ON VALUES OF THE KATZ TWO-VARIABLE p-ADIC L-FUNCTION

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Abstract. We develop a framework that enables us to study a broad class of special values of the Katz two-variable p-adic L-function, including certain special values lying outside the range of p-adic interpolation.

Contents

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1. Introduction

In this article we shall study a wide class of special values of the Katz two-variable p-adic L-function by extending the techniques and results of [1, 2, 30, 31]

Let K be an imaginary quadratic field, and let E/K be an elliptic curve with complex multiplication by the ring of integers O_K of K; then K is necessarily of class number one. Let $p > 3$ be a prime of good, ordinary reduction for E. We may write $p = \mathfrak{p}\mathfrak{p}^*$, with $\mathfrak{p}O_K = \pi O_K$ and $\mathfrak{p}^* = \pi^* O_K$.

Let

$$\psi : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(E_{\pi^\infty}) \xrightarrow{\sim} O_{K,\mathfrak{p}}^\times \xrightarrow{\sim} \mathbf{Z}_p^\times,$$
$$\psi^* : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(E_{\pi^{*\infty}}) \xrightarrow{\sim} O_{K,\mathfrak{p}^*}^\times \xrightarrow{\sim} \mathbf{Z}_p^\times$$

denote the natural $\mathbf{Z}_p^\times$-valued characters of $\mathrm{Gal}(\overline{K}/K)$ arising via Galois action on E_{π^∞} and $E_{\pi^{*\infty}}$ respectively. We may identify ψ with the Grossecharacter associated to E (and ψ^* with the complex conjugate $\overline{\psi}$ of this Grossencharacter), as described, for example, in [30, p. 325].

Set $\mathfrak{K}_\infty := K(E_{p^\infty})$, and let $\mathcal{O}$ denote the completion of the ring of integers of the maximal unramified extension of $K_\mathfrak{p}$. For any extension L/K we set

$$\Lambda(L) := \Lambda(\mathrm{Gal}(L/K)) := \mathbf{Z}_p[[\mathrm{Gal}(L/K)]],$$

and $\Lambda(L)_\mathcal{O} := \mathcal{O}[[\mathrm{Gal}(L/K)]]$.

The Katz two-variable p-adic L-function $\mathcal{L}_\mathfrak{p} \in \Lambda(\mathfrak{K}_\infty)_\mathcal{O}$ satisfies a p-adic interpolation formula that may be described as follows (see [30, Theorem 7.1] for the version given here, and also [13, Theorem II.4.14]. Note also that, as the notation indicates, $\mathcal{L}_\mathfrak{p}$ depends upon a choice of prime $\mathfrak{p}$ lying above p). For all pairs of integers $j, k \in \mathbf{Z}$ with $0 \le -j < k$, and for all characters $\chi : \mathrm{Gal}(K(E_p)/K) \to \overline{K}^\times$, we have

$$\mathcal{L}_\mathfrak{p}(\psi^k \psi^{*j} \chi) = A \cdot L(\psi^{-k}\overline{\psi}^{-j}\chi^{-1}, 0). \tag{1.1}$$

Here $L(\psi^{-k}\overline{\psi}^{-j}\chi^{-1}, s)$ denotes the complex Hecke L-function, and A denotes an explicit, non-zero factor whose precise description we shall not need.

Suppose that

$$\phi : \mathrm{Gal}(\overline{K}/K) \to \mathbf{Z}_p^{\times};$$

$$\rho : \mathrm{Gal}(\overline{K}/K) \to \mathbf{Z}_p,$$

are characters, with ρ non-trivial. We define

$$L_{\mathfrak{p}}(\phi; \rho, s) := \mathcal{L}_{\mathfrak{p}}(\phi \cdot \rho^s).$$

This function captures the behaviour of $\mathcal{L}_{\mathfrak{p}}$ at ϕ in the direction cut out by ρ.

Suppose that $L_{\mathfrak{p}}(\phi; \rho, s)$ is not identically zero as a function of s. Then if ϕ lies within the range of p-adic interpolation of $\mathcal{L}_{\mathfrak{p}}$, the behaviour of $\mathcal{L}_{\mathfrak{p}}$ at ϕ in the ρ direction (i.e. the behaviour of $L_{\mathfrak{p}}(\phi; \rho, s)$ at $s = 0$) is often predicted by various p-adic generalisations of conjectures of Birch and Swinnerton-Dyer type due to several people. On the other hand, the behaviour of $\mathcal{L}_{\mathfrak{p}}$ outside the range on interpolation is much less well-understood. Variants of the p-adic Birch and Swinnerton-Dyer conjecture involving special values of $\mathcal{L}_{\mathfrak{p}}$ lying outside the range of interpolation were first introduced and studied by Rubin in [30, 31], with some later work by the present author in [1, 2]. (See also [7] for more recent subsequent work related to this topic, using a very different approach.) In this paper we shall generalise the framework introduced in [1, 2]; this will enable us to analyse a broad class of special values of $\mathcal{L}_{\mathfrak{p}}$ in a uniform manner.

