*(G) & \to & H^*(M) \\ P & \to & P(\dfrac{\sqrt{-1}}{2\pi} R^E). \end{array} \tag{2.22}$$

Here $I^*(G)$ denotes the space of invariant polynomials of G.

Example: Let $G = GL(k, \mathbb{C})$. Then $\mathbf{g} = gl(k, \mathbb{C})$ and the adjoint action $\mathrm{Ad}_g A = g^{-1} A g$ is just conjugation. Clearly, $\mathrm{Tr}(A^r)$ (for any $r \in \mathbb{N}$) is an invariant polynomial (of degree r). It turns out that invariant polynomials of $GL(k, \mathbb{C})$ are generated by $\mathrm{Tr}(A^r)$ $(r = 1, \cdots, k)$. The same is true for $GL(k, \mathbb{R})$.

However, for $G = SO(k)$, there is one additional invariant polynomial—the Pfaffian, which defines the Euler class.

Lemma 3.2.4 *There is an invariant polynomial of $SO(k)$, called the Pfaffian,*

$$\mathrm{Pf}: \ so(k) \to \mathbb{R}$$

such that, for any $A \in so(k)$,

$$\mathrm{Pf}(A)^2 = \det A.$$

Proof: The Pfaffian is defined to be zero in odd dimension k. We now assume k is even. Recall that we have a 2-form associated to a skew symmetric matrix. Thus, setting $l = k/2$, we define for $A = (A_{ij}) \in so(k)$

$$\mathrm{Pf}(A) e_1 \wedge \cdots \wedge e_k = \frac{1}{l!} \left(\frac{1}{2} A_{ij} e_i \wedge e_j \right)^l = \left[e^{\frac{1}{2} A_{ij} e_i \wedge e_j} \right]_{(k)} \tag{2.23}$$

where the subscript (k) means taking the top degree (k-th) component.

We first show that Pfaffian is invariant. Let $g \in SO(k)$. Then

$$\begin{aligned} \mathrm{Pf}(g^{-1} A g) e_1 \wedge \cdots \wedge e_k &= \left[e^{\frac{1}{2} (g^{-1} A g)_{ij} e_i \wedge e_j} \right]_{(k)} \\ &= \left[e^{g^* (\frac{1}{2} A_{ij} e_i \wedge e_j)} \right]_{(k)} \\ &= \mathrm{Pf}(A) e_1 \wedge \cdots \wedge e_k, \end{aligned}$$

where we have used that $g^{-1} = g^t$ and $\det g = 1$.

Now any $A \in so(k)$ can be written $A = g^{-1}Dg$, where $g \in SO(k)$ and D is in block diagonal form

$$D = \begin{pmatrix} \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & a_l \\ -a_l & 0 \end{pmatrix} \end{pmatrix}.$$

Therefore $\mathrm{Pf}(A) = \mathrm{Pf}(D) = a_1 \cdots a_l$ and hence $\mathrm{Pf}(A)^2 = \det(A)$ as claimed. ■

Note that in fact

$$\mathrm{Pf}(A) = \frac{1}{2^l l!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) A_{\sigma(1)\sigma(2)} \cdots A_{\sigma(2l-1)\sigma(2l)}.$$

Remark It turns out that the invariant polynomials of $SO(k)$ are generated by $\mathrm{Tr}(A^r)$ together with the Pfaffian.

Consider an oriented real vector bundle $E \to M$ of rank k. By giving E a metric and taking an orthogonal connection, we get a $so(k)$-valued curvature 2-form $R^E = (R^E_{ij})$.

Definition: We define $\mathrm{Pf}(\frac{R^E}{2\pi})$ by the invariant polynomial Pf. That is

$$\mathrm{Pf}(\frac{R^E}{2\pi}) = \frac{1}{(4\pi)^l l!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) R^E_{\sigma(1)\sigma(2)} \cdots R^E_{\sigma(2l-1)\sigma(2l)}.$$

In particular we have $\mathrm{Pf}(\frac{R}{2\pi})$ for an even dimensional Riemannian manifold. In dimension 2, we find

$$\mathrm{Pf}(\frac{R}{2\pi}) = \frac{1}{4\pi}(R_{12} - R_{21}) = \frac{1}{2\pi} K d\mathrm{vol},$$

where K denotes the Gaussian curvature.

For $\dim M = 4$, one finds

$$\mathrm{Pf}(\frac{R}{2\pi}) = \frac{1}{32\pi^2} \sum_{\sigma \in S_4} \mathrm{sign} R_{\sigma(1)\sigma(2)} R_{\sigma(3)\sigma(4)} \tag{2.24}$$

In physicists' notation, $\mathrm{Pf}(\frac{R}{2\pi}) = \frac{1}{32\pi^2}\epsilon^{abcd} R_{ab} R_{cd}$.

Using the special structure in dimension 4, this formula can be made more explicit. In fact, the curvature decomposes

$$R = \begin{pmatrix} W^+ + \frac{S}{12}\mathrm{Id} & Z \\ Z & W^- + \frac{S}{12}\mathrm{Id} \end{pmatrix}, \tag{2.25}$$

where $W = W^+ + W^-$ is the Weyl curvature, Z the traceless Ricci, and S the scalar curvature. It can be verified that

$$\mathrm{Pf}(\frac{R}{2\pi}) = \frac{1}{8\pi^2}(|W|^2 - |Z|^2 + \frac{1}{24}S^2) d\mathrm{vol}. \tag{2.26}$$

In comparison, the Pontryagin classes depend only on the Weyl curvature part W. For example, in dimension 4,

$$p_1(M) = \frac{1}{4\pi^2}(|W^+|^2 - |W^-|^2)d\mathrm{vol}. \tag{2.27}$$

In fact, one has

Theorem 3.2.5 *The Pontryagin classes $p_i(M)$ of M depend only on the Weyl curvature W, and is thus conformally invariant.*

Proof:
Remark If $E \to M$ is a complex vector bundle, it has an underlying real vector bundle $E^{\mathbb{R}}$, which is canonically oriented by the complex structure. A unitary connection R^E on E gives rise to an orthogonal connection on $E^{\mathbb{R}}$ via the inclusion $U(k) \hookrightarrow SO(2k)$. Since the canonical inclusion $A \in GL(k,\mathbb{C}) \hookrightarrow A^{\mathbb{R}} \in GL(2k,\mathbb{R})$ has the property that $\det A^{\mathbb{R}} = |\det A|^2$, we deduce that

$$\mathrm{Pf}(E^{\mathbb{R}}) = (-1)^k c_k(E). \tag{2.28}$$

3.3 Atiyah-Singer index theorems

Let M be a closed spin manifold. The Dirac operator

$$D: \ C^\infty(M,S) \to C^\infty(M,S)$$

is self adjoint elliptic first order differential operator. By standard elliptic theory,

$$D: \ L^2(M,S) \to L^2(M,S)$$

is self adjoint Fredholm operator. Here $L^2(M,S)$ denotes the space of L^2 sections of S. Moreover, $\ker D \subset C^\infty(M,S)$. Thus, $\mathrm{ind} D = 0$.

But, when the dimension $n = \dim M$ is even, the spinor bundle S splits $S = S^+ \oplus S^-$, given by $c(\omega_{\mathbb{C}}) = (\sqrt{-1})^{[\frac{n+1}{2}]} c(e_1)\cdots c(e_n)$.

Lemma 3.3.1 *The Dirac operator D anti-commutes with $c(\omega_{\mathbb{C}})$.*

Proof: This is clear, since $D = c(e_i)\nabla^S_{e_i}$ and $c(\omega_{\mathbb{C}})$ anti-commutes with the Clifford multiplication $c(e_i)$ and is parallel with respect to ∇^S. ∎

Hence, with respect to the decomposition $S = S^+ \oplus S^-$,

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \tag{3.29}$$

and $(D^+)^* = D^-$. In physics literature $D^\pm$ are usually referred as the chiral Dirac operators. The index of the chiral Dirac operator D^+,

$$\mathrm{ind} D^+ = \dim\ker D^+ - \dim\ker D^- \in \mathbb{Z}$$

is given by the Aityah-Singer index theorem.

