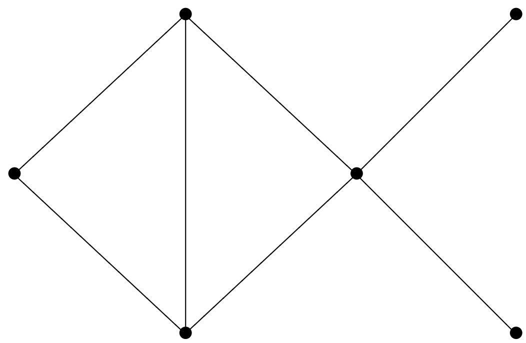
MATH 137A Final Exam

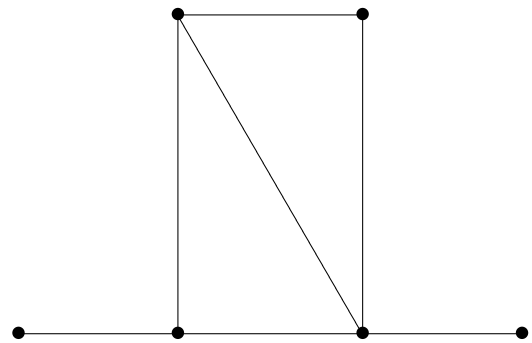
Thursday, March 17, 2011

SOLUTIONS

1. Give two different reasons why the graphs below are not isomorphic.



A



B

Solution: Both A and B are simple graphs with six vertices and seven edges. Both graphs contain exactly two 3-cliques. Both graphs contain two vertices of degree 1, one vertex of degree 2, two vertices of degree 3, and one vertex of degree 4.

However:

(a) The vertex of degree 4 in graph A has neighbors with degree $1, 1, 3$ and 3, while the vertex of degree 4 in B has neighbors with degree $1, 2, 3$ and 3.

(b) In graph A, every vertex of degree 3 has a vertex of degree 2 as a neighbor. In graph B, there exists a vertex of degree 3 with no vertex of degree 2 as a neighbor.

(c) The graph B contains a Hamiltonian path, a walk that includes every vertex of B exactly once. Graph A contains no such path.

(d) If we remove the vertex of degree 4 from A we get a graph with three components. Removing any vertex in graph B results in a graph of no more than 2 components.

(e) The shortest distance between the leaves of A is 2. The shortest distance between the leaves of B is 3.

2. Let G be a graph with exactly 10 vertices and 27 edges. Suppose that each vertex has degree $3, 5$, or 7, and that there are exactly 2 vertices of degree 5. How many vertices of degree 7 does G have? Justify your answer.

   Solution:

   Let x, y and z denote the number of vertices in G of degree 3, 5 and 7, respectively. We are interested in the value of z.

   Since all possible degrees are accounted for, we have

   $$x + y + z = 10.$$

   From the statement of the problem, we have

   $$y = 2.$$

   Finally, by the Hand Shaking Theorem, we have

   $$3x + 5y + 7z = 2 \times 27.$$

   The three equations in three unknowns reduce to a system of two equations in two unknowns:

   $$x + z = 8$$

   $$3x + 7z = 44.$$

   Hence, $z = 5$.

3. Let G be a simple connected graph. The **square** of G, denoted G^2, is defined to be the graph with the same vertex set as G in which two vertices u and v are adjacent if and only if the distance between u and v in G is 1 or 2.

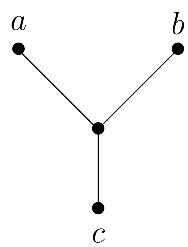
   In other words, in the square of a graph G, vertices that were adjacent remain so. And in addition, vertices that were connected by a path of length 2 become adjacent.

   (a) Show that the square of $K_{1,3}$ is K_4.

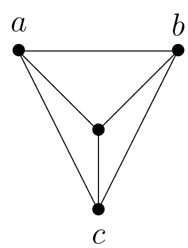
   (b) Find two more graphs whose square is K_4.

Solution:

(a) Consider the following labeled representation of $K_{1,3}$.

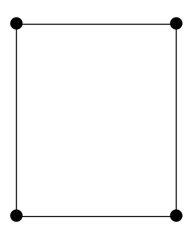
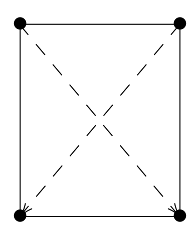


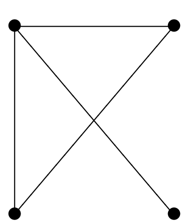
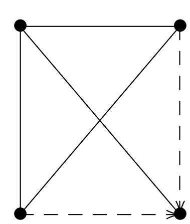
By definition, $K_{1,3}^2$ will keep the existing edges of $K_{1,3}$ and then join a to b, a to c and b to c.