For any integer $r \geq 0$, we write

$$\mathcal{L}_{\mathfrak{p}}^{(r)}(\phi; \rho) := \frac{1}{r!} \left(\frac{d}{ds} \right)^r L_{\mathfrak{p}}(\phi; \rho, s) \Bigg|_{s=0}.$$

Let χ_{cyc} denote the p-adic cyclotomic character of $\mathrm{Gal}(\overline{K}/K)$, and define $\phi^* := \phi^{-1}\chi_{\mathrm{cyc}}$. Set $T := \mathbf{Z}_p(\phi)$ and $W := T \otimes_{\mathbf{Z}_p} (\mathbf{Q}_p/\mathbf{Z}_p)$. Define T^* and W^* analogously. Write K_∞^ρ for the $\mathbf{Z}_p$- extension of K cut out by ρ.

In order to study the behaviour of $\mathcal{L}_{\mathfrak{p}}$ at ϕ in the ρ direction, we introduce an Iwasawa module that is naturally associated to $L_{\mathfrak{p}}(\phi; \rho, s)$ via a theorem of Rubin, viz. the two-variable main conjecture (see [32]). The Iwasawa module $X_{\mathfrak{p}}(K_\infty^\rho, W)$ that we consider is the Pontryagin dual of a certain *restricted Selmer group* $\Sigma_{\mathfrak{p}}(K^{\rho}_{\infty}, W)$. The two-variable main conjecture shows that a characteristic power series of $X_{\mathfrak{p}}(K^{\rho}_{\infty}, W)$ may be viewed as being an algebraic p-adic L-function associated to $L_{\mathfrak{p}}(\phi; \rho, s)$.

We define corresponding compact restricted Selmer groups $\Sigma_{\mathfrak{p}}(K,T) \subseteq H^1(K,T)$ and $\Sigma_{\mathfrak{p}^*}(K,T^*) \subseteq H^1(K,T^*)$, and we construct a p-adic height pairing

$$[-,-]^{(\rho)}_{K,\mathfrak{p}} : \Sigma_{\mathfrak{p}^*}(K,T^*) \times \Sigma_{\mathfrak{p}}(K,T) \to \mathbf{Z}_p$$

together with an associated regulator $\mathcal{R}^{(\rho)}_{K,\mathfrak{p}}$.

Set

$$n_{\mathfrak{p}}(\phi) := \mathrm{rank}_{\mathbf{Z}_p}[\Sigma_{\mathfrak{p}}(K,T)],$$

and let $\Sigma_{\mathfrak{p}}(K,W)_{/\,\mathrm{div}}$ denote the quotient of $\Sigma_{\mathfrak{p}}(K,W)$ by its maximal divisible subgroup. Write $\Sigma_{\mathfrak{p}}(K,T)_{\mathrm{tors}}$ for the torsion subgroup of $\Sigma(K,T)$. The following result is a special case of Theorem 6.3 of the main text.

Theorem A. *Suppose that $\mathcal{R}^{(\rho)}_{K,\mathfrak{p}} \neq 0$ and that $\phi \neq \chi_{\mathrm{cyc}}$ is of infinite order.*

(a) We have

$$\mathrm{ord}_{s=0}\, L_{\mathfrak{p}}(\phi; \rho, s) = n_{\mathfrak{p}}(\phi).$$

(b) Let S be any finite set of finite places of K containing all places lying above p and all places of bad reduction of E, and let $\mathcal{N}$ be the maximal abelian extension of K that is unramified outside S. Suppose in additiion that $H^2(G(\mathcal{N}/K^{\rho}_{\infty}), W) = 0$, and that the $\Lambda(K^{\rho}_{\infty})$-module $X(K^{\rho}_{\infty}, W)$ has no finite submodules. Then

$$\mathcal{L}^{(n_{\mathfrak{p}}(\phi))}_{\mathfrak{p}}(\phi; \rho) \sim |\Sigma_{\mathfrak{p}}(K,W)_{/\,\mathrm{div}}| \cdot |\Sigma_{\mathfrak{p}}(K,T)_{\mathrm{tors}}| \cdot \mathcal{R}^{(\rho)}_{K,\mathfrak{p}},$$

where the symbol '$\sim$' denotes equality up to multiplication by a p-adic unit.

By using duality theorems to study $n_{\mathfrak{p}}(\phi)$ (see Theorem 7.3 of the main text), we show the following result.