Theorem 3.3.2 (Atiyah-Singer index theorem for chiral Dirac operator) *For an even dimensional closed spin manifold M,*

$$\mathrm{ind} D^+ = \int_M \hat{A}(\frac{R}{2\pi}) = \hat{A}(M).$$

The proof of this index theorem and other index theorems in this section will be deferred to the next Chapter. We note here however that we can now deduce the topological obstruction to positive scalar curvature metrics.

Corollary 3.3.3 (Lichnerowicz) *If M is a closed spin manifold with a metric of positive scalar curvature, then $\hat{A}(M) = 0$.*

Proof: This clearly follows form the vanishing theorem for harmonic spinors and the Atiyah-Singer index theorem for chiral Dirac operator. ■

Example: The so called $K3$ surface is a quartic in $\mathbb{CP}^3$, which is spin, but $\hat{A} = 2$. Thus $K3$ can not have any metrics of positive scalar curvature.

Coupling with gauge fields leads us to the twisted Dirac operator. Thus, let $E \to M$ be an hermitian vector bundle with unitary connection ∇^E. Denote by F the curvature of ∇^E. When $\dim M$ is even, the twisted Dirac operator

$$D_E: \; C^\infty(M, S \otimes E) \to C^\infty(M, S \otimes E)$$

also has the off diagonal structure with respect to the splitting $S \otimes E = S^+ \otimes E \oplus S^- \otimes E$:

$$D_E = \begin{pmatrix} 0 & D_E^- \\ D_E^+ & 0 \end{pmatrix}. \tag{3.30}$$

Theorem 3.3.4 (Atiyah-Singer index theorem for twisted Dirac operator) *With the assumptions above and for closed M, we have*

$$\mathrm{ind} D_E^+ = \int_M \hat{A}(\frac{R}{2\pi}) \wedge \mathrm{ch}(\frac{\sqrt{-1}F}{2\pi}).$$

Using Chern roots we note that

$$\begin{aligned} \mathrm{ch}(\frac{\sqrt{-1}F}{2\pi}) &= \sum e^{x_i} = k + \sum x_i + \frac{1}{2!}\sum x_i^2 + \cdots \\ &= \mathrm{rank}\, E + \mathrm{Tr}(\frac{\sqrt{-1}F}{2\pi}) + \frac{1}{2!}\mathrm{Tr}(\frac{\sqrt{-1}F}{2\pi})^2 + \cdots \\ &= \mathrm{rank}\, E + c_1(E) + \frac{1}{2}(c_1^2(E) - c_2(E)) + \cdots . \end{aligned}$$

Hence, when $\dim M = 4$ and $E = L$ a line bundle, we have

$$\begin{aligned} \mathrm{ind} D_L^+ &= -\frac{1}{24} p_1(M) + \frac{1}{2} c_1^2(L) \qquad (3.31) \\ &= \frac{1}{192\pi^2}\int_M \mathrm{Tr}(R \wedge R) - \frac{1}{8\pi^2}\int_M F \wedge F. \end{aligned}$$

Remark If M is only spin^c, then its spin^c Dirac operator D^c is formally a twisted Dirac operator: $D^c = D_{L^{1/2}}$, where L is the canonical line bundle of the spin^c structure. It turns out that this formal twisting does give the right formula for the index of spin^c Dirac operator, i.e.,

$$\mathrm{ind} D^c_+ = \int_M \hat{A}(\frac{R}{2\pi}) \wedge e^{\frac{1}{2}c_1(L)}.$$

This is discussed in the following section.

3.4 Supertrace and Clifford module

In dealing with the index of Dirac operator, we have made use of the $\mathbb{Z}_2$-grading of the spinor bundle S given by $c(\omega_{\mathbb{C}})$. The essential feature here is that

$$(c(\omega_{\mathbb{C}})^2 = 1, \quad \text{and} \quad Dc(\omega_{\mathbb{C}}) + c(\omega_{\mathbb{C}})D = 0.$$

This turns out to play an essential role which can be generalized as follows.
Definition: [Super vector bundle] A vector bundle $E \to M$ is called super, or $\mathbb{Z}_2$-graded, if there is a bundle endomorphism $\sigma: \ E \to E$ such that $\sigma^2 = 1$.

The endomorphism σ defines a splitting $E = E^+ \oplus E^-$ as before. Elements of E^+ are called even while those of E^- are said to be odd. Conversely, any splitting gives rise to such an endomorphism.

Associated to a super vector bundle E, we have a so called supertrace.
Definition: [Supertrace] The supertrace

$$\mathrm{Tr}_s: \ C^\infty(M, \mathrm{End}(E)) \to C^\infty(M)$$

is defined by

$$\mathrm{Tr}_s(A) = \mathrm{Tr}(\sigma A).$$

In other words, if we write $A \in C^\infty(M, \mathrm{End}(E))$ in block forms with respect to the splitting $E = E^+ \oplus E^-$,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{4.32}$$

then

$$\mathrm{Tr}_s(A) = \mathrm{Tr}(A_{11}) - \mathrm{Tr}(A_{22}).$$

Just as ordinary trace, the supertrace extends to

$$\begin{array}{rcl} \mathrm{Tr}_s: \ \Omega^*(M, \mathrm{End}(E)) & \to & \Omega^*(M) \\ \omega \otimes A & \to & \omega \mathrm{Tr}_s(A). \end{array}$$

The $\mathbb{Z}_2$-grading of E induces a $\mathbb{Z}_2$-grading on $\mathrm{End}(E)$. In fact $\mathrm{End}^\pm(E)$ is the endomorphisms that commutes (anti-commutes, resp.) with σ. That is, in terms of (4.32), $\mathrm{End}^+(E)$ consists of diagonal block forms and $\mathrm{End}^-(E)$ the off diagonals. Write $\deg A = 0$ if $A \in \mathrm{End}^+(E)$, and $\deg A = 1$ if $A \in \mathrm{End}^-(E)$. This extends to $A \in \Omega^*(M, \mathrm{End}(E))$ where the degree will be the combined degree of differential forms (mod 2) and that of endomorphisms.

Definition: [Supercommutator] The supercommutator for homogeneous elements $A, B \in \Omega^*(M, \mathrm{End}(E))$ is

$$[A, B] = AB - (-1)^{\deg A \deg B} BA$$

and extends by linearity to all elements. Here the product is in terms of both the wedge product on differential forms and composition in endomorphisms.

Lemma 3.4.1 *The supertrace vanishes on the supercommutators: for any $A, B \in \Omega^*(M, \mathrm{End}(E))$,*

$$\mathrm{Tr}_s[A, B] = 0.$$

Proof: Check on the homogeneous elements. (Note: (super)trace of odd elements are zero; hence one only needs to check when both are even or both are odd.) ■

Remark Recall that for an elliptic differential operator $P : C^\infty(M, E) \oplus C^\infty(M, F)$ on a compact manifold M, its index can be written via heat equation method as

$$\mathrm{ind} P = \mathrm{Tr}(e^{-tP^*P}) - \mathrm{Tr}(e^{-tPP^*}).$$

Now $\tilde{E} = E \oplus F$ is $\mathbb{Z}_2$-graded with the obvious grading. Put

$$\tilde{P} = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}.$$

Then the heat equation formula can be rewritten as

$$\mathrm{ind} P = \mathrm{Tr}_s(e^{-t\tilde{P}^2}). \tag{4.33}$$

In particular, we have

$$\mathrm{ind} D^+ = \mathrm{Tr}_s(e^{-tD^2}). \tag{4.34}$$

We now consider a Clifford module $\mathcal{E} \to M$ with a Clifford connection $\nabla^{\mathcal{E}}$. Then its curvature decomposes as

$$R^{\mathcal{E}} = R^S + F^{\mathcal{E}/S}, \tag{4.35}$$

where

$$R^S(e_i, e_j) = \frac{1}{4} R_{ijkl} c(e_k) c(e_l)$$

is the part coming from the spinors (formally). Here $F^{\mathcal{E}/S}$ in fact commutes with all the Clifford multiplications, and is called the twisting curvature of $\mathcal{E}$. (Here is the proof: define $F^{\mathcal{E}/S} = R^{\mathcal{E}} - R$ and show that it commutes with all the Clifford multiplications using the compatibility of connections with Clifford multiplications.)