Hence $K_{1,3}^2$ is isomorphic to K_4.

(b) Two more examples are

C_4 $(\cong K_{2,2})$  $\rightarrow$ 

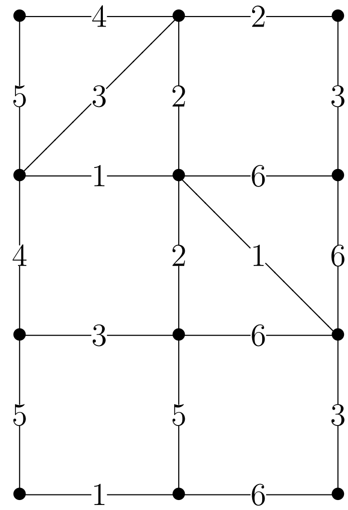
and the graph G  $\rightarrow$ 

4. What is the number of regular binary trees on 9 vertices?

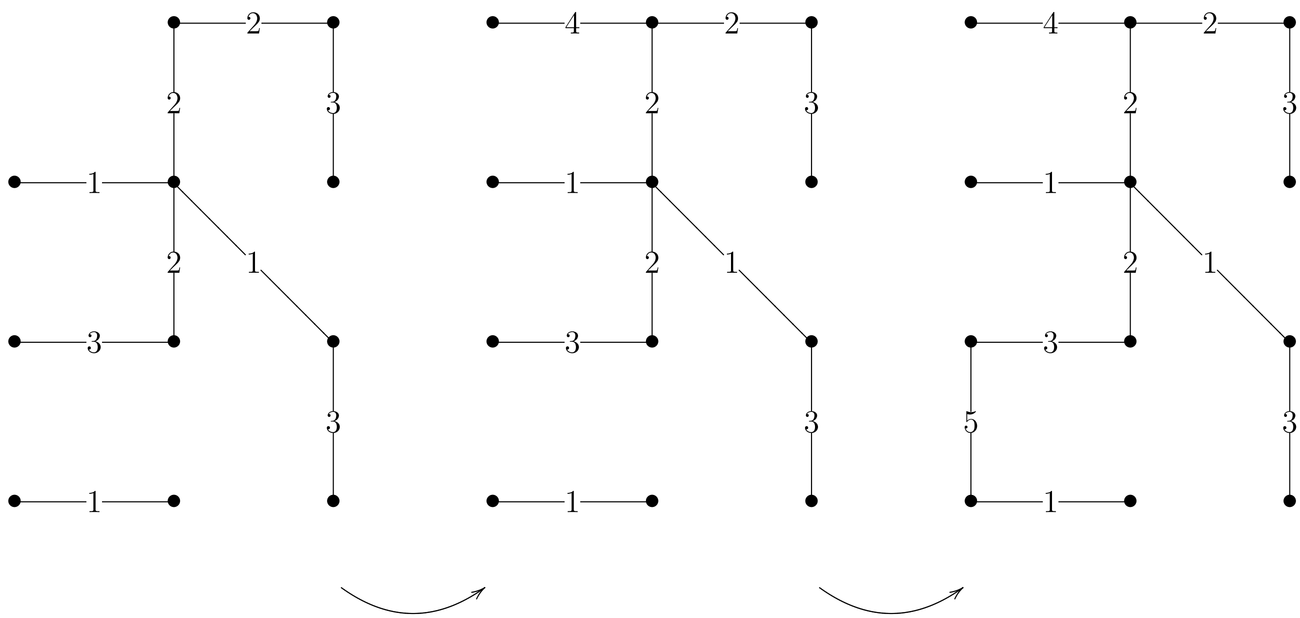
Solution: A binary tree is an ordered rooted tree in which each vertex has at most two children. A regular binary tree is a binary tree where every vertex has an even number of children. We proved in class that the number of regular binary trees on $2n+1$ vertices is C_n, the n^{th} Catalan number. As $9 = 2 \times 4 + 1$, the number of regular binary trees on 9 vertices is

$$C_4 = \frac{1}{4+1}\binom{2 \times 4}{4} = \frac{1}{5} \times 70 = 14.$$

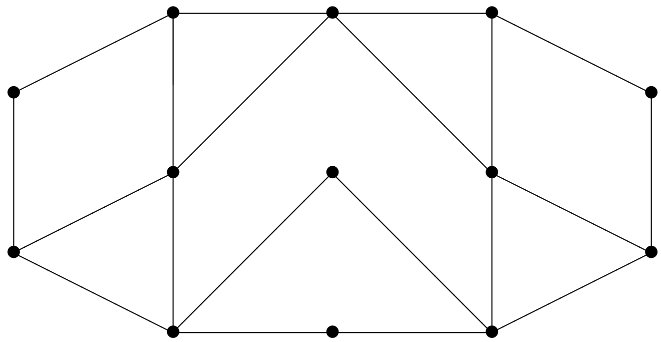
5. Find the weight of a minimum cost spanning tree for the weighted graph below.