Theorem B. *Suppose that ϕ is of infinite order, with $\phi \neq \chi_{\mathrm{cyc}}$. Suppose also that* $\mathrm{loc}_{\mathfrak{p}}(\Sigma_{\mathfrak{p}}(K,T))$ *is of infinite order. Then*

$$n_{\mathfrak{p}}(\phi^*) = n_{\mathfrak{p}}(\phi) - 1.$$

Hence, if in addition $\mathcal{R}^{(\rho)}_{K,\mathfrak{p}^*} \neq 0$*, then*

$$\mathrm{ord}_{s=0}[L_{\mathfrak{p}}(\phi^*;\rho,s)] = \mathrm{ord}_{s=0}[L_{\mathfrak{p}}(\phi;\rho,s)] - 1.$$

In order to be able to obtain exact special value formulae for the functions $L_{\mathfrak{p}}(\phi;\rho,s)$, we need to impose a further condition on the characters ρ that we consider.

For any character $\eta : \mathrm{Gal}(\overline{K}/K) \to \mathbf{Z}_p^{\times}$ write

$$\langle \eta \rangle : \mathrm{Gal}(\overline{K}/K) \to 1 + p\mathbf{Z}_p \simeq \mathbf{Z}_p$$

for the composition of η with the natural projection map $\mathbf{Z}_p^{\times} \to 1 + p\mathbf{Z}_p$.

Definition C. *Suppose that* $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}^*\}$*. Say that*

$$\rho : \mathrm{Gal}(\overline{K}/K) \to \mathbf{Z}_p$$

is locally Lubin-Tate (LLT) at $\mathfrak{q}$ *if* $\rho = \langle \eta \rangle$ *for some* $\eta : \mathrm{Gal}(\overline{K}/K) \to \mathbf{Z}_p^{\times}$ *which is surjective and totally ramified at* $\mathfrak{q}$*.* □

When ρ is LLT at $\mathfrak{p}$, it is possible to obtain exact expressions for $\mathcal{L}_{\mathfrak{p}}^{(n_{\mathfrak{p}}(\phi))}(\phi;\rho)$ by applying suitable explicit reciprocity laws to certain canonical elements in $\Sigma_{\mathfrak{p}}(K,T)$ that are constructed using twisted Euler systems of elliptic units (see Section 10 below).

Let us illustrate a special case of our results in the setting of characters associated to CM modular forms of higher weight. Consider the characters

$$\phi_k := \psi^{k+1}\psi^{*-k}, \quad \phi_k^* := \psi^{-k}\psi^{*(k+1)}, \quad (k \geq 0).$$

The p-adic character ϕ_k is naturally associated to the CM modular form of weight $2k+2$ attached to the Grossencharacter ψ^{k+1}, and it lies within the range of interpolation of $\mathcal{L}_{\mathfrak{p}}$. The behaviour of $L_{\mathfrak{p}}(\phi_k;\rho,s)$ at $s=0$ is conjecturally well-understood in terms of generalisations of the p-adic Birch and Swinnerton-Dyer conjecture to the case of modular forms of higher weight. On the other hand, the p-adic character ϕ_k^* lies outside the range of interpolation of $\mathcal{L}_{\mathfrak{p}}$, and the behaviour of $L_{\mathfrak{p}}(\phi_k^*;\rho,s)$ at $s=0$ for arbitrary k is less well understood. When $k=0$ (so $\phi_0=\psi$) and $n_{\mathfrak{p}}(\psi) \geq 1$, the function $L_{\mathfrak{p}}(\psi^*;\langle\psi^*\rangle,s)$ with was first studied by K. Rubin, who formulated a version of the p-adic Birch and Swinnerton-Dyer conjecture in this setting (see [30, 31]). This work was subsequently extended to the cover the case $n_{\mathfrak{p}}(\psi) = 0$ by the present author (see [1, 2]). When $k \geq 1$, the function $L_{\mathfrak{p}}(\phi_k^*; \rho, s)$ has not, to the best of our knowledge, previously been studied.

Set

$$T_k := \mathbf{Z}_p(\phi_k), \quad T_k^* := \mathbf{Z}_p(\phi_k^*),$$
$$V_k := T_k \otimes \mathbf{Q}_p, \quad V_k^* := T_k^* \otimes \mathbf{Q}_p.$$

Write

$$\exp^*_{V_k} : H^1(K_{\mathfrak{p}^*}, V_k) \to \mathbf{Q}_p,$$
$$\exp^*_{V_k^*} : H^1(K_{\mathfrak{p}}, V_k^*) \to \mathbf{Q}_p$$

for the Bloch-Kato dual exponential maps associated to V_k and V_k^*, and

$$\log_{V_k} : H^1(K_{\mathfrak{p}}, V_k) \to \mathbf{Q}_p,$$
$$\log_{V_k^*} : H^1(K_{\mathfrak{p}^*}, V_k^*) \to \mathbf{Q}_p$$

for the corresponding Bloch-Kato logarithm maps.

The following result is a special case of Theorem 10.9 (see also Theorem 11.1) of the main text.

