HOMEWORK 1

SOLUTIONS

(1) Let $A = \{3, 4, 5\}, B = \{3, 4\}, C = \{4\}$. Find $D = A \triangle B \triangle C$.

Solution: Recall from class that $A \triangle B = (A \setminus B) \cup (B \setminus A)$. That is, $A \triangle B$ contains all elements that lie in A but not in B and all elements that lie in B but not in A. Note that we also have $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Thus

$$\begin{aligned}
(A \triangle B) \triangle C &= \Big[(A \triangle B) \setminus C\Big] \cup \Big[C \setminus (A \triangle B)\Big] \\
&= \Big[\big[(A \setminus B) \cup (B \setminus A)\big] \setminus C\Big] \cup \Big[C \setminus \big[(A \cup B) \setminus (A \cap B)\big]\Big] \\
&= \big[A \setminus (B \cup C)\big] \cup \big[B \setminus (A \cup C)\big] \cup \big[C \setminus (A \cup B)\big] \cup \big[C \cap A \cap B\big].
\end{aligned}$$

On the other hand

$$\begin{aligned}
A \triangle (B \triangle C) &= \Big[A \setminus (B \triangle C)\Big] \cup \Big[(B \triangle C) \setminus A\Big] \\
&= \Big[A \setminus \big[(B \cup C) \setminus (B \cap C)\big]\Big] \cup \Big[\big[(B \setminus C) \cup (C \setminus B)\big] \setminus A\Big] \\
&= \big[A \setminus (B \cup C)\big] \cup \big[A \cap B \cap C\big] \cup \big[B \setminus (C \cup A)\big] \cup \big[C \setminus (B \cup A)\big].
\end{aligned}$$

A close inspection shows that the last term in both of the above equations are the same.

(2) Suppose 70% of Californians like cheese, 80% like apples and 10% like neither. What percentage of Californians like both cheese and apples?

Solution: First let's ask a different question. How many Californians like apples or like cheese? If we add the percentages together we get $70\% + 80\% = 150\%$. This is clearly an over-counting. The problem is that we have counted those who like both apples and cheese twice.

Since 10% of Californians like neither apples nor cheese, we know that 90% either like apples or like cheese (including those who like both). Hence, the *difference* between the percentages, $150\% - 90\% = 60\%$ answers the original question.

Put another way, for finite sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$. For the given problem, this translates to $90 = 70 + 80 - x$. Thus $x = 60$.

(3) Use the Principle of Mathematical Induction to prove that for $n \in \mathbb{N}$, $n^3 - n$ is always divisible by 3.

Solution: Let $P(n)$ be the statement: $n^3 - n$ is always divisible by 3.

Since $1^3 - 1 = 1 - 1 = 0$, $P(1)$ is true. Now assume $P(n)$ is true. That is, assume that $n^3 - n$ is always divisible by 3. We want to show that $P(n+1)$ is true, i.e., $(n+1)^3 - (n+1)$ is always divisible by 3.

By expansion, we have

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 = n^3 - n + 3n^2 + 3n.$$

Clearly, $3n^2 + 3n$ is divisible by 3. By the induction hypothesis, $n^3 - n$ is also divisible by 3. Thus $P(n+1)$ is true.

Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers.

(4) Find a surjective function from $\mathbb{N}$ to $\mathbb{Z}$. Find an injective function from $\mathbb{Z}$ to $\mathbb{N}$.

Solution: There are, of course, many correct answers for this question. For the first case, we may choose the function