Example: If M is spin and $\mathcal{E} = S \otimes W$, then $F^{\mathcal{E}/S} = F^W$.
Remark The Bochner-Lichnerowicz-Weitzenbock formula can be rewritten using twisting curvature as

$$\mathcal{D}^2 = \nabla^*\nabla + c(F^{\mathcal{E}/S}) + \frac{R}{4}, \tag{4.36}$$

where

$$c(F^{\mathcal{E}/S}) = \frac{1}{2}F^{\mathcal{E}/S}(e_i, e_j)c(e_i)c(e_j),$$

and R here denotes the scalar curvature of M.

Assume now that the Dirac operator $\mathcal{D}$ of the Clifford module $\mathcal{E}$ is self adjoint with respect to some metric of $\mathcal{E}$ and the Riemannian metric on M. Further, let σ be a $\mathbb{Z}_2$-grading of $\mathcal{E}$ which anti-commutes with $\mathcal{D}$. Denotes by Tr_s the supertrace with respect to σ.

Theorem 3.4.2 (Aityah-Singer index theorem for Clifford module) *Let M be an even dimensional oriented closed manifold. Then with the assumptions above, we have*

$$\mathrm{ind}\mathcal{D}^+ = \int_M \hat{A}(\frac{R}{2\pi}) \wedge \mathrm{Tr}_s^{\mathcal{E}/S}(e^{\frac{\sqrt{-1}F^{\mathcal{E}/S}}{2\pi}}),$$

where the relative supertrace $\mathrm{Tr}_s^{\mathcal{E}/S}$ is defined by

$$\mathrm{Tr}_s^{\mathcal{E}/S}(A) = 2^{-n/2}\mathrm{Tr}_s(c(\omega_{\mathbb{C}})A).$$

Example: Continue from the previous example, for M spin and $\mathcal{E} = S \otimes W$,

$$\mathrm{Tr}_s^{\mathcal{E}/S}(e^{\frac{\sqrt{-1}F^{\mathcal{E}/S}}{2\pi}}) = \mathrm{Tr}(e^{\frac{\sqrt{-1}F^{W}}{2\pi}})$$

is the Chern character of W.

3.5 Classical examples

1). Gauss-Bonnet-Chern formula—The Atiyah-Singer index theorem applied to the Clifford module $\mathcal{E} = \Lambda^* M$ gives us the famous Gauss-Bonnet-Chern formula for the Euler number.

As we have seen before, for $\mathcal{E} = \Lambda^* M$, the Dirac operator $\mathcal{D} = c(e_i)\nabla^{\mathcal{E}}_{e_i} = d + d^*$. Here the Clifford multiplication is given by

$$c(e_i) = e_i \wedge - i_{e_i}.$$

Now $\sigma = (-1)^{\deg}$ defines a $\mathbb{Z}_2$-grading that clearly anti-commutes with $d + d^*$. Using Hodge theory, we compute

$$\begin{aligned}
\mathrm{ind}\mathcal{D}^+ &= \dim\ker(d+d^*)|_{\Omega^{even}} - \dim\ker(d+d^*)|_{\Omega^{odd}} \\
&= \dim\ker(d^*d+dd^*)|_{\Omega^{even}} - \dim\ker(d^*d+dd^*)|_{\Omega^{odd}} \\
&= \sum_{i=even} b_i(M) - \sum_{i=odd} b_i(M) = \chi(M).
\end{aligned}$$

To compute the right hand side of the Atiyah-Singer index theorem, we first figure out the twisting curvature. Now

$$R^{\mathcal{E}}(e_i, e_j) = -R_{ijkl} e_k \wedge \circ\, i_{e_l}.$$

To express this in terms of the Clifford multiplication, we introduce another Clifford multiplication:

$$\tilde{c}(e_i) = e_i \wedge + i_{e_i}. \tag{5.37}$$

One verifies easily that

$$\tilde{c}(e_i)\tilde{c}(e_j) + \tilde{c}(e_j)\tilde{c}(e_i) = 2\delta_{ij}. \tag{5.38}$$

Moreover, it commutes with the previous Clifford multiplication

$$c(e_i)\tilde{c}(e_j) = \tilde{c}(e_j)c(e_i). \tag{5.39}$$

Thus,

$$\begin{aligned} R^{\mathcal{E}}(e_i, e_j) &= -\frac{1}{4} R_{ijkl}(c(e_k) + \tilde{c}(e_k))(\tilde{c}(e_l) - c(e_l)) \\ &= R^S(e_i, e_j) - \frac{1}{4} R_{ijkl}\tilde{c}(e_k)\tilde{c}(e_l). \end{aligned}$$

That is, we have

$$F^{\mathcal{E}/S}(e_i, e_j) = -\frac{1}{4} R_{ijkl}\tilde{c}(e_k)\tilde{c}(e_l).$$

Or as 2-form valued endomorphism,

$$F^{\mathcal{E}/S} = -\frac{1}{4} R_{ijkl} e_i \wedge e_j\, \tilde{c}(e_k)\tilde{c}(e_l).$$

We now turn to the relative supertrace.

Lemma 3.5.1 *Let n be even. Then*

$$(-1)^{\deg} = c(\omega_{\mathbb{C}})\tilde{c}(\omega_{\mathbb{C}}),$$

where

$$\tilde{c}(\omega_{\mathbb{C}}) = (\sqrt{-1})^{n/2}\tilde{c}(e_1)\cdots\tilde{c}(e_n).$$

Proof: Clearly $c(\omega_{\mathbb{C}})1 = \tilde{c}(\omega_{\mathbb{C}})1$ when n is even. Hence

$$\begin{aligned} c(\omega_{\mathbb{C}})\tilde{c}(\omega_{\mathbb{C}})(e_{i_1} \wedge \cdots \wedge e_{i_p}) &= c(\omega_{\mathbb{C}})\tilde{c}(\omega_{\mathbb{C}})(c(e_{i_1})\cdots c(e_{i_p})1) \\ &= (-1)^p c(e_{i_1})\cdots c(e_{i_p}) c(\omega_{\mathbb{C}})\tilde{c}(\omega_{\mathbb{C}})1 \\ &= (-1)^p c(e_{i_1})\cdots c(e_{i_p}) c(\omega_{\mathbb{C}})c(\omega_{\mathbb{C}})1 \\ &= (-1)^p c(e_{i_1})\cdots c(e_{i_p})1 = (-1)^p e_{i_1} \wedge \cdots \wedge e_{i_p}. \end{aligned}$$

■

It follows that the relative supertrace is really the supertrace defined by $\tilde{c}(\omega_{\mathbb{C}})$. We now denote this supertrace, defined by $\tilde{c}(\omega_{\mathbb{C}})$ on $\mathcal{E} = \Lambda^* M$, simply by Tr_s.

Lemma 3.5.2 *For* $\mathrm{I}=\{i_1,\cdots,i_p\}\subset\{1,2\cdots,n\}$, *define* $\tilde c(e_{\mathrm{I}})=\tilde c(e_{i_1})\cdots\tilde c(e_{i_p})$. *Then*

$$\mathrm{Tr}_s(\tilde c(e_{\mathrm{I}}))=\begin{cases}0, & \text{if } \mathrm{I}\neq\{1,2\cdots,n\},\\ 2^n(-\sqrt{-1})^{n/2}, & \text{if } (i_1,\cdots,i_p)=(1,\cdots n).\end{cases}$$

Proof: If $\mathrm{I}\neq\{1,2\cdots,n\}$, we have a $j\notin\mathrm{I}$. Then

$$\tilde c(e_{\mathrm{I}})=-\frac{1}{2}[\tilde c(e_j),\tilde c(e_j)\tilde c(e_{\mathrm{I}})]$$

is a supercommutator. The first part follows. The second part is trivial. ∎

Remark This simple algebraic fact plays significant role in the proof of local index theorems and is the reason for the so-called "fantastic cancellations".
Remark Note the equivalent statement that trace vanishes on all $\tilde c(e_{\mathrm{I}})$ except the scalars.