Theorem D. *Suppose that $\mathcal{L}_{\mathfrak{p}}(\phi_k) \neq 0$ and that ρ is LLT at $\mathfrak{p}^*$.*

(a) We have $n_{\mathfrak{p}^*}(\phi_k) = n_{\mathfrak{p}}(\phi_k^*) = 1$.

(b) Let $y \in \Sigma_{\mathfrak{p}^*}(K, V_k)$ and $y^* \in \Sigma_{\mathfrak{p}}(K, V_k^*)$ be elements of infinite order. Then

$$\frac{[y^*, y]^{(\rho)}_{K,\mathfrak{p}^*}}{\exp^*_{V_k^*,\mathfrak{p}}(y^*) \cdot \exp^*_{V_k,\mathfrak{p}^*}(y)} \doteq \frac{\mathcal{L}^{(1)}_{\mathfrak{p}^*}(\phi_k; \rho)}{\mathcal{L}_{\mathfrak{p}}(\phi_k)}.$$

Here the symbol '$\doteq$' denotes equality up to multiplication by an explicit non-zero factor involving periods and Euler factors.

In particular, we have that $\mathcal{L}^{(1)}_{\mathfrak{p}^*}(\phi_k; \rho) \neq 0$ if and only if $[y^*, y]^{(\rho)}_{K,\mathfrak{p}^*} \neq 0$. □

Theorem D, (which is a generalisation of [2, Theorem A] to modular forms of higher weight), relates the non-zero value of $\mathcal{L}_{\mathfrak{p}}$ at the point ϕ_k lying within the range of interpolation to the derivative of $\mathcal{L}_{\mathfrak{p}}$ at the point ϕ_k^* lying outside the range on interpolation. It may be viewed as being an analogue of the well-known exceptional zero phenomenon observed in the work of Mazur, Tate and Teitelbaum in the setting of elliptic curves without complex multiplication (see [17, especially Conjecture 2], [21, especially page 38]). We remark that this is currently the only such result in the present setting of which we are aware.

If $\mathcal{L}_{\mathfrak{p}}(\phi_k) = 0$, then Theorems 10.9 and 11.3 below also yield the following generalisation of [30, Theorem 10.1] to the higher weight case, which again illustrates the phenomenon alluded to above.

Theorem E. *Suppose that*

$$\mathrm{ord}_{s=1} \mathcal{L}_{\mathfrak{p}}(\phi_k; \rho, s) = 1, \quad \mathcal{L}_{\mathfrak{p}^*}(\phi_k) \neq 0,$$

and that ρ *is LLT at* $\mathfrak{p}$. *Suppose also that* $\mathcal{R}^{(\rho)}_{K,\mathfrak{p}} \neq 0$ *and that* $\mathrm{loc}_{\mathfrak{p}}(\Sigma_{\mathfrak{p}}(K, T_k))$ *is of infinite order.*

Let $y \in \Sigma_{\mathfrak{p}}(K, V_k)$ *and* $y^* \in \Sigma_{\mathfrak{p}^*}(K, V_k^*)$ *be elements of infinite order. Then*

$$\frac{[y^*, y]^{(\rho)}_{K,\mathfrak{p}}}{\log_{V_k^*,\mathfrak{p}^*}(y^*) \cdot \log_{V_k,\mathfrak{p}}(y)} \doteq \frac{\mathcal{L}^{(1)}_{\mathfrak{p}}(\phi_k; \rho)}{\mathcal{L}_{\mathfrak{p}^*}(\phi_k)}.$$

□

A brief outline of the contents of this paper is as follows. In Section 2, we establish certain notation and conventions that will apply throughout this paper. We then recall some general facts about twists of Iwasawa modules and derivatives of their characteristic power series in Section 3, and we explain how these facts may be applied to the Katz two-variable p-adic L-function. In Section 4, we define various Selmer groups that we need, and we prove a control theorem for restricted Selmer groups.

We construct the p-adic height pairing on restricted Selmer groups in Section 5. In Section 6, under the assumption that this p-adic height pairing is non-degenerate, we prove a very general leading term formula for a characteristic power series of a restricted Selmer group over an arbitrary finite extension of K. We compare different restricted Selmer groups over K in Section 7, and we describe the relationship between the leading terms of the relevant characteristic power series (see Theorem 7.6).

In Sections 8 and 9, we recall various results we need concerning formal groups, explicit reciprocity laws, and the Katz two-variable p-adic L-function. In Section 10, we apply our previous results to construct canonical elements in restricted Selmer groups over K using Euler systems of twisted elliptic units, and to prove a very general exact leading term formula for $\mathcal{L}_{\mathfrak{q}}$ (see especially Theorem 10.9). Finally, in Section 11, we specialise the results of Section 10 to prove results (see Theorems 11.1 and 11.3) which imply Theorems D and E above.

