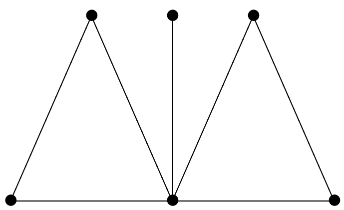
HOMEWORK 2

SOLUTIONS

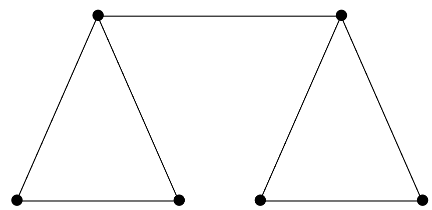
(1) Draw two graphs on six vertices that are 1-isomorphic but are not isomorphic.

Solution: The graphs given below are clearly 1-isomorphic. However, G has a vertex of degree 5 while G' does not. Therefore, the graphs are not isomorphic.

G



G'



(2) Let $(\vec{G}, c; s, t)$ be a network and f a flow. Show that if S is a source vertex cut, then $\text{val}(f) = f^+(S) - f^-(S) \leq c^+(S)$.

Solution: A source vertex cut is a set $S \subseteq V(\vec{G})$ such that $s \in S$ and $t \notin S$. A flow is also a weight for a graph, so by Corollary 6.40 in the text

$$f^+(S) - f^-(S) = \sum_{u \in S}(f^+(u) - f^-(u)) = f^+(s) - f^-(s) + \left(\sum_{u \in S \setminus \{s\}} (f^+(u) - f^-(u)) \right).$$

By definition,

$$\text{val}(f) = f^+(s) - f^-(s).$$

And by Kirchhoff's Law,

$$f^+(u) = f^-(u) \ \forall \ u \in V(\vec{G}) \setminus \{s, t\}.$$

Therefore, we have

$$f^+(S) - f^-(S) = \text{val}(f).$$

Finally, by the definition of a flow, $f^+(S) \leq c^+(S)$ and $f^-(S) \geq 0$. Thus,

$$f^+(S) - f^-(S) \leq c^+(S).$$

(3) Let $(\vec{G}, c; s, t)$ be a network, f a flow, and p an augmenting path of $\vec{G}$ from s to t with a tolerance of $\delta > 0$. Let f' be given by

$$f'(e) = \begin{cases} f(e) + \delta & \text{if } \eta(e) = \eta(e_i) = (u_{i-1}, u_i), \\ f(e) - \delta & \text{if } \eta(e) = \eta(e_i) = (u_i, u_{i-1}), \\ f(e) & \text{if } e \text{ is not in } p. \end{cases}$$

Show that f' is a flow and $\mathrm{val}(f') = \mathrm{val}(f) + \delta$.

Solution: By the definition of a flow and the tolerance of an augmenting path, we have that

$$0 \leq f'(e) = f(e) + \delta \leq f(e) + c(e) - f(e) = c(e)$$

for any edge e in the augmenting path with the same orientation as the network. If e is an edge on the augmenting path with the opposite orientation as that of the network, then

$$0 = f(e) - f(e) \leq f'(e) = f(e) - \delta \leq f(e) \leq c(e).$$

And so, f' satisfies condition 1 of a flow.

We now show that f' satisfies Kirchhoff's condition. Let $u \notin \{s, t\}$ be a vertex of the augmenting path p. Then there are exactly two (consecutive) edges of p incident with u.

Suppose both edges have u as head with respect to the network. Then both edges contribute to the value of $f'^{-}(u)$. In this case the network orientation and augmenting path orientation agree for one edge and disagree for the other. So one edge increases $f'^{-}(u)$ over $f^{-}(u)$ by δ and the other decreases $f'^{-}(u)$ from $f^{-}(u)$ by δ. Since $f^{-}(u) = f^{+}(u)$, we have that $f'^{-}(u) = f'^{+}(u)$. If the two edges share u as a tail, they contribute a net of zero to $f'^{+}(u)$ over $f^{+}(u)$, again maintaining the Kirchhoff condition.