Lemma 3.5.3 *For* $A=(A_{ij})\in so(n)$, *we have*

$$\mathrm{Tr}_s(e^{\frac{1}{2}A_{ij}\tilde c(e_i)\tilde c(e_j)})=2^{n/2}(-1)^{n/2}\hat A^{-1}(2\sqrt{-1}A)\mathrm{Pf}(2\sqrt{-1}A).$$

Proof: First of all, the same proof for Pfaffian shows that the left hand side is invariant under the adjoint action of $SO(n)$. Thus we can assume that A is in block diagonal form,

$$A=\begin{pmatrix}\begin{pmatrix}0 & y_1\\ -y_1 & 0\end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix}0 & y_{n/2}\\ -y_{n/2} & 0\end{pmatrix}\end{pmatrix}.$$

Hence,

$$\begin{aligned}\mathrm{Tr}_s(e^{\frac{1}{2}A_{ij}\tilde c(e_i)\tilde c(e_j)}) &= \mathrm{Tr}_s(e^{\sum y_i\tilde c(e_{2i-1})\tilde c(e_{2i})})=\mathrm{Tr}_s(\prod_{i=1}^{n/2}e^{y_i\tilde c(e_{2i-1})\tilde c(e_{2i})})\\ &= \prod_{i=1}^{n/2}(\cos y_i+\sin y_i\tilde c(e_{2i-1})\tilde c(e_{2i}))\\ &= 2^n(-\sqrt{-1})^{n/2}\prod_{i=1}^{n/2}\sin y_i\\ &= 2^{n/2}(-1)^{n/2}\hat A^{-1}(2\sqrt{-1}A)\mathrm{Pf}(2\sqrt{-1}A).\end{aligned}$$

Here we have used the previous lemma. ∎

Remark Using the previous remark, we also have

$$\begin{aligned}\mathrm{Tr}(e^{\frac{1}{2}A_{ij}\tilde c(e_i)\tilde c(e_j)}) &= 2^n\prod\cos y_i\\ &= 2^n\hat A^{-1}(2\sqrt{-1}A)\mathcal{L}(2\sqrt{-1}A),\end{aligned}$$

where

$$\mathcal{L}(x) = \frac{x/2}{\tanh x/2}.$$

Noting that the equality in the lemma extends to even differential form valued endomorphisms, we plug in $A = -\frac{\sqrt{-1}}{4\pi}R$, i.e., $2\sqrt{-1}A = \frac{R}{2\pi}$, to obtain the Gauss-Bonnet-Chern formula.

2). Hirzebruch signature fromula—Our second classical example concerns the signature of a closed oriented manifold, a topological invariant, defined as follows.

Let M be a $4k$ dimensional closed oriented manifold. The cup product defines a bilinear form on the middle dimensional cohomology,

$$\begin{aligned} B:\ H^{2k}(M,\mathbb{R}) \times H^{2k}(M,\mathbb{R}) &\rightarrow \mathbb{R} \\ (\alpha,\beta) &\rightarrow (\alpha\cup\beta)[M] = \int_M \alpha\wedge\beta. \end{aligned} \tag{5.40}$$

The second interpretation uses de Rham theory. By Poincaré duality, B is non-degenerate. It is also symmetric. The signature of M is defined to be the signature of this symmetric bilinear form B,

$$\tau(M) = \operatorname{sign} B. \tag{5.41}$$

Note that

$$\tau(M) = \dim V_+ - \dim V_-$$

where $V_\pm$ are, respectively, the maximal subspace of $H^{2k}(M,\mathbb{R})$ on which B is positive (resp. negative) definite.

Remark The signature is clearly a homotopy invariant.

It turns out that the signature $\tau(M)$ is the index of a geometric differential operator. To see this, we first recall that the Poncaré duality for a closed oriented Riemannian manifold of dimension n can also be expressed in terms of the Hodge $*$-operator,

$$*:\ \Lambda^p M \rightarrow \Lambda^{n-p} M.$$

It is characterized by the equation

$$\alpha\wedge *\beta = \langle\alpha,\beta\rangle d\mathrm{vol}.$$

We also note the following basic properties:

i). $*^2 = (-1)^{(n+1)p}$ on $\Lambda^p M$.
ii). $d^* = (-1)^{n(p+1)+1} * d *$ on $\Omega^p(M)$.
iii). $*$ is parallel with respect to the Levi-Civita connection.

Now let $n = 4k$ and set $\sigma = (-1)^{p(p-1)/2+k} *$ on $\Lambda^p M$. Then

$$\sigma:\ \Lambda^* M \rightarrow \Lambda^* M$$

is a $\mathbb{Z}_2$-grading: $\sigma^2 = 1$. Thus

$$\Lambda^* M = \Lambda^*_+ M \oplus \Lambda^*_- M \tag{5.42}$$

decomposes into the ± 1-eigenspaces of σ. The forms in $\Lambda^*_\pm M$ are called self-dual (resp. anti-self-dual).

Also, $d^* = -\sigma d \sigma$, which implies that $d + d^*$ anti-commutes with σ,

$$\sigma(d + d^*) + (d + d^*)\sigma = 0.$$

Hence σ commutes with $\Delta = dd^* + d^*d$. Therefore, σ induces an automorphism on the space of harmonic forms $\mathcal{H}^* = \oplus_p \mathcal{H}^p$, $\mathcal{H}^p = \ker \Delta_p$:

$$\sigma : \ \mathcal{H}^* \to \mathcal{H}^*.$$

Thus,

$$\mathcal{H}^* = \mathcal{H}^*_+ \oplus \mathcal{H}^*_-.$$

Note that

$$\sigma = * : \ \Lambda^{2k} M \to \Lambda^{2k} M$$

leaves the middle degree forms invariant. Thus

$$\mathcal{H}^{2k} = \mathcal{H}^{2k}_+ \oplus \mathcal{H}^{2k}_-.$$

Lemma 3.5.4 *We have*

$$\tau(M) = \dim \mathcal{H}^{2k}_+ - \dim \mathcal{H}^{2k}_-.$$

Proof: If $\alpha \neq 0 \in \mathcal{H}^{2k}_+$, then $\alpha = *\alpha$. Therefore,

$$\begin{aligned} B(\alpha, \alpha) &= \int_M \alpha \wedge \alpha \\ &= \int_M \alpha \wedge *\alpha \\ &= \int_M |\alpha|^2 d\mathrm{vol} > 0. \end{aligned}$$

Similarly, $\alpha \neq 0 \in \mathcal{H}^{2k}_-$, then $\alpha = - * \alpha$. Hence,

$$B(\alpha, \alpha) = -\int_M |\alpha|^2 d\mathrm{vol} < 0.$$

The lemma follows. ∎

Coming back to the operator $d + d^*$, since it anti-commutes with σ, it will be of off-diagonal form with respect to the splitting (5.42).

Definition: [Signature operator] The operator

$$A = (d + d^*) : \ C^\infty(M, \Lambda^*_+ M) \to C^\infty(M, \Lambda^*_- M)$$

is called the signature operator.

Lemma 3.5.5 *The index of the signature operator is the signature,*

$$\mathrm{ind}A = \tau(M).$$

Proof: We have

$$\mathrm{ind}A = \dim\ker A - \dim\ker A^* = \dim \mathcal{H}^*_+ - \dim \mathcal{H}^*_-.$$

Now,

$$\mathcal{H}^* = \mathcal{H}^{2k} \oplus (\oplus_{p\neq 2k}\mathcal{H}^p)$$

and the splitting is invariant under σ. Therefore,

$$\mathcal{H}^*_\pm = \mathcal{H}^{2k}_\pm \oplus (\oplus_{p\neq 2k}\mathcal{H}^p)_\pm.$$

We now see that the contributions from dimensions other than the middle dimension cancel out. In fact,

$$\begin{array}{rcl}\oplus_{p<2k}\mathcal{H}^p & \to & (\oplus_{p\neq 2k}\mathcal{H}^p)_\pm \\ \alpha & \to & \dfrac{1}{2}(\alpha \pm \sigma\alpha)\end{array}$$

defines an isomorphism. Hence

$$\dim(\oplus_{p\neq 2k}\mathcal{H}^p)_\pm = \dim \oplus_{p<2k}\mathcal{H}^p.$$

Thus,

$$\mathrm{ind}A = \dim \mathcal{H}^{2k}_+ - \dim \mathcal{H}^{2k}_- = \tau(M).$$

■

Applying the Atiyah-Singer index theorem, we arrive at

Theorem 3.5.6 (Hirzebruch signature theorem) *Let M be a closed oriented manifold of dimension $4k$. Then*

$$\tau(M) = \int_M L(\frac{R}{2\pi}).$$

Proof: To compute the right hand side of the Atiyah-Singer index theorem, we follow the same route as the Gauss-Bonnet-Chern and arrive at

$$\mathrm{ind}A = 2^{n/2}\int_M \mathcal{L}(\frac{R}{2\pi}) = \int_M L(\frac{R}{2\pi}),$$

where $\mathcal{L}(x) = \frac{x/2}{\tanh x/2}$ while $L(x) = \frac{x}{\tanh x}$. The apparent difference in the formulas produce the same result when integrated over M, since we only take the top degree forms and the factor $2^{n/2}$ clearly compensates for the change of scale by $\frac{1}{2}$. ■

In particular, in dimension 4,

$$\tau(M) = \frac{1}{3}p_1(M).$$

Thus,

$$\tau(M) = -8\hat{A}(M)$$

for 4-manifolds. Now $\hat{A}(M)$ is an integer if M is spin. Therefore, the signature of a 4 dimensional spin manifold is a multiple of 8. In fact, one has

Theorem 3.5.7 (Rohlin) *The signature of a 4 dimensional spin manifold is a multiple of 16,*

$$\tau(M^4) \equiv 0 \mod 16.$$

When $\dim M = 8$, the signature formula becomes

$$\tau(M) = \frac{1}{45}(7p_2 - p_1^2)(M^8).$$

This formula played a crucial role in the discovery of exotic sphere.