2. Notation and conventions

If L is any field, we write O_L for its ring of integers and L^{ab} for its maximal abelian extension. We let $\overline{L}$ denote an algebraic closure of L.

Let K be an imaginary quadratic field of class number one, and let E/K be a fixed elliptic curve with complex multiplication by the ring of integers O_K of K. We write $\mathfrak{f}$ for the conductor of E. We fix a prime $p > 3$ of good, ordinary reduction for E, so that $pO_K = \mathfrak{p}\mathfrak{p}^*$. Let

$$\psi : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(E_{\mathfrak{p}^\infty}) \xrightarrow{\sim} O_{K,\mathfrak{p}}^\times \xrightarrow{\sim} \mathbf{Z}_p^\times,$$
$$\psi^* : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(E_{\mathfrak{p}^{*\infty}}) \xrightarrow{\sim} O_{K,\mathfrak{p}^*}^\times \xrightarrow{\sim} \mathbf{Z}_p^\times$$

denote the natural $\mathbf{Z}_p^\times$-valued characters of $\mathrm{Gal}(\overline{K}/K)$ arising via Galois action on $E_{\mathfrak{p}^\infty}$ and $E_{\mathfrak{p}^{*\infty}}$ respectively. We may identify ψ with the Grossecharacter associated to E (and ψ^* with the complex conjugate $\overline{\psi}$ of this Grossencharacter), as described, for example, in [30, p. 325].

The symbol $\mathfrak{q}$ will always denote a prime of O_K lying above p, and we write $i_{\mathfrak{q}} : \overline{K} \hookrightarrow \overline{K}_{\mathfrak{q}}$ for the natural embedding afforded by $\mathfrak{q}$.

We write $\chi_{\mathrm{cyc}} : \mathrm{Gal}(\overline{K}/K) \to \mathbf{Z}_p^\times$ for the p-adic cyclotomic character of $\mathrm{Gal}(\overline{K}/K)$. If $\chi : \mathrm{Gal}(\overline{K}/K) \to \mathbf{Z}_p^\times$ is any character of $\mathrm{Gal}(\overline{K}/K)$, we set $\chi^* := \chi^{-1}\chi_{\mathrm{cyc}}$ and $\mathbf{Z}_p(\chi) := \mathbf{Z}_p \otimes \chi$. We write $\langle \chi \rangle : \mathrm{Gal}(\overline{K}/K) \to 1 + p\mathbf{Z}_p$ for the composition of χ with the natural surjection $\mathbf{Z}_p^\times \to 1 + p\mathbf{Z}_p$.

Throughout this paper $\phi : \mathrm{Gal}(\overline{K}/K) \to \mathbf{Z}_p^\times$ denotes a character of infinite order. We let

$$T := \mathbf{Z}_p(\phi), \quad T^* := \mathbf{Z}_p(\phi^*);$$

so T and T^* are free, rank one $\mathbf{Z}_p$-modules on which $\mathrm{Gal}(\overline{K}/K)$ acts via ϕ and ϕ^* respectively. Set

$$V := T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p, \quad W := V/T,$$

$$V^* := T^* \otimes_{\mathbf{Z}_p} \mathbf{Q}_p, \quad W^* := V^*/T^*,$$

and write W_{p^n} and $W^*_{p^n}$ for the p^n-torsion subgroups of W and W^* respectively. We view $T = \varprojlim W_{p^n}$, $T^* = \varprojlim W^*_{p^n}$, (where the inverse limits are taken with respect to the obvious multiplication-by-p maps), and we let $w = [w_n]$, $w^* = [w^*_n]$ denote fixed generators of T and T^* respectively, chosen to satisfy the condition 2.1 below.

For each integer $n \geq 1$, we let

$$e_n : W_{p^n} \times W^*_{p^n} \to \mu_{p^n}$$

denote the pairing afforded by Cartier duality via viewing W_{p^n} and $W^*_{p^n}$ as group schemes over $\mathrm{Spec}(K)$. This pairing satisfies the identity

$$e_n(p \cdot w_n, w^*_n) = e_n(w_n, p \cdot w^*_n).$$

We fix once and for all a generator $\zeta = [\zeta_n]$ of $\mathbf{Z}_p(1)$, and we assume that w and w^* are chosen to satisfy

$$e_n(w_n, w^*_n) = \zeta_n \tag{2.1}$$

for all $n \geq 1$.

Hence, if, for example, $\phi = \psi^i \cdot \psi^{*j}$, with $i, j \in \mathbf{Z}$, then

$$V = V(\phi) = V(\psi)^{\otimes i} \otimes V(\psi^*)^{\otimes j},$$

and so

$$e(V) := w^{\otimes i} \otimes w^{*\otimes j}$$

is a basis of V.