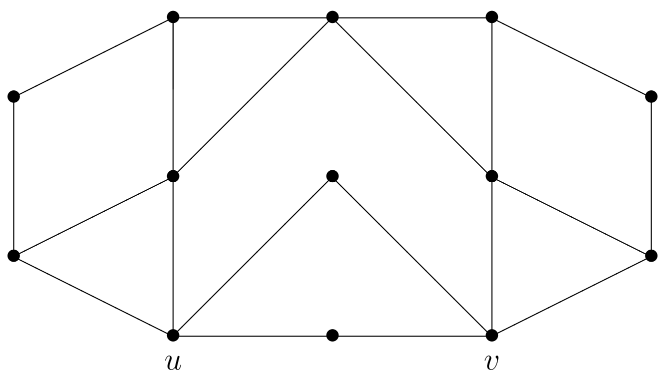
Solution: We employ Kruskal's algorithm. First, choose all edges weighted 1 2 or 3, except for the one edge of weight 3 that would create a cycle. Then choose the single edge of weight 4 that does not create a cycle. Finally, add the one of the edges of weight 5 that will not create a cycle. All $12 - 1 = 11$ edges are thus accounted for and we have a minimum cost spanning tree of weight 27.



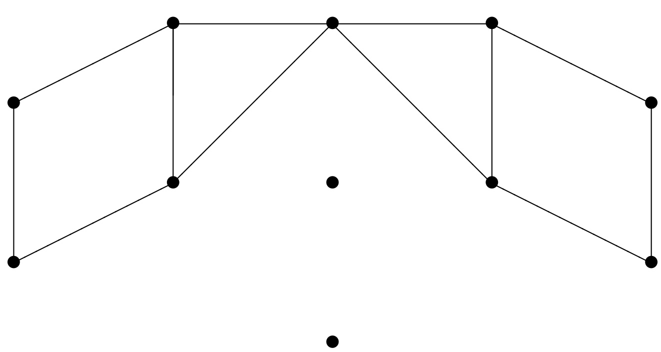
6. Determine whether the given graph is Hamiltonian. If it is, find a Hamiltonian cycle. If it is not, prove it is not.



Solution: A necessary condition for a graph to be Hamiltonian is that for each nonempty $S \subseteq V(G)$, the number of components of $G - S$ is less that or equal to the number of elements of S. Let G be the graph given above and let $S = \{u, v\}$ be the set of vertices as indicated below.



Then the graph $G - S$ has 3 components.



Since $3 \not\leq 2$, the graph G is not Hamiltonian.

7. Give a clear and careful proof of the following: A connected bipartite graph has a unique bipartition (except for, of course, interchanging the two bipartition sets).

   Solution: Let G be a connected bipartite graph. Let $X, Y, X', Y' \subseteq V(G)$, where X, Y and X', Y' are distinct bipartitions of the vertex set of G.

   We then have that

   (a) $X \cup Y = V(G) = X' \cup Y'$

   (b) $X \cap Y = \emptyset = X' \cap Y'$

   (c) $G[X], G[Y], G[X']$ and $G[Y']$ are all null graphs.

   Let $A = (X \cap X') \cup (Y \cap Y')$ and $B = (Y \cap X') \cup (X \cap Y')$. Note that the set A cannot be empty. If it were, it would mean that $X \subseteq Y'$ and $Y \subseteq X'$. This, in turn, implies that $X = Y'$ and $Y = X'$, i.e., the bipartition sets have been switched. Similarly, $B \neq \emptyset$.

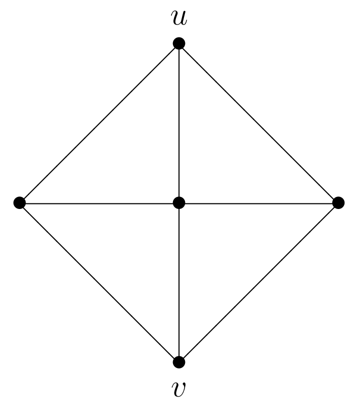
   Now let $v \in A \subseteq V(G)$. Then either $v \in X \cap X'$ or $v \in Y \cap Y'$. If $v \in X \cap X'$, then v cannot be adjacent to any vertex in $Y \cap X'$, since $G[X']$ is a null graph. Furthermore, v cannot be adjacent to any vertex in $X \cap Y'$, since $G[X]$ is a null graph. Hence, v is not adjacent to any vertex in B. We reach the same conclusion when $v \in Y \cap Y'$.