Remark Thinking about spaces one step more complicated than the spheres, Milnor considered closed manifolds M^{2n} of dimension $2n$ which is $(n-1)$-connected. By Poincaré duality, all cohomology groups are trivial except the top, the bottom and the middle:

$$H^0(M) \cong \mathbb{Z}, \quad H^n(M) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}, \quad H^{2n}(M) \cong \mathbb{Z}.$$

Homotopically, such spaces have a simple structure:

$$M^{2n} \simeq_{h.} (S^n \vee \cdots \vee S^n) \cup_f e^{2n},$$

where e^{2n} is a $2n$-cell and f attaching map. Clearly, signature plays an important role in the study of such spaces.

Assuming now that $\dim H^n(M) = 1$, then $\tau(M) = \pm 1$. By fixing the orientation, we can assume that $\tau(M) = 1$. In dimension 4 ($2n = 4$), $\mathbb{CP}^2$ is the only space satisfying these assumptions. Things are more interesting in dimension 8 ($2n = 8$). In this case, $M^8 \simeq_{h.} S^4 \cup_f e^8$, and Whitney showed that the homotopic S^4 in fact can be made to be smoothly embedded,

$$S^4 \hookrightarrow M^8.$$

Thus M^8 is made by attaching an 8-cell e^8 to (the disk bundle of) the normal bundle $\nu(S^4)$.

This leads Milnor to studying rank 4 vector bundles $E \to S^4$. Such bundles can be constructed as follows. Decomposing S^4 into the upper and lower hemispheres,

$$D^4 = D^4_+ \cup D^4_-,$$

E can be obtained by gluing two trivial bundles on $D^4_\pm$ by the gluing map (transition function)

$$f: \; S^3 \to SO(4).$$

Now, $\mathbb{R}^4 = \mathbb{H}$ can be thought as the quaternions, and $S^3 \subset \mathbb{H}$ the unit ones. Setting

$$f(x)y = x^i y x^j$$

gives us a family of rank 4 vector bundles $E_{i,j}$.

Let $N_{i,j}$ be the sphere bundle of $E_{i,j}$. It is the boundary of the disk bundle, to which the 8-cell is attached. Thus, we must require $N_{i,j}$ to be a 7-sphere, if it were to be used to construct the 8 dimensional manifolds considered above. It turns out that for $N_{i,j}$ to be a homotopic sphere if and only if $i+j=\pm 1$. So take $j = i-1$. In this case, Morse theory shows that it is in fact a topological sphere.

The question now is: is it also a differentiable sphere? If so, one would be able to construct a smooth manifold M^8 as before, whose signature is one,

$$1 = \tau(M^8) = \frac{1}{45}(7p_2 - p_1^2)(M^8).$$

Or equivalently,

$$p_2(M^8) = \frac{p_1^2(M) + 45}{7}.$$

The first Pontryagin class can be computed explicitly, using the isomorphism $H^4(M^8) \to H^4(S^4)$,

$$p_1(M^8) = 2(i-j) = 2(2i-1).$$

Hence, if $N_{i,i-1}$ is a differentiable sphere, the associated 8-manifold M^8 would have

$$p_2(M^8) = \frac{4(2i-1)^2 + 45}{7}.$$

Keeping in mind that Pontryagin classes are integral, we gingerly plug in experimental values of i:

$$\begin{aligned} i = 1: &\quad p_2 = 7 \\ i = 2: &\quad p_2 = \tfrac{81}{7}! \end{aligned}$$

The rest is history.

3). Riemann-Roch Theorem—Let M now be a closed Kähler manifold of $\dim_{\mathbb{C}} = n$. If $E \to M$ is a holomorphic vector bundle with an hermitian metric and unitary connection ∇^E, its arithmetic genus is defined as

$$\chi(M,E) = \sum_{q=0}^{n} (-1)^q \dim H^q(M, \mathcal{O}(E)). \tag{5.43}$$

Now recall that $\mathcal{E} = \Lambda^{0,*}M \otimes E$ is a Clifford module with associated Dirac operator $\mathcal{D} = \sqrt{2}(\bar\partial + \bar\partial^*)$. Further $\sigma = (-1)^q$ on $\Lambda^{0,q}M \otimes E$ defines a $\mathbb{Z}_2$-grading on the Clifford module $\mathcal{E}$. From Delbeault theorem and the Hodge theory we deduce that

$$\mathrm{ind}\mathcal{D}^+ = \chi(M, E).$$

Applying the Atiyah-Singer index theorem we arrive at the Riemann-Roch-Hirzebruch theorem.

Theorem 3.5.8 (Riemann-Roch-Hirzebruch theorem) *With the assumptions as above, one has*

$$\chi(M, E) = \int_M Td(\frac{\sqrt{-1}R^{1,0}}{2\pi}) \wedge ch(\frac{\sqrt{-1}F^E}{2\pi}),$$

where $R^{1,0}$ is the curvature of the holomorphic tangent bundle $T^{1,0}M$ and the Todd genus is defined in (2.16).

The derivation is similar to the previous two examples so rather than going through it we look at a special case of the theorem: M a Riemann surface (i.e., $n = 1$) and $E = L$ a holomorphic line bundle. Then

$$Td(\frac{\sqrt{-1}R^{1,0}}{2\pi}) = 1 - \frac{\sqrt{-1}R^{1,0}}{4\pi}, \quad ch(\frac{\sqrt{-1}F^E}{2\pi}) = 1 - \frac{\sqrt{-1}F^E}{2\pi}.$$

Thus the Riemann-Roch-Hirzebruch theorem becomes

$$\dim H^0(M, \mathcal{O}(L)) - \dim H^1(M, \mathcal{O}(L)) = -\frac{\sqrt{-1}}{4\pi}\int_M (R^{1,0} + 2F^L).$$

To see what the right hand gives, we let L be the trivial line bundle first. In this case, $\dim H^0(M, \mathcal{O}(L)) = 1$ and $\dim H^1(M, \mathcal{O}(L)) = g$ is the genus of M. Hence the above formula gives

$$1 - g = -\frac{\sqrt{-1}}{4\pi}\int_M R^{1,0}.$$

On the other hand

$$-\frac{\sqrt{-1}}{2\pi}\int_M F^L = \deg L$$

is the degree of the holomorphic line bundle L. Therefore, the Riemann-Roch-Hirzebruch theorem can finally be stated as

$$\dim H^0(M, \mathcal{O}(L)) - \dim H^1(M, \mathcal{O}(L)) = 1 - g + \deg L.$$

This is the classical Riemann-Roch formula.

Remark If M is complex but not Kähler, then $\mathcal{D} = \sqrt{2}(\bar{\partial} + \bar{\partial}^*) + l.o.t.$. By the stability of index, the Riemann-Roch-hirzebruch formula still holds.