We write

$$\mathcal{K}_n := K(W_{p^n}), \quad \mathcal{K}^*_n := K(W^*_{p^n}), \quad \mathfrak{K}_n := \mathcal{K}_n \cdot \mathcal{K}^*_n,$$

and

$$\mathcal{K}_\infty := \cup_{n\geq 1}\mathcal{K}_n, \quad \mathcal{K}^*_\infty := \cup_{n\geq 1}\mathcal{K}^*_n, \quad \mathfrak{K}_\infty := \cup_{n\geq 1}\mathfrak{K}_n.$$

We denote the unique $\mathbf{Z}_p$-extension contained in $\mathcal{K}_\infty$ by K_∞. For any finite extension F/K, we set

$$\mathcal{F}_n := F \cdot \mathcal{K}_n, \quad \mathcal{F}_n^* := F \cdot \mathcal{K}_n^*, \quad \mathfrak{F}_n := F \cdot \mathfrak{K}_n,$$

and

$$\mathcal{F}_\infty := F \cdot \mathcal{K}_\infty, \quad F_\infty := F \cdot K_\infty, \quad \mathcal{F}_\infty^* := F \cdot \mathcal{K}_\infty^*, \quad \mathfrak{F}_\infty := F \cdot \mathfrak{F}_\infty.$$

The symbol $\mathcal{O}$ denotes the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$.

For any extension L of K or $K_\mathfrak{q}$ with Galois group G, we set

$$\Lambda(L) := \Lambda(G) := \mathbf{Z}_p[[G]], \quad \Lambda(L)_\mathcal{O} := \mathcal{O}[[G]].$$

If

$$\chi : \mathrm{Gal}(\overline{L}/L) \to \mathbf{Z}_p^\times$$

is a character of infinite order, we sometimes write L_∞^χ/L (or some variant thereof) for the extension of L cut out by χ. We set

$$\mathcal{I}_\chi(L) := \mathrm{Ker}[\chi : \Lambda(L_\infty^\chi) \to \mathbf{Z}_p],$$

and we define a generator $\vartheta_\chi = \vartheta_\chi(L)$ of $\mathcal{I}_\chi(L)$ as follows. We choose a topological generator $\gamma_\chi = \gamma_\chi(L)$ and we set

$$\vartheta_\chi(L) := \chi(\gamma_\chi^{-1})\gamma_\chi - 1. \tag{2.2}$$

For any extension L/K we write $\mathcal{M}^\mathfrak{q}(L)$ for the maximal abelian pro-p extension of L which is unramified away from $\mathfrak{q}$, and we set $\mathcal{X}^\mathfrak{q}(L) := \mathrm{Gal}(\mathcal{M}^\mathfrak{q}(L)/L)$. We let $\mathcal{B}^\mathfrak{q}(L)$ denote the maximal abelian pro-p extension of L which is unramified away from $\mathfrak{q}$ and totally split at all places of L lying above $\mathfrak{q}^*$, and we write $\mathcal{Y}^\mathfrak{q}(L) := \mathrm{Gal}(\mathcal{B}^\mathfrak{q}(L)/L)$.

If M is any $\mathbf{Z}_p$-module, then M_{div} denotes the maximal divisible submodule of M, and we set $M_{/\,\mathrm{div}} := M/M_{\mathrm{div}}$. We write M_{tors} for the torsion submodule of M, and $M^\wedge$ for the Pontryagin dual of M. If M is a torsion $\mathbf{Z}_p$-module, then we write $T_p(M)$ for the p-adic Tate module of M.

Suppose that L_1 is a local field or a number field. If M is a $\mathbf{Z}_p$-module of finite rank on whch $\mathrm{Gal}(\overline{L}_1/L_1)$ acts, and $M_{\mathbf{Q}_p} := M \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, then for any (possibly infinite) extension L/L_1, we set

$$H^1_{\mathrm{Iw}}(L, M) := \varprojlim H^1(L', M), \quad H^1_{\mathrm{Iw}}(L, M_{\mathbf{Q}_p}) := \varprojlim H^1(L', M_{\mathbf{Q}_p}),$$

where the inverse limit is over all subfields $L' \subseteq L$ finite over L_1 and is taken with respect to the obvious corestriction maps.

3. Characteristic power series

Our goal in this section is to recall (following [1], but in slightly greater generality) some basic facts concerning twists of Iwasawa modules and derivatives of characteristic power series. We shall then apply these results to twists of the two-variable p-adic L-function $\mathcal{L}_{\mathfrak{q}}$ (where $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}^*\}$) by characters of $\mathrm{Gal}(\mathfrak{K}_\infty/K)$ of infinite order. This will later enable us to apply the two-variable main conjecture to relate special values of $\mathcal{L}_{\mathfrak{q}}$ to the arithmetic of certain Selmer groups.