   So we have that no vertex in A can have a vertex in B as a neighbor. Since

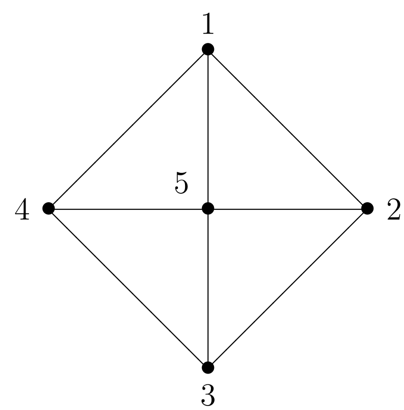
$$\begin{aligned}
A \cup B =& [(X \cap X') \cup (Y \cap Y')] \cup [(Y \cap X') \cup (X \cap Y')] \\
=& [(X \cap X') \cup (Y \cap X')] \cup [(Y \cap Y') \cup (X \cap Y')] \\
=& [(X \cup Y) \cap X'] \cup [(Y \cup X) \cap Y'] \\
=& [V(G) \cap X'] \cup [V(G) \cap Y'] \\
=& X' \cup Y' \\
=& V(G)
\end{aligned}$$

   the graph G is disconnected. Contradiction.

8. Compute the total resistance between u and v in the electrical network of resistors corresponding to the graph below, where each resistor is one ohm.



Solution: Let G be the graph corresponding to the given electrical network. We compute $\tau(G)$ using the Matrix-Tree Theorem. First label graph G as follows.



Then

$$\mathcal{T}(G) = D(G) - A(G) = \begin{pmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

By the Matrix-Tree Theorem, $\tau(G)$ is equal to the $(5,5)$-cofactor of $\mathcal{T}$. Thus

$$\tau(G) = \det \begin{pmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix}.$$

To make our lives easier, we first subtract column 2 from column 4.

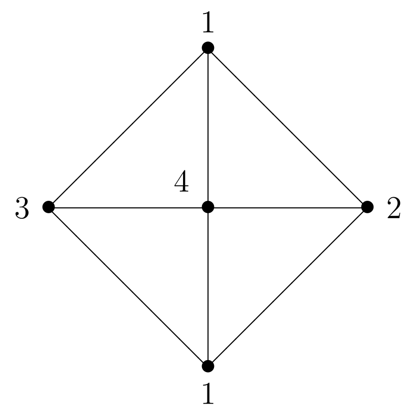
$$\tau(G) = \det \begin{pmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & -3 \\ 0 & -1 & 3 & 0 \\ -1 & 0 & -1 & 3 \end{pmatrix}.$$

Then we expand the determinant along the first row.

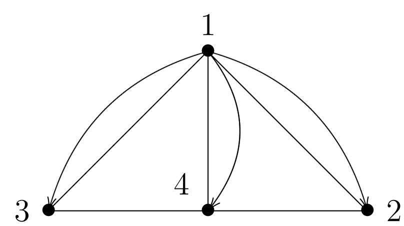
$$\begin{aligned}\tau(G) =&(-1)^{1+1}\cdot 3\cdot \det\begin{pmatrix} 3 & -1 & -3 \\ -1 & 3 & 0 \\ 0 & -1 & 3 \end{pmatrix} + (-1)^{1+2}\cdot(-1)\cdot\det\begin{pmatrix} -1 & -1 & -3 \\ 0 & 3 & 0 \\ -1 & -1 & 3 \end{pmatrix}\\ =&3\times 21 - 18\\ =&45\end{aligned}$$

Now we are interested in $\tau_{uv}(G')$, or rather $\tau(G'')$. That is, the number of spanning trees of the graph obtained from G by identifying the vertices u and v.

We can visualize G'' simply by giving u and v the same label in G.



Or equivalently,



Then

$$\mathcal{T}(G'') = D(G'') - A(G'') = \begin{pmatrix} 6 & -2 & -2 & -2 \\ -2 & 3 & 0 & -1 \\ -2 & 0 & 3 & -1 \\ -2 & -1 & -1 & 4 \end{pmatrix}.$$

Hence

$$\tau(G'') = \det\begin{pmatrix} 6 & -2 & -2 \\ -2 & 3 & 0 \\ -2 & 0 & 3 \end{pmatrix} = 30.$$

From the results shown in class, we now conclude that the total resistance of the network between u and v is given by.

$$R_t = \frac{\tau(G'')}{\tau(G)} = \frac{30}{45} = \frac{2}{3}.$$