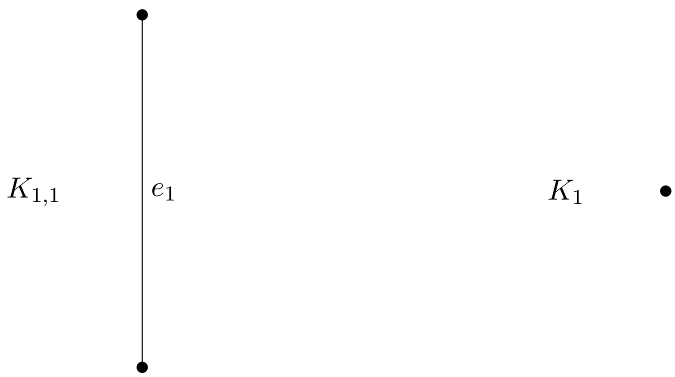
HOMEWORK 3

SOLUTIONS

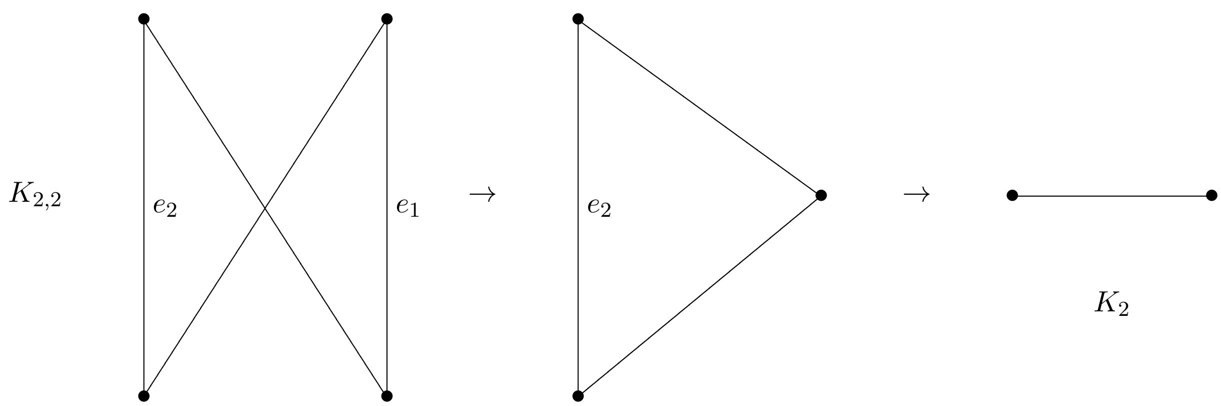
(1) Show that for each $n \in \mathbb{N}$ the complete graph K_n is a contraction of $K_{n,n}$.

Solution: We describe the process for several small values of n. In this way, we can discern the inductive step.

Clearly, K_1, which is just one vertex, is a simple contraction of $K_{1,1}$, which is simply one edge along with its endvertices.

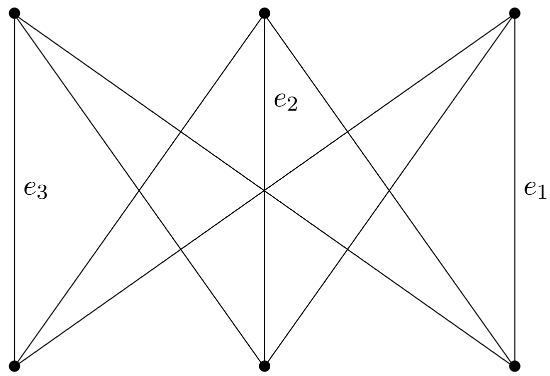


Now consider the graph $K_{2,2}$, represented in the usual way. A simple contraction on the right vertical edge followed by a simple contraction on the left vertical edge produces a graph that consists of a single edge, i.e., K_2.



We continue on to $K_{3,3}$. We perform simple contractions on the vertical edges in sequence, starting from the right.