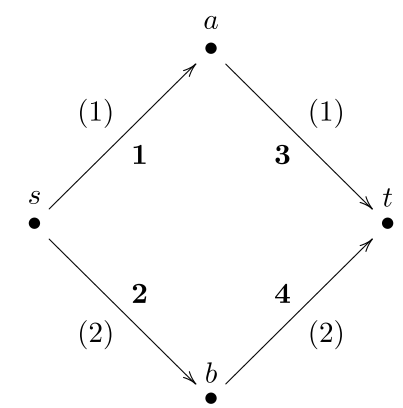
Now, say that one edge has u as head and the other has u as tail. Then one edge contributes to $f'^{+}(u)$ and the other contributes to $f'^{-}(u)$. Furthermore, the network and augmenting path orientations agree for both edges or disagree for both edges. In either case, the change of $f'^{+}(u)$ from $f^{+}(u)$ and of $f'^{-}(u)$ from $f^{-}(u)$ is the same. Both quantities are increased by δ or decreased by δ. So again, the Kirchhoff condition for f' is maintained and we have established condition 2 for f'.

Finally, we show that $\mathrm{val}(f') = \mathrm{val}(f) + \delta$. Let e be the first edge of p. This is the only edge of p that has s as an endvertex. The network and augmenting path orientations of e coincide and e contributes δ to $f'^{+}(s)$ over $f^{+}(s)$. Therefore,

$$\mathrm{val}(f') = f'^{+}(s) - f'^{-}(s) = \delta + f^{+}(s) - f^{-}(s) = \delta + \mathrm{val}(f).$$

(4) Give an example of a network $\vec{G}$ with a unique maximum flow f.

Solution: In the example below, the capacity of the network is in bold and the flow f is given by the numbers in parentheses.



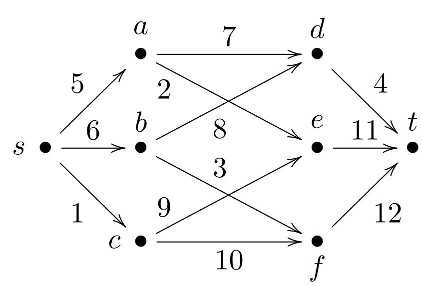
The value of the flow f is $\text{val}(f) = f^+(s) - f^-(s) = 3 - 0 = 3$. And from the list

$$\begin{aligned} c^+(\{s\}) &= 1 + 2 = 3 \\ c^+(\{s, a\}) &= 2 + 3 = 5 \\ c^+(\{s, b\}) &= 1 + 4 = 5 \\ c^+(\{s, a, b\}) &= 3 + 4 = 7 \end{aligned}$$

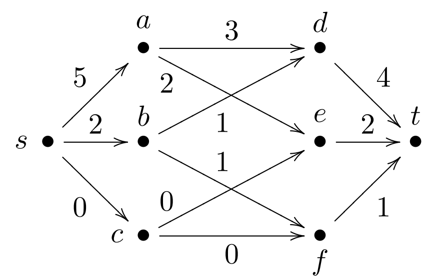
we see that the capacity of a minimum source vertex cut is also 3. Therefore, the flow f is maximum.

Now, let g be another maximum flow. The edges (s, a) and (s, b) must have the same image under g as under f. Otherwise, $\text{val}(g) \neq 3$. But, once $g((s, a))$ and $g((s, b))$ are given, $g((a, t))$ and $g((b, t))$ are predetermined by Kirchhoff's Law. Hence, $g = f$.

(5) Use the Ford-Fulkerson Algorithm to find a maximum flow for the network $\vec{G}$ given below. Prove that your flow f is maximum by finding a source vertex cut S such that $\text{val}(f) = c^+(S)$.



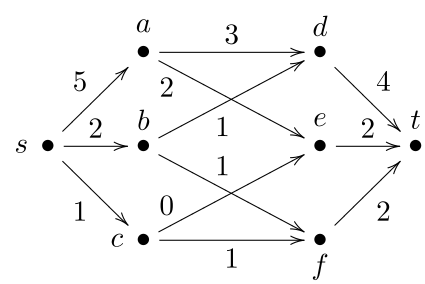
Solution: To put just any flow on a network, we simply reduce capacities keeping Kirchhoff's condition in mind. The flow f is given below:



The simple path $p = (s, c, f, t)$ is clearly an augmenting path for f. Since the network and path orientations agree for each edge, all we have to worry about is whether the image under the flow is strictly less than capacity. This is obviously the case.

That path and network orientation coincide means that ε is capacity minus image under f for each edge. Therefore, we have $\varepsilon = 1, 10$, and 11, respectively, for the three edges of p. Thus, the tolerance of p is $\delta = \min \varepsilon = 1$.

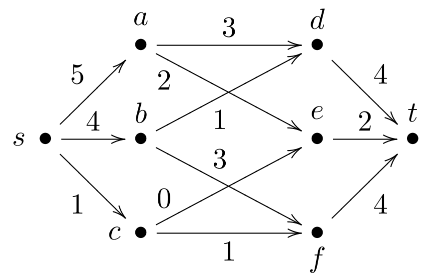
By Problem 3, we can create a new flow f' with a larger value than f by adding 1 to each of the edges of p.