The classical examples all come from some complexes. This is generalized by Atiyah-Singer to the so called elliptic complex, which consists of vector bundles $E_0, \cdots, E_m$ together with

$$D: \ 0 \to C^\infty(E_0) \overset{D_0}{\to} C^\infty(E_1) \overset{D_1}{\to} \cdots \overset{D_{m-1}}{\to} C^\infty(E_m) \to 0$$

such that
1). each D_i is a differential operator of order p;
2). $D_{i+1}D_i = 0$ ($D^2 = 0$);
3). for any nonzero cotangent vector $\xi \in T_x^*M$,

$$0 \to (E_0)_x \xrightarrow{\sigma(D_0)(\xi)} (E_1)_x \xrightarrow{\sigma(D_1)(\xi)} \cdots \xrightarrow{\sigma(D_{m-1})(\xi)} (E_m)_x \to 0$$

is an exact sequence.

Define the index of the elliptic complex by

$$\mathrm{ind} D = \sum_{i=0}^{m} (-1)^i \dim H^i(D),$$

where

$$H^i(D) = \ker D_i / \mathrm{Im} D_{i-1}.$$

The Atiyah-Singer index theorem in this case states

$$\mathrm{ind} D = (-1)^n \left(\frac{1}{e(T^*M)} (\sum_{i=0}^{m} (-1)^i ch(E_i)) Td(TM \otimes \mathbb{C}) \right) [M]. \tag{5.44}$$

A single elliptic operator $P: \ C^\infty(E) \to C^\infty(F)$ corresponds to a (much shortened) elliptic complex

$$0 \to \ C^\infty(E) \xrightarrow{P} C^\infty(F) \to 0.$$

3.6 Proof of the local index theorem

A great theorem such as Atiyah-Singer index theorem has great many different proofs, each having its own merits. We indicate the proof of local index theorem modelled after Getzler. We will concentrate on the Dirac operator and make remarks for the general Clifford module.

The outline is as follows:
Step 1—McKean-Singer formula for the index;
Step 2—localization; usually via the asymptotic expansion of heat kernel;
Step 3—Getzler's rescaling;
Step 4—heat kernel for generalized harmonic oscillators.

We now start with **Step 1**. Thus, let M^n be a closed even dimensional spin manifold, and

$$D: \ C^\infty(M,S) \to C^\infty(M,S)$$

the Dirac operator acting on spinor fields. Recall that $\sigma = c(\omega_{\mathbb{C}})$ is a $\mathbb{Z}_2$-grading on S, and the heat equation method gives

$$\mathrm{ind} D^+ = \mathrm{Tr}_s(e^{-tD^2}).$$

Since D is a self adjoint elliptic differential operator, the spectrum of D^2 consists of eigenvalues of finite multiplicity. Let $\varphi_k \in C^\infty(M,S)$ be a spectral decomposition of D^2. That is, φ_k is an orthonormal basis of $L^2(M,S)$ which are eigensections of D^2 with eigenvalue λ_k,

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots .$$

Then the heat operator

$$e^{-tD^2} : \ C^\infty(M,S) \to C^\infty(M,S)$$

is given by

$$\begin{aligned} (e^{-tD^2} s)(x) &= \sum_{k=1}^{\infty} e^{-t\lambda_k}(s,\varphi_k)\varphi_k(x) \\ &= \int_M K(t,x,y)s(y)d\mathrm{vol}, \end{aligned}$$

where

$$K(t,x,y) = \sum_{k=1}^{\infty} e^{-t\lambda_k}\varphi_k(x) \otimes \varphi_k^*(y)$$

is called the heat kernel of D^2. Here $\varphi_k^*(y)s = \langle \varphi_k^*(y), s(y)\rangle$ is the metric dual. Hence,

$$K(t,x,y) : \ S_y \to S_x.$$

We record the basic properties of heat kernel in the following proposition.

Proposition 3.6.1 *1. The heat kernel depends smoothly on $x, y \in M$ and $t \in \mathbb{R}^+ = (0,\infty)$. That is,*

$$K(t,x,y) \in C^\infty(\mathbb{R}^+ \times M \times M, \pi_1^* S \otimes \pi_2^* S^*),$$

where π_1 (resp. π_2) is the projection onto the first (resp. second) M factor.
2. The heat kernel $K(t,x,y)$ satisfies the heat equation with respect to either variable,

$$(\partial_t + D^2)K(t,x,y) = 0.$$

3. The heat kernel converges to the delta function as t goes to zero.

Consequently, using Lidskii's theorem, we have

$$\mathrm{ind}D^+ = \int_M \mathrm{Tr}_s(K(t,x,x)d\mathrm{vol}. \tag{6.45}$$

This is the McKean-Singer formula. Note that $K(t,x,x) : S_x \to S_x$, i.e. $K(t,x,x) \in \mathrm{End}(S_x)$.

Example: In $\mathbb{R}^n$, $D^2 = \Delta \mathrm{I}$ with $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$. Since $u = e^{-t\Delta} f$ is the solution of the initial value problem

$$\begin{cases} (\partial_t + \Delta)u = 0, \quad t > 0, \\ u|_{t=0} = f, \end{cases}$$

one can solve using any standard method (Fourier transform, for example) to obtain

$$e^{-t\Delta}f(x) = u(x) = \int_{\mathbb{R}^n} K(t,x,y)f(y)dy,$$

where

$$K(t,x,y) = \frac{1}{(4\pi t)^{n/2}}e^{-|x-y|^2/4t}.$$

In particular, the heat kernel along the diagonal

$$K(t,x,x) = \frac{1}{(4\pi t)^{n/2}}$$

is singular as $t \to 0$. Note however, in taking supertrace, we have the difference of two heat kernels. The "fantastic cancellation" conjectured by McKean-Singer says that, in fact, all singular terms cancel out, leaving us with the local index density:

$$\lim_{t\to 0} \mathrm{Tr}_s(K(t,x,x))dx = \hat{A}(\frac{R}{2\pi}). \tag{6.46}$$

This is now referred as the local index theorem.

Step 2: For small time, the Euclidean heat kernel approximates any heat kernel. This leads to the asymptotic expansion for heat kernel.

Theorem 3.6.2 *We have an uniform asymptotic expansion as $t \to 0$*

$$K(t,x,x) \sim \frac{1}{(4\pi t)^{n/2}}[a_0(x) + a_1(x)t + \cdots a_k(x)t^k + \cdots]$$

where $a_k(x)$ is a universal polynomial in the derivatives of coefficients of D^2 at x.

In particular, the limit as t goes to zero of $\mathrm{Tr}_s(K(t,x,x))$ depends only on the operator D^2 near x. This enables us to localize the problem to a neighborhood of x.

We now fix a point x_0 and let x be the normal coordinates around x_0. Thus $x = 0$ at x_0. We trivialize the spinor bundle S in the normal neighborhood by parallel translation along radial geodesics from x_0. In fact, by extending everything trivially outside the normal neighborhood, we can assume that $M = \mathbb{R}^n$ with a metric which is Euclidean outside a compact set.

For simplicity, we will first see what the usually rescaling will give us. The usual rescaling is obtained by dilating the coordinates. Namely, given $\epsilon > 0$, for any section $a(t,x)$,

$$(\delta_\epsilon a)(t,x) = a(\epsilon t, \epsilon^{\frac{1}{2}}x).$$

Then, clearly, we have

$$\lim_{t\to 0} \mathrm{Tr}(a(t,0)) = \lim_{\epsilon\to 0} \mathrm{Tr}(\delta_\epsilon a)(t,0).$$

We will of course apply this to $a(t,x) = K(t,x,0)$, the heat kernel for D^2. On the other hand, $\epsilon^{\frac{n}{2}}\delta_\epsilon K$ is the heat kernel K_ϵ for

$$D_\epsilon^2 = \epsilon\delta_\epsilon D^2 \delta_\epsilon^{-1}.$$

Here the factor $\epsilon^{\frac{n}{2}}$ is from the volume form (or the measure). Therefore

$$\lim_{t\longrightarrow 0} t^{\frac{n}{2}}\,\mathrm{Tr}(a(t,0)) = t^{\frac{n}{2}} \lim_{\epsilon\longrightarrow 0} \mathrm{Tr}(\epsilon^{\frac{n}{2}}(\delta_\epsilon a)(t,0)).$$