Suppose that F/K is any finite extension. Let $\mathcal{G}_F := \mathrm{Gal}(\mathfrak{F}_\infty/F)$, and suppose that $\eta : \mathcal{G}_F \to \mathbf{Z}_p^\times$ is any character. Then there is a twisting map

$$\mathrm{Tw}_\eta : \Lambda(\mathcal{G}_F) \to \Lambda(\mathcal{G}_F)$$

associated to η which is induced by the map $g \mapsto \eta(g)g$ for all $g \in \mathcal{G}_F$. It is easy to see that if $f \in \Lambda(\mathcal{G}_F)$, then

$$f(\eta) = [\mathrm{Tw}_\eta(f)](\mathbf{1}).$$

If M is any finitely generated $\Lambda(\mathcal{G}_F)$-module with characteristic power series $f_M \in \Lambda(\mathcal{G}_F)$, then a routine computation shows that $\mathrm{Tw}_\eta(f_M)$ is a characteristic power series of $M(\eta^{-1}) := M \otimes \eta^{-1}$.

Suppose now that

$$\rho : \mathcal{G}_F \to \mathbf{Z}_p$$

is a character of infinite order, and set $\mathcal{H} = \mathcal{H}(\rho) := \mathrm{Ker}(\rho)$. Write

$$G_{\mathcal{H}} := \mathcal{G}/\mathcal{H} \simeq \mathbf{Z}_p.$$

Then there is a natural quotient map

$$\Pi_{G_{\mathcal{H}}} : \Lambda(\mathcal{G}_F) \to \Lambda(G_{\mathcal{H}}),$$

and we write $\mathcal{I}_{G_{\mathcal{H}}}$ for the kernel of $\Pi_{G_{\mathcal{H}}}$.

The element $\Pi_{G_{\mathcal{H}}}(\mathrm{Tw}_{\eta}(f_M))$ is a characteristic power series of the $\Lambda(G_{\mathcal{H}})$-module $M(\eta^{-1}) \otimes_{\Lambda(\mathcal{G}_F)} \Lambda(G_{\mathcal{H}})$, and there is an isomorphism

$$M(\eta^{-1}) \otimes_{\Lambda(\mathcal{G}_F)} \Lambda(G_{\mathcal{H}}) \simeq M(\eta^{-1})/\mathcal{I}_{G_{\mathcal{H}}} M(\eta^{-1})$$

of $\Lambda(G_{\mathcal{H}})$-modules.

We deduce that we may study the behaviour of f_M at η in the direction cut out by ρ by studying the behaviour of the element $\Pi_{G_{\mathcal{H}}}(\mathrm{Tw}_{\eta}(f_M))$ at the identity character $\mathbf{1}$ of $G_{\mathcal{H}}$.

Now set

$$M_1 := M(\eta^{-1}) \otimes_{\Lambda(\mathcal{G}_F)} \Lambda(G_{\mathcal{H}}),$$
$$f_{M_1} := \Pi_{G_{\mathcal{H}}}(\mathrm{Tw}_{\eta}(f_M)).$$

Let $\gamma_{\mathcal{H}}$ be a fixed topological generator of $G_{\mathcal{H}}$. We may identify $\Lambda(G_{\mathcal{H}})$ with the power series ring $\mathbf{Z}_p[[X]]$ in one variable in the usual way via the map $\gamma_{\mathcal{H}} \mapsto 1 + X$.

Let $I_{G_{\mathcal{H}}}$ denote the augmentation ideal of $\Lambda(G_{\mathcal{H}})$, and suppose that $n \geq 0$ is the smallest integer such that the image of f_{M_1} in $I^n_{G_{\mathcal{H}}}/I^{n+1}_{G_{\mathcal{H}}}$ is non-zero. It is not hard to check that f_{M_1} is a characteristic power series of the $\Lambda(G_{\mathcal{H}})$-module M_1, and that

$$((\gamma_{\mathcal{H}} - 1)^{-n} f_{M_1})(\mathbf{1}) = \left. \frac{f_{M_1}(X)}{X^n} \right|_{X=0}, \tag{3.1}$$

where $\mathbf{1}$ denotes the identity character of $G_{\mathcal{H}}$.

Set

$$\vartheta_{\rho} := \rho(\gamma_{\mathcal{H}})^{-1}\gamma_{\mathcal{H}} - 1.$$

(This makes sense as ρ factors through $\mathcal{H}$.)

Then if $m \geq 0$ is any integer, it follows from the definitions that we have

$$(\vartheta_{\rho}^{-m} f_{M_1})(\rho) = [(\gamma_{\mathcal{H}} - 1)^{-m}\, \mathrm{Tw}_{\rho}(f_{M_1})](\mathbf{1}), \tag{3.2}$$

where $\mathrm{Tw}_\rho : \Lambda(G_{\mathcal{H}}) \to \Lambda(G_{\mathcal{H}})$ is the twisting map associated to ρ viewed as a character of $G_{\mathcal{H}}$.