We now need to find an augmenting path for f'. If we start at s we are forced to go to b, since the other edges are already at capacity. We could then go from b to d. Once at d we can't head straight to t, because that edge is at capacity. We are allowed to go backwards to a, as $3 > 0$. But now we are stuck. We can't go backwards to s since paths do not repeat vertices and we can't go forward to e because that edge is at capacity.

The problems started when we moved on our network from b to d. So let's try b to f. This works, as does f to t. Hence $r = (s, b, f, t)$ is an augmenting path for f'. The respective ε for the three edges are $4, 2$ and 10. Therefore, the tolerance of r is $\delta = 2$.

We use this δ, as before, to create a new flow f''.



The flow f'' has no augmenting path. Starting out s, we are still forced to b. We could go to d, but we already know that wont get us anywhere. And we can no longer go to f since that edge is now at capacity. Hence, f'' is a maximum flow.

To check our work, we note that $\text{val}(f'') = 10 = C^+(\{s, a, b, d\})$. Thus, by Theorem 6.44 from the text, $S = \{s, a, b, d\}$ is a minimum source vertex cut and f'' is a maximum flow.

(6) Prove Euler's Formula by induction on the number of vertices.

Solution: If a connected graph G has only 1 vertex, then all of its edges are loops. Each loop corresponds to a face, its interior. Every face, except for the infinite face, corresponds to a loop. Thus,

$$n - m + f = 1 - m + (m + 1) = 2.$$

Now, assume G has $n > 1$ vertices. Since G is connected, there exists one edge that is not a loop. By contracting that edge, we get a new graph with n' vertices, m' edges and f' faces. A contraction reduces the number of edges and vertices by one, but does not change the number of faces. So we have

$$n' - m' + f' = (n - 1) - (m - 1) + f = n - m + f.$$

Hence, the number of vertices minus the number of edges plus the number of faces remains the same as we contract each nonloop edge. We continue until all edges are loops, whence we get 2.

(7) Let G be a plane graph with n vertices, m edges, f faces and k components. Show that

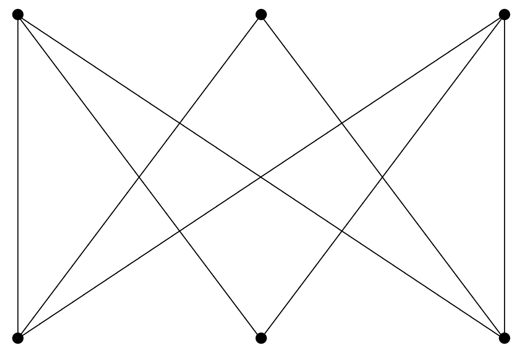
$$n - m + f = k + 1.$$

Solution: Let $H_1 \cdots H_k$ be the components of G. Connect these components in a "path". That is, introduce an edge from a vertex in H_1 to a vertex in H_2, another edge from a vertex in H_2 to a vertex in H_3, and so on. The new graph is connected, has the same number of vertices and faces as G and $k - 1$ additional edges. Thus,

$$n - (m + k - 1) + f = 2 \Rightarrow n - m - k + 1 + f = 2 \Rightarrow n - m + f = k + 1.$$

(8) Let e be an edge of $K_{3,3}$. Show that $K_{3,3} - e$ is planar.

Solution: We systematically deform $K_{3,3} - e$ to reveal a planar embedding.



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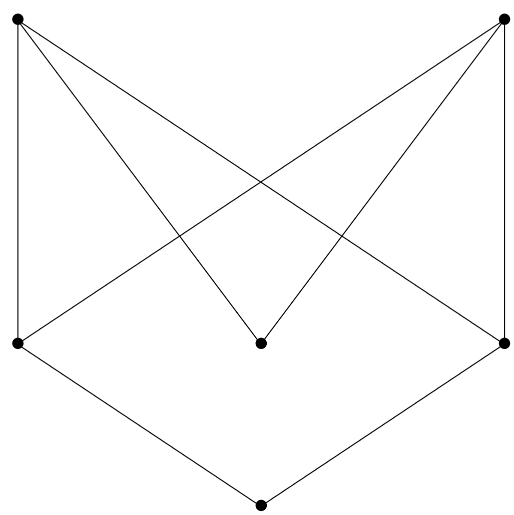

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