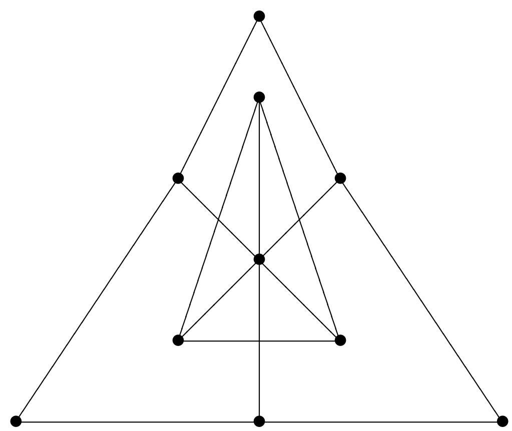
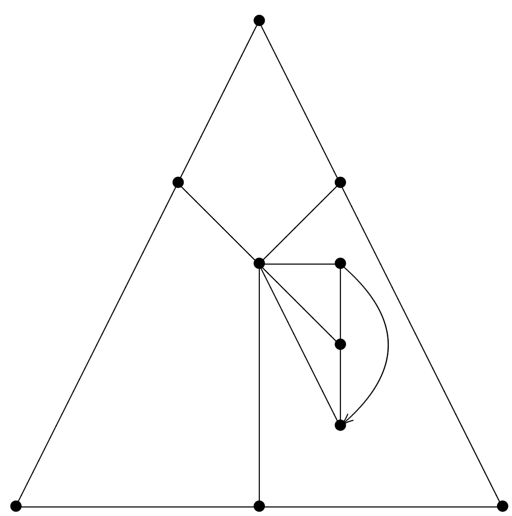
HOMEWORK 3

SOLUTIONS

(1) Show that the graph given below is planar.



Solution: To get a plane graph isomorphic to the given graph, we "push" the inner triangle into the lower right corner of the larger triangle and turn it inside out.



(2) Let G be a plane graph. If $e \in G$, what is the plane dual of the contraction $(G \cdot e)^*$ in terms of the plane dual G^* of G?

Solution: Contracting by an edge $e \in E(G)$ does not change the faces of G. Hence, $(G \cdot e)^*$ has the same vertex set as G^*. Furthermore, as $G \cdot e$ has one less edge than G, $(G \cdot e)^*$ has one less edge than G^*. In short,

$$(G \cdot e)^* = G^* - e^*.$$

(3) Let G be a simple connected graph with at least 11 vertices. Prove that either G or its complement $\overline{G}$ must be nonplanar.

Solution: Let G be a graph on n vertices and assume that both G and $\overline{G}$ are planar. Let m and m' be the number of edges in G and $\overline{G}$, respectively. The union of the two graphs is the complete graph on n vertices. Thus,

$$m + m' = \binom{n}{2} = \frac{n(n-1)}{2}.$$

By Corollary 7.15 in the text, $m, m' \leq 3n - 6$. Therefore,

$$m + m' \leq 6n - 12.$$

We then have

$$\frac{n(n-1)}{2} = m + m' \leq 6n - 12 \Rightarrow n^2 - 13n + 24 \leq 0 \Rightarrow n < 11.$$

(4) Let G be a simple connected planar graph with less than 12 vertices. Prove that G has a vertex of degree at most 4.

Solution: Suppose G has n vertices and m edges and assume that the degree of each vertex is greater than or equal to 5. Again by Corollary 7.15, and the Hand-Shaking Theorem,

$$5n \leq \sum_{u \in V(G)} d_G(u) = 2|E(G)| \leq 6n - 12 \Rightarrow n \geq 12.$$

(5) Let G be a simple connected planar graph with less than 30 edges. Prove that G has a vertex of degree at most 4.

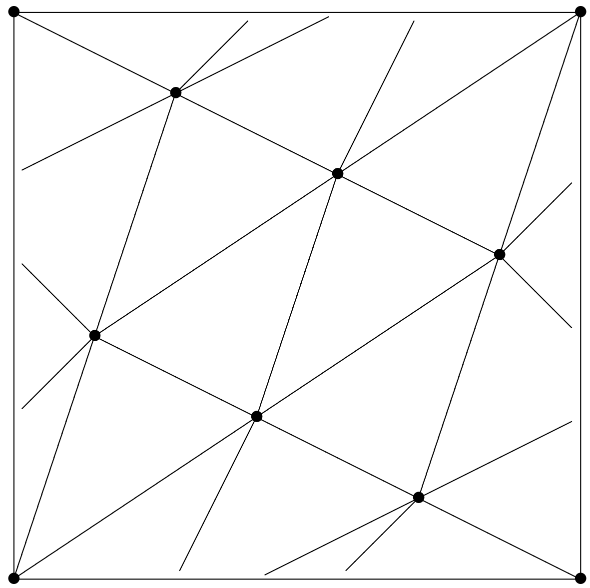
Solution: Again assume that the degree of each vertex is greater than or equal to 5. By the Hand-Shaking-Theorem and our assumption on the number of edges

$$5n \leq \sum_{u \in V(G)} d_G(u) = 2|E(G)| < 2 \times 30 = 60 \Rightarrow n < 12.$$

Problem 5 now follows from Problem 4.

(6) Show that K_7 is toroidal.

Solution: A torus embedding of K_7 is given below. By Kuratowski's theorem, K_7 is not planar. Thus, K_7 is toroidal.



(7) Prove that K_n is toroidal if, and only if $n \in \{5, 6, 7\}$.

Solution: We have seen in class planar embeddings of K_1, K_2, K_3 and K_4. Therefore, none of these graphs are planar.

We now have toroidal embeddings of K_5 and K_7. Since, these graphs are not planar, they are toroidal. The graph K_6 must also be toroidal, since its genus cannot be less than the genus of K_5 or greater than genus of K_7.

By Theorem 7.55 from the text,

$$\gamma(K_n) \geq \frac{(n-3)(n-4)}{12}.$$

So when $n \geq 8$, $\gamma(K_n) \geq \frac{(n-3)(n-4)}{12} = \frac{20}{12} > 1$. Hence, K_n is not toroidal for those values of n.

(8) What is the maximum number of edges in a graph with n vertices and genus γ? Justify, your answer.

Solution: The generalization of Euler's Formula gives

$$n - m + f = 2 - 2\gamma.$$

In class, we saw that

$$f \leq \frac{2}{3}m.$$

Therefore we have

$$n - m + \frac{2}{3}m \geq 2 - 2\gamma.$$

Hence

$$-\frac{1}{3}m \geq 2 - 2\gamma - n.$$

Thus

$$m \leq 3n + 6\gamma - 6.$$