$K_{3,3}$



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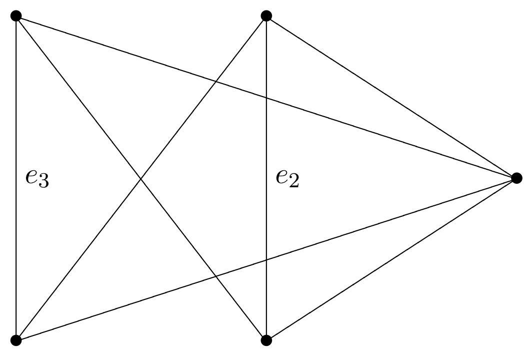

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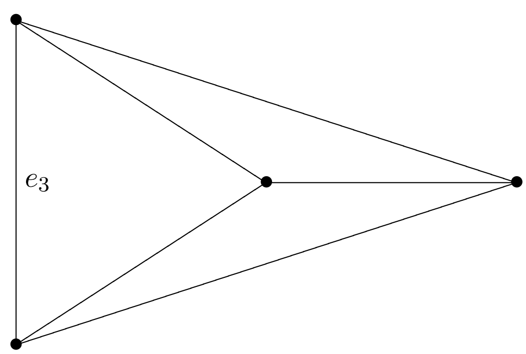

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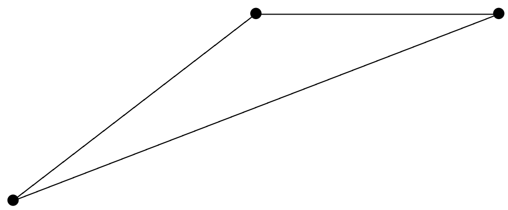
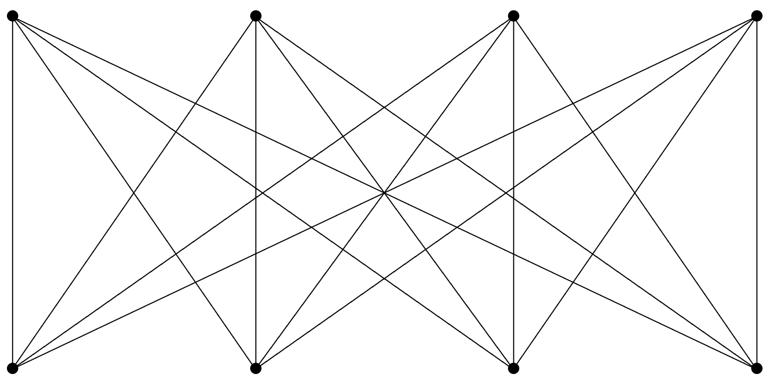
K_3



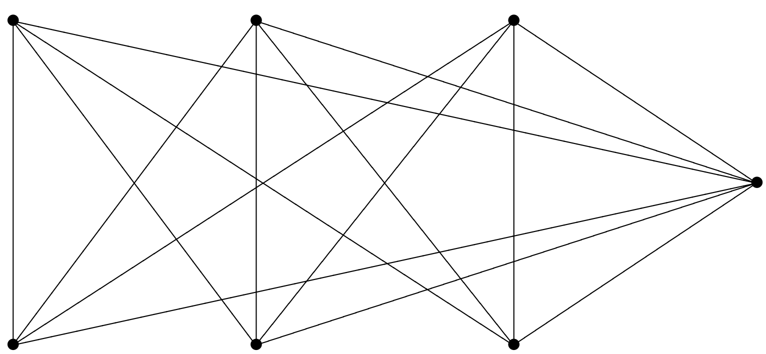
The pattern has now becomes a little bit clearer. When we performed a simple contraction on the edge e_1 of $K_{3,3}$ the result was a graph with a single vertex, a K_1, on the right connected to every vertex of a $K_{2,2}$ on the left. After performing a simple contraction on the edge e_2, the resulting graph had two adjacent vertices on the right, a K_2, that were each connected to every vertex of a $K_{1,1}$ on the left. The last simple contraction, on edge e_3, produced a graph on three vertices, where every pair of vertices are adjacent. This, of course, is the definition of K_3.

To check our analysis, we conclude with the case when $n = 4$.