Now,

$$D_\epsilon^2 = \Delta Id_S + O(\epsilon^{\frac{1}{2}}).$$

This can be easily seen from the Lichnerowicz formula and the local formulas

$$\delta_\epsilon a(t,x)\delta_\epsilon^{-1} = a(\epsilon t, \epsilon^{\frac{1}{2}}x); \qquad \delta_\epsilon \partial_{x_i}\delta_\epsilon^{-1} = \epsilon^{-\frac{1}{2}}\partial_{x_i},$$

where the smooth function $a(t,x)$ acts as multiplication. Since the coefficients in the asymptotic expansion for heat kernel are universal polynomials in the derivatives of the coefficients of the operator at the given point, it follows that

$$\lim_{\epsilon\longrightarrow 0} \mathrm{Tr}(\epsilon^{\frac{n}{2}}(\delta_\epsilon K)(t,0)) = \lim_{\epsilon\longrightarrow 0} \mathrm{Tr} K_\epsilon(t,0,0) = \mathrm{Tr} K_0(t,0,0).$$

Here $K_0(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} Id_S$ is the heat kernel for $D_0^2 = \Delta Id_S$. Combining all these, we obtain

$$\lim_{t\longrightarrow 0} t^{\frac{n}{2}}\,\mathrm{Tr}(K(t,0,0)) = \frac{\mathrm{rank} S}{(4\pi)^{n/2}}.$$

This, up to a Tauberian theorem, is the Weyl's asymptotic estimate on eigenvalues of Laplacian. It is essentially the leading term a_0 in the asymptotic expansion of the heat kernel. It is of course far away from the local index theorem, which concerns the n-th term a_n.

Step 3: Getzler's rescaling is a modification of the usual rescaling that captures the essence of the "fantastic cancellation" through supertrace. Roughly speaking, it is a rescaling that incorporates the Clifford degrees. Recall that $K(t,x,x) \in \mathrm{End}(S_x)$ are linear combinations of Clifford multiplications $c(e_I)$ where $I = i_1, \cdots, i_p, i_1 < \cdots < i_p$ are multi-indices. In fact, since we have trivialized the spinor bundle S, this continues to hold true for $K(t,x,0)$. Now, roughly speaking, for any

$$a(t,x) = \sum_I a_I(t,x)c(e_I),$$

the Getzler's rescaling is defined as

$$(\delta_\epsilon a)(t,x) = \sum_I a_I(\epsilon t, \epsilon^{\frac{1}{2}}x)\epsilon^{-|I|/2}c(e_I).$$

Here $|I| = p$ is the order of the multi-index.

Similar to Lemma (3.5.2), we have

Lemma 3.6.3 *For* $\mathrm{I} = \{i_1, \cdots, i_p\} \subset \{1, 2 \cdots, n\}$,

$$\mathrm{Tr}_s(c(e_\mathrm{I})) = \begin{cases} 0, & \textit{if } \mathrm{I} \neq \{1, 2 \cdots, n\}, \\ (-2\sqrt{-1})^{n/2}, & \textit{if } (i_1, \cdots, i_p) = (1, \cdots n). \end{cases}$$

Proof: The proof is the same, with $c(e_I)$ replacing $\tilde{c}(e_I)$ (also note that $\dim S = 2^{n/2}$). ■

Thus, we have

Lemma 3.6.4 *For any* $a(t,x) = \sum_I a_I(t,x)c(e_I)$,

$$\lim_{t\to 0} \mathrm{Tr}_s(a(t,0)) = \lim_{\epsilon\to 0} \mathrm{Tr}_s(\epsilon^{\frac{n}{2}}(\delta_\epsilon a)(t,0)).$$

Once again, $K_\epsilon(t,x,y) = \epsilon^{\frac{n}{2}}(\delta_\epsilon K)(t,x,y)$ is the heat kernel for

$$D_\epsilon^2 = \epsilon \delta_\epsilon D^2 \delta_\epsilon^{-1},$$

provided we define what is the rescaling on the spinors. To this end, we embed S into $\Lambda^* M$, in which the Clifford multiplication becomes

$$c(e_i) = e_i \wedge - i_{e_i}.$$

We now rescale $\Lambda^* M$ by its degree:

$$\delta_\epsilon(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \epsilon^{-p/2} e_{i_1} \wedge \cdots \wedge e_{i_p}.$$

This rescales S accordingly.

And once again, to compute D_ϵ^2, we have

$$\delta_\epsilon f(t,x)\delta_\epsilon^{-1} = f(\epsilon t, \epsilon^{\frac{1}{2}} x); \qquad \delta_\epsilon \partial_{x_i} \delta_\epsilon^{-1} = \epsilon^{-\frac{1}{2}} \partial_{x_i},$$

where $f(t,x)$ is a scalar function. However, we also have now

$$\delta_\epsilon c(e_i) \delta_\epsilon^{-1} = c_\epsilon(e_i),$$

where

$$c_\epsilon(e_i) = \epsilon^{-\frac{1}{2}} e_i \wedge - \epsilon^{\frac{1}{2}} i_{e_i}.$$

In particular, we note that

$$\lim_{\epsilon\to 0} \epsilon^{\frac{1}{2}} c_\epsilon(e_i) = e_i \wedge .$$

Now we define the Getzler's recaling on the kernel functions accordingly. For any $a(t,x) = \sum_I a_I(t,x)c(e_I)$,

$$(\delta_\epsilon a)(t,x) = \sum_I a_I(\epsilon t, \epsilon^{\frac{1}{2}} x) c_\epsilon(e_I).$$

Then, we have

Lemma 3.6.5 *For any* $a(t,x) = \sum_I a_I(t,x)c(e_I)$,

$$\lim_{t\to 0} \mathrm{Tr}_s(a(t,0)) = (-2\sqrt{-1})^{n/2}[\lim_{\epsilon\to 0} \epsilon^{\frac{n}{2}}(\delta_\epsilon a)(t,0)]_{(n)},$$

where $[\ ]_{(n)}$ *means taking the degree* n *term (the top degree term).*

Proof: We have $\lim_{t\to 0}\mathrm{Tr}_s(a(t,0)) = (-2\sqrt{-1})^{n/2}a_{12\cdots n}(0,0)$. On the other hand, $\lim_{\epsilon\to 0}\epsilon^{\frac{n}{2}}(\delta_\epsilon a)(t,0) = a_{12\cdots n}(0,0)e_1\wedge\cdots\wedge e_n$. Comparing the two we have the desired identity. ∎

By the Lichnerowicz formula,

$$\begin{aligned} D^2 = &\quad \nabla^*\nabla + \tfrac{R}{4} \\ = &\quad -g^{ij}(\partial_{x_i} + \tfrac{1}{4}\tilde{\omega}_{ab}(\partial_{x_i})c(e_a)c(e_b))(\partial_{x_j} + \tfrac{1}{4}\tilde{\omega}_{ab}(\partial_{x_j})c(e_a)c(e_b)) \\ &\quad +g^{ij}\Gamma_{ij}^k(\partial_{x_k} + \tfrac{1}{4}\tilde{\omega}_{ab}(\partial_{x_k})c(e_a)c(e_b)) + \tfrac{R}{4}. \end{aligned}$$

Here $x_1,\cdots,x_n$ are local coordinates near the origin 0, which we can choose so that

$$g_{ij} = \delta_{ij} + O(|x|^2),$$

and $e_1,\cdots,e_n$ are orthonormal basis which we choose to be obtained from parallel translating $\partial_{x_1},\cdots,\partial_{x_n}$ at 0 along radial geodesics. From Riemannian geometry, we have

$$\tilde{\omega}_{ab}(\partial_{x_i}) = -\frac{1}{2}R_{ijab}x_j + O(|x|^2),$$

where R_{ijab} is the curvature tensor at the origin 0 It follows then

$$D_\epsilon^2 = \mathcal{L} + O(\epsilon),$$

where

$$\mathcal{L} = -\sum_i(\partial_{x_i} - \frac{1}{4}\Omega_{ij}x_j)^2$$

is the generalized harmonic oscillator. Here

$$\Omega_{ij} = \frac{1}{2}R_{ijab}e^a\wedge e^b$$

is the curvature 2-form at the origin. And we conclude that K_ϵ converges to K_0, the heat kernel of the generalized harmonic oscillator $\mathcal{L}$.

Step 4: It turns out that the heat kernel for $\mathcal{L}$ can be computed explicitly. We first look at the case of the harmonic oscillator.