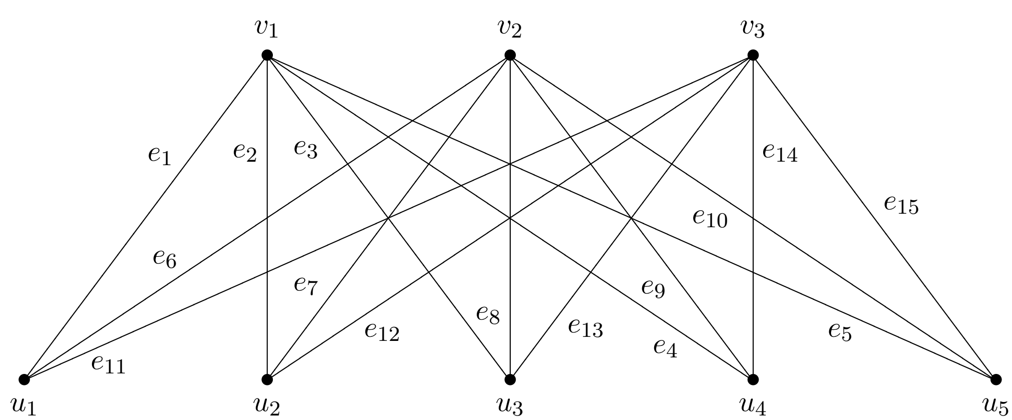
$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ (1-x)/2 & \text{if } x \text{ is odd.} \end{cases}$$

An example of a function that illustrates the second statement is

$$g(x) = \begin{cases} 2x+1 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0. \end{cases}$$

(5) Write an explicit description of the edgemap for the complete bipartite $(3,5)$-graph.

Solution: A labeled complete bipartite $(3,5)$-graph, $K_{3,5}$, is given below.



From this we have:

$$\begin{aligned}
\phi(e_1) &= \{v_1, u_1\} \\
\phi(e_2) &= \{v_1, u_2\} \\
\phi(e_3) &= \{v_1, u_3\} \\
\phi(e_4) &= \{v_1, u_4\} \\
\phi(e_5) &= \{v_1, u_5\} \\
\phi(e_6) &= \{v_2, u_1\} \\
\phi(e_7) &= \{v_2, u_2\} \\
\phi(e_8) &= \{v_2, u_3\} \\
\phi(e_9) &= \{v_2, u_4\} \\
\phi(e_{10}) &= \{v_2, u_5\} \\
\phi(e_{11}) &= \{v_3, u_1\} \\
\phi(e_{12}) &= \{v_3, u_2\} \\
\phi(e_{13}) &= \{v_3, u_3\} \\
\phi(e_{14}) &= \{v_3, u_4\} \\
\phi(e_{15}) &= \{v_3, u_5\}
\end{aligned}$$

(6) Is there a simple graph on 6 vertices such that the vertices all have distinct degree? If not, why not? If so, draw one.

Solution: The answer to the question is no.

Since the graph is simple (no loops and no multiple edges) and has 6 vertices, the degree of any vertex can be no more than 5. Therefore, if we are to have 6 distinct degrees for the 6 vertices we have no choice but to use $0, 1, 2, 3, 4$ and 5.

The presence of 0 means that one vertex is isolated. So the problem can be restated: Is there a graph with 5 vertices where one vertex has degree 1, another vertex has degree 2, another vertex has degree 3, another vertex has degree 4 and the last vertex has degree 5. This is clearly not possible since a simple graph with 5 vertices cannot have any vertex with degree 5.

(7) Let G be a k-regular graph, where k is an odd number. Prove that the number of edges in G is a multiple of k.

Solution: Assume the graph G contains m vertices. Applying the Hand-Shaking Theorem, we have

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d_G(v) = \frac{1}{2} \sum_{i=1}^{m} k = \frac{1}{2} mk$$

Since $|E(G)|$ must be a natural number and k is odd, m must be even and therefore $\frac{1}{2}m$ must be a natural number. Hence $|E(G)| = nk$, for $n \in \mathbb{N}$. Thus $|E(G)|$ is a multiple of k.

(8) Prove that it is impossible to have a group of nine people at a party such that each one knows exactly five of the others in the group.

Solution: The graph of this problem has nine vertices, one for each member of the group attending the party. Two vertices are connected by an edge if the people they represent know each other. In graph theory terms, each vertex has degree five. The graph must be a simple graph since the word "other" in the statement of the problem excludes loops and there is no reason for multiple edges.

The sum of the degrees of the vertices is $9 \times 5 = 45$. But, by the Hand-Shaking Theorem, this sum is also equal to twice the number of edges in the graph. In other words, the sum must be an even number. As 45 is not even, we have a contradiction.