$K_{4,4}$



$\downarrow$



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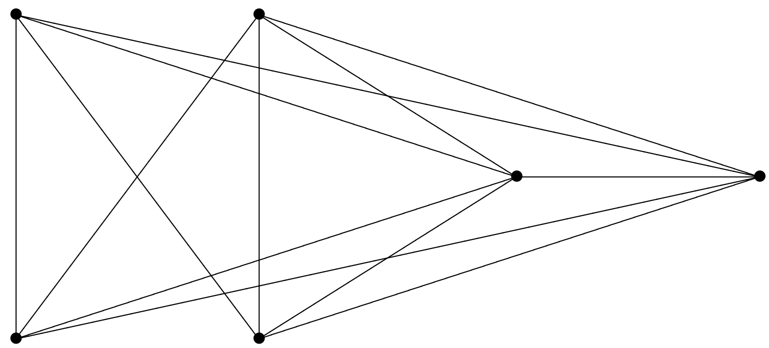

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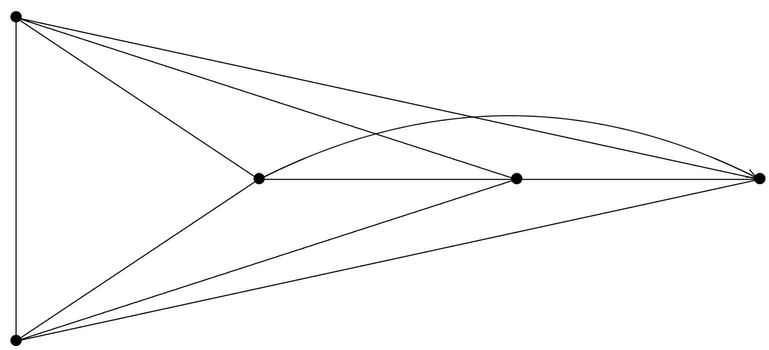

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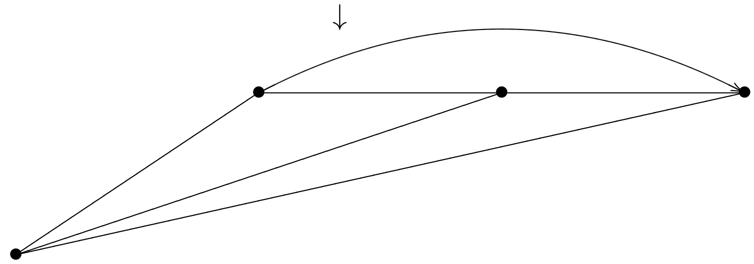
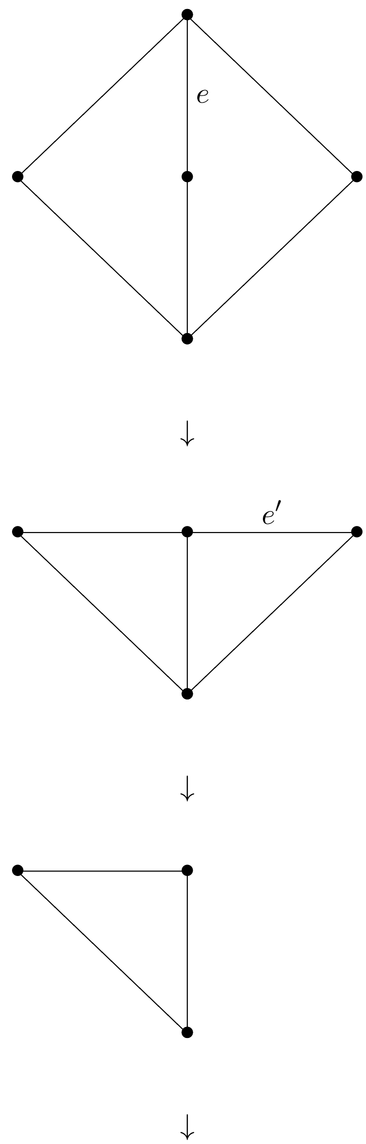


(2) For $n \in \mathbb{N}$, can K_n be a contraction of $K_{m,n}$ if $m < n$?

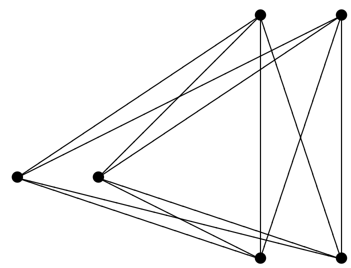
Solution: It depends. The graph K_3 is a contraction of $K_{2,3}$, as we see below.