On the real line $\mathbb{R}$, the operator

$$H = -\frac{d^2}{dx^2} + \lambda^2x^2$$

is called the harmonic oscillator. Without loss of generality, we let $\lambda = 1$, the general case being a change of variable. Note that

$$H = aa^+ - 1 = a^+a + 1,$$

with annihilation and creation operator given respectively by

$$a = \frac{1}{i}(\frac{d}{dx} + x), \qquad a^+ = \frac{1}{i}(\frac{d}{dx} - x).$$

They satisfy

$$[H, a] = -2a, \quad [H, a^+] = 2a^+, \quad [a, a^+] = 2.$$

The eigenfunctions of H are obtained by applying the creation operator a^+ to the vacuum $u_0 = e^{-x^2/2}$ which is solved from the equation $au_0 = 0$. Thus $u_k = i^k(a^+)^k u_0 = H_k(x)e^{-x^2/2}$ is an eigenfunction of H with eigenvalue $2k+1$, where $H_k(x)$ is the k-th Hermite polynomial. Since u_k forms a complete basis of the L^2 space, this gives the spectral decomposition of H. In particular

$$\mathrm{Tr}(e^{-tH}) = \sum_{k=0}^{\infty} e^{-(2k+1)t} = e^{-t}\frac{1}{1-e^{-2t}} = \frac{1}{2\sinh t}.$$

Lemma 3.6.6 (Mehler's formula) *The heat kernel of H is given by*

$$K(t,x,y) = \frac{1}{(4\pi t)^{1/2}}(\frac{2t}{\sinh 2t})^{1/2}\exp(-\frac{1}{2\tanh 2t}(x^2+y^2) + \frac{1}{\sinh 2t}xy).$$

Proof: We present two proofs here.
First proof: Since H is quadratic both in differentiation and multiplication and is self adjoint, we start with the ansatz

$$K(t,x,y) = \exp(a(t)x^2/2 + b(t)xy + a(t)y^2/2 + c(t)).$$

Then

$$(\partial_t + H)K(t,x,y) = [a'(t)x^2/2 + b'(t)xy + a'(t)y^2/2 + c'(t) - (a(t)x + b(t)y)^2 - a(t) + x^2]K(t,x,y) = 0$$

gives us ODE's

$$\frac{1}{2}a' = a^2 - 1 = b^2, \qquad c' = a,$$

which yields

$$a(t) = -\coth(2t+C), \quad b(t) = \frac{1}{\sinh(2t+C)}, \quad c(t) = -\frac{1}{2}\log(\sinh(2t+C)) + D.$$

Initial condition then gives $C = 0$ and $D = \log(2\pi)^{-1/2}$. This is the Mehler's formula.
Second proof: The second proof explores the algebraic structure in harmonic oscillator. Let

$$X = -\partial_x^2, \quad Y = x^2, \quad Z = x\partial_x + \partial_x x.$$

Then $H = X + Y$ and

$$[X, Y] = -2Z, \quad [X, Z] = 4X, \quad [Y, Z] = -4Y.$$

That is X, Y, Z generates the Lie algebra $sl(2,\mathbb{R})$. In fact the isomorphism is given by

$$X \to 2n_+, \quad Y \to 2n_-, \quad Z \to -2\alpha,$$

where

$$n_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad n_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the standard generators for $sl(2,\mathbb{R})$.

Now it can be computed that

$$e^{-t(2n_+ + 2n_-)} = e^{-2\sigma_1(t)n_+} e^{-2\sigma_3(t)\alpha} e^{-2\sigma_2(t)n_-},$$

with

$$\sigma_1(t) = \sigma_2(t) = \frac{1}{2}\tanh 2t, \quad e^{2\sigma_3(t)} = \cosh 2t.$$

Thus,

$$e^{-tH} = e^{-\sigma_1(t)X} e^{\sigma_3(t)Z} e^{-\sigma_2(t)Y}.$$

Since

$$\begin{aligned} e^{-\sigma_1 X} u(x) &= (4\pi\sigma_1)^{-1/2} \int e^{-(x-y)^2/4\sigma_1} u(y)dy, \\ e^{-\sigma_2 Y} u(x) &= e^{-\sigma_2 x^2} u(x), \\ e^{\sigma_3 Z} u(x) &= e^{\sigma_3} u(e^{2\sigma_3} x), \end{aligned}$$

composing according to the above formulas, we obtain

$$e^{-tH} u(x) = \int K(t,x,y) u(y) dy,$$

with

$$K(t,x,y) = \frac{\exp([-\frac{1}{2}(\cosh 2t)(x^2+y^2) + xy]/\sinh 2t)}{(2\pi \sinh 2t)^{1/2}},$$

which is the Mehler's formula. ∎

By a change of variable, we have that the heat kernel for $-\frac{d^2}{dx^2} + \lambda^2 x^2$ is

$$K(t,x,0) = \frac{1}{(4\pi t)^{1/2}} (\frac{2\lambda t}{\sinh 2\lambda t})^{1/2} \exp(-\frac{\lambda}{2\tanh 2\lambda t} x^2).$$

On $\mathbb{R}^n$, we define a generalized harmonic oscillator to be

$$H = -\sum_i (\partial_{x_i} + \frac{1}{4}\Omega_{ij} x_j)^2,$$

where $\Omega = (\Omega_{ij})$ is an $n \times n$ skew-symmetric matrix.

Lemma 3.6.7 *The heat kernel for the generalized harmonic oscillator H is given by*

$$K(t,x,0)=\frac{1}{(4\pi t)^{n/2}}\det{}^{1/2}\left(\frac{t\Omega/2}{\sinh t\Omega/2}\right)\exp\left(-\frac{1}{4t}\left(\frac{t\Omega/2}{\tanh t\Omega/2}\right)_{ij}x_ix_j\right).$$

Proof: One verifies the heat equation and initial condition for $K(t,x,0)$. We just look at the heat equation, the initial condition being easy:

$$(\partial_t+H)K(t,x,0)=0.$$

Since both sides are analytic with respect to the coefficients Ω_{ij}, it suffices to prove the result when Ω is a real matrix (our final Ω is a 2-form valued matrix). In a suitable orthonormal basis, Ω can be block diagonalized into 2×2 blocks. Thus, the problem reduces to the case when $n=2$, and $\Omega_{12}=\lambda,\ \Omega_{21}=-\lambda$. In this case,

$$H=-[\partial_{x_1}^2+\partial_{x_1}^2+(\frac{\lambda}{4})^2(x_1^2+x_2^2)]-\frac{\lambda}{2}(x_2\partial_{x_1}-x_1\partial_{x_2}).$$

and

$$K(t,x,0)=\frac{1}{4\pi t}\frac{t\lambda/2}{\sin t\lambda/2}\exp\left(-\frac{1}{4t}\frac{t\lambda/2}{\tan t\lambda/2}|x|^2\right).$$

By the previous lemma, $K(t,x,0)$ is the heat kernel for the first part of H, $-[\partial_{x_1}^2+\partial_{x_1}^2+(\frac{\lambda}{4})^2(x_1^2+x_2^2)]$. To see that it is also the heat kernel of H, we simply note that $K(t,x,0)$ is a function of $|x|^2$, and hence is annihilated by the infinitesimal rotation $-\frac{\lambda}{2}(x_2\partial_{x_1}-x_1\partial_{x_2})$. ■

Combining all the discussion above, we have proved the local index theorem for chiral Dirac operator.

Theorem 3.6.8 *Let M^n be a closed spin manifold of even dimension and $K(t,x,y)$ the heat kernel for D^2. Then*

$$\lim_{t\to 0}\mathrm{Tr}_s(K(t,x,x))dx=\hat{A}(\frac{\Omega}{2\pi}).$$

Proof: We compute

$$\begin{aligned}\lim_{t\to 0}\mathrm{Tr}_s(K(t,x,x)) &= (-2\sqrt{-1})^{n/2}[\lim_{\epsilon\to 0}\epsilon^{\frac{n}{2}}(\delta_\epsilon K)(t,0)]_{(n)}\\ &= (-2\sqrt{-1})^{n/2}[K_0(t,0,0)]_{(n)}\\ &= (-2\sqrt{-1})^{n/2}[\frac{1}{(4\pi t)^{n/2}}\det{}^{1/2}\left(\frac{t\Omega/2}{\sinh t\Omega/2}\right)]_{(n)}\\ &= [\det{}^{1/2}\left(\frac{\Omega/4\pi}{\sinh\Omega/4\pi}\right)]_{(n)}.\end{aligned}$$

■

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