However, K_4 is clearly not a contraction of $K_{1,4}$. A main consideration is the cardinality of the edge sets. The graph K_n has $\frac{n(n-1)}{2}$ edges while the graph $K_{m,n}$ has nm edges. If $m < \frac{(n-1)}{2}$ then K_n has more edges than $K_{m,n}$, and therefore cannot be contraction of it.

(3) The complete tripartite graph $K_{r,s,t}$ consists of three disjoint sets of vertices (of sizes r, s and t), with an edge joining two vertices if and only if they lie in different sets. Draw $K_{2,2,2}$. What is the number of edges of $K_{2,3,4}$?

Solution: A representation of $K_{2,2,2}$ is given below.



To calculate the number of edges of $K_{2,3,4}$, we appeal to the Hand-Shaking Theorem. The two vertices that lie in the set of cardinality two both have degree $3+4=7$. Similarly, three vertices have degree 6 and four vertices have degree 5. Therefore, the sum of the degrees of the vertices of $K_{2,3,4}$ is

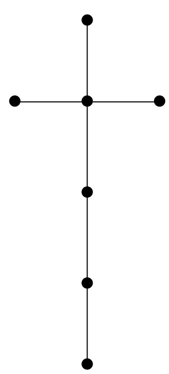
$$2 \times 7 + 3 \times 6 + 4 \times 5 = 52.$$

Thus, the total number of edges is $52 \div 2 = 26$.

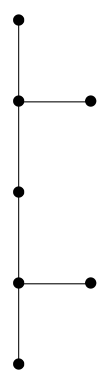
(4) There are exactly 11 unlabeled trees on seven vertices. Draw these eleven trees, making sure that no two are isomorphic.

Solution: Below are 11 trees on seven vertices.

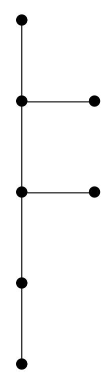
$A:$



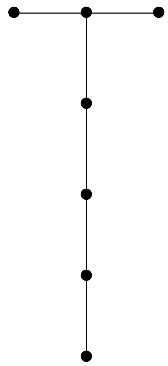
$B:$



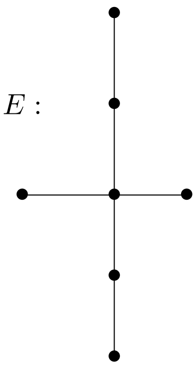
$C:$



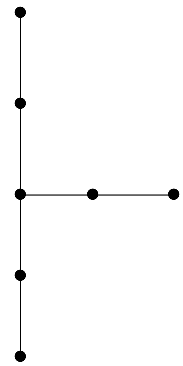
D :



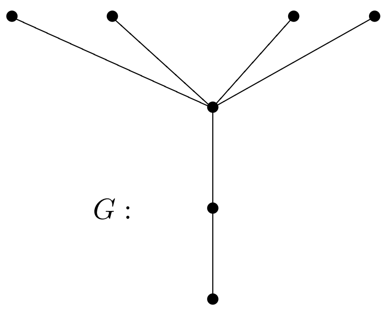
E :



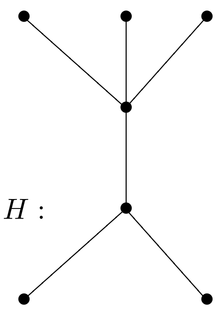
F :



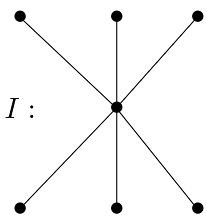
G :



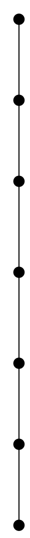
H :



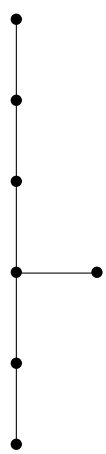
I :



J :



K :



Graph I is the only graph with a vertex of degree 6. Graph G is the only graph with a vertex of degree 5. Graph J is the only graph with exactly two leaves.

Graphs A, E and H all have exactly one vertex of degree 4. But the vertex of degree 4 in A has neighbors with degree $1, 1, 1$ and 2. The vertex of degree 4 in E has neighbors with degree $1, 1, 2$ and 2. And the vertex of degree 4 in H has neighbors with degree $1, 1, 1$ and 3.

Graphs D, F and K all have exactly one vertex of degree 3. But the vertex of degree 3 in D has neighbors with degree $1, 1$ and 2. The vertex of degree 3 in F has neighbors with degree $2, 2$ and 2. And the vertex of degree 3 in K has neighbors with degree $1, 2$ and 2.

Graphs B and C both have exactly one vertex of degree 2. But the vertex of degree 2 in B has neighbors with degree 3 and 3. And the vertex of degree 2 in C has neighbors with degree 1 and 3.

Hence, no two of the given graphs are isomorphic.

(5) Show that every tree containing a vertex of degree k contains at least k leaves.

Solution: Proof by contradiction. Assume that T is a tree that has a vertex of degree k and *less than* k leaves.

WLOG we may also assume that T has n vertices. Thus, T contains $n-1$ edges. Applying, yet again, the Hand-Shaking Theorem yields the following equation

$$\sum_{v \in V(T)} d(v) = 2n - 2.$$

Let us now make the left-hand side of the above equation as small as possible. That is, we want to assign to each vertex in T the smallest degree allowed, given our assumptions. There has to be one vertex of degree k, so we have control over the degree of the remaining $n-1$ vertices.

Since T is a tree, and therefore is connected, no vertex has degree 0. So, the smallest degree we can assign is 1. The maximum number of vertices that can have this degree (i.e., the maximum number of leaves in T) is $k-1$. We then assign a degree of 2 to the remaining $n-(k-1)-1 = n-k$ vertices.

Therefore, we have

$$2n - 2 = \sum_{v \in V(T)} d(v) \geq 2 \times (n-k) + 1 \times (k-1) + k \times 1 = 2n - 1.$$

Which implies that

$$-2 \geq -1.$$

Contradiction.

(6) For two points in $\mathbb{R}^2$, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be given by

$$d(P_1, P_2) = |x_2 - x_1| + |y_2 - y_1|.$$

Show that d is a metric on $\mathbb{R}^2$.

Solution: The fact that d is a metric follows immediately from the properties of absolute value.

(a) *Positive definiteness.* The sum of two nonnegative numbers is again nonnegative. Furthermore, such a sum can only be zero if both terms are zero. In that case $x_1 = x_2$ and $y_1 = y_2$, which implies that $P_1 = P_2$.

(b) *Symmetry.* We have

$$d(P_1, P_2) = |x_2 - x_1| + |y_2 - y_1| = |x_1 - x_2| + |y_1 - y_2| = d(P_2, P_1).$$

(c) *Triangle Inequality.* Let $P_3 = (x_3, y_3)$. Then

$$\begin{aligned} d(P_1, P_3) &= |x_3 - x_1| + |y_3 - y_1| \\ &= |x_3 - x_2 + x_2 - x_1| + |y_3 - y_2 + y_2 - y_1| \\ &\leq |x_3 - x_2| + |x_2 - x_1| + |y_3 - y_2| + |y_2 - y_1| \\ &= |x_2 - x_1| + |y_2 - y_1| + |x_3 - x_2| + |y_3 - y_2| \\ &= d(P_1, P_2) + d(P_2, P_3). \end{aligned}$$

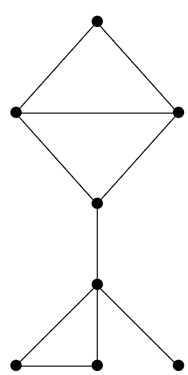
(7) For all $n \in N$ what is the eccentricity of each vertex of K_n? How many centers does K_n have?

Solution: Since all of the vertices of K_n are adjacent to each other, the distance between any two distinct pair of vertices is 1. Hence, the eccentricity of any vertex $v \in K_n$ (i.e., the distance from v to the vertex farthest from v) is also 1.

The center of K_n is the vertex having minimum eccentricity. Thus, all n vertices of K_n are centers of K_n.

(8) Draw all spanning trees of the graph G.

G :



Solution: We observe that the vertical edge that lies second from the bottom will remain in any spanning tree of G. Hence, the problem separates into finding the spanning trees of the subgraph generated by the top four vertices and the spanning trees of the subgraph generated by the bottom four vertices and then combining them.