Let us now explain how (3.2) is related to derivatives of certain p-adic analytic functions (see [30, §7]). Then the map from $\mathbf{Z}_p$ to $\mathbf{C}_p$ given by

$$s \mapsto f_M(\eta \cdot \rho^s) = f_{M_1}(\rho^s)$$

defines an analytic function $f(s)$ on $\mathbf{Z}_p$. Define

$$\mathrm{ord}_\eta^{(\rho)}(f_M) := \mathrm{ord}_{s=0}\, f(s),$$

and set

$$\mathbf{D}^{(m;\rho)} f_M(\eta) := \frac{1}{m!}\left(\frac{d}{ds}\right)^m f_M(\eta \cdot \rho^s)\Bigg|_{s=0} = \frac{1}{m!}\left(\frac{d}{ds}\right)^m f_{M_1}(\rho^s)\Bigg|_{s=0}.$$

We write

$$f_M^{(m;\rho)}(\eta) := \mathbf{D}^{(m;\rho)} f_M(\eta),$$

and we extend these definitions to all of $\Lambda(\mathcal{G}_F)$ in the obvious manner. A routine calculation shows that we have

$$\mathbf{D}^{(m;\rho)}(\vartheta_\rho^m(\mathbf{1})) = \left\{\frac{\log_p(\rho(\gamma_{\mathcal{H}}))}{\rho(\gamma_{\mathcal{H}})}\right\}^m,$$

and

$$\mathbf{D}^{(m;\rho)}(\vartheta_\rho^m f_M)(\eta) = \{\log_p[\rho(\gamma_{\mathcal{H}})]/\rho(\gamma_{\mathcal{H}})\}^m f_{M_1}(\mathbf{1}) \tag{3.3}$$

We can now see from (3.1), (3.2) and (3.3) that if we set

$$m_1 := \mathrm{ord}_\eta^{(\rho)}(f_M),$$

then we may write $f_{M_1} = \vartheta_\rho^{m_1} F_\rho$ with $F_\rho \in \Lambda(G_{\mathcal{H}})$, and we have

$$\begin{aligned} f_M^{(m_1)}(\eta) &= \lim_{s\to 0} \frac{f_{M_1}(\rho^s)}{s^{m_1}} \\ &= \mathbf{D}^{(m_1)}(\vartheta_\rho^{m_1} F_\rho)(\mathbf{1}) \\ &= [\{\log_p[\rho(\gamma_{\mathcal{H}})]/\rho(\gamma_{\mathcal{H}})\}^{m_1} F_\rho](\mathbf{1}) \\ &= \{\log_p[\rho(\gamma_{\mathcal{H}})]/\rho(\gamma_{\mathcal{H}})\}^{m_1} \cdot \frac{f_{M_1}(X)}{X^{m_1}}\Bigg|_{X=0}. \end{aligned} \tag{3.4}$$

We shall now apply the above discussion to the case in which $F = K$ and $M = \mathcal{X}_{\mathfrak{p}}(\mathfrak{K}_\infty)$.

Set $\mathcal{D}_\infty := \mathfrak{K}_\infty^{\mathcal{H}}$, the fixed field of $\mathcal{H}$, and write $D_\infty := \mathcal{D}_\infty^{\Delta_\mathcal{H}}$; then

$$\mathrm{Gal}(D_\infty/K) = G_\mathcal{H} \simeq \mathbf{Z}_p.$$

Recall that the two-variable main conjecture asserts that if $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}^*\}$, then $\mathcal{X}_\mathfrak{q}(\mathfrak{K}_\infty)$ is a torsion $\Lambda(\mathfrak{K}_\infty)$-module, and that the Katz two-variable p-adic L-function $\mathcal{L}_\mathfrak{q}$ is a characteristic power series of $\mathcal{X}_\mathfrak{q}(\mathfrak{K}_\infty)$ in $\Lambda(\mathfrak{K}_\infty)_\mathcal{O}$. We therefore see from the discussion above that $\mathrm{Tw}_\eta(\mathcal{L}_\mathfrak{q}) \in \Lambda(\mathfrak{K}_\infty)_\mathcal{O}$ is a characteristic power series of $\mathcal{X}_\mathfrak{q}(\mathfrak{K}_\infty)(\eta^{-1})$.

Let $\mathcal{I}_{D_\infty}$ denote the kernel of the natural map $\Lambda(\mathfrak{K}_\infty) \to \Lambda(D_\infty)$. Fix any characteristic power series $H_{\mathfrak{q},\eta}^{(\eta)}$ of the $\Lambda(D_\infty)$-module

$$\mathcal{X}_\mathfrak{q}(\mathfrak{K}_\infty)(\rho^{-1}) \otimes_{\Lambda(\mathfrak{K}_\infty)} (\Lambda(\mathfrak{K}_\infty)/\mathcal{I}_{D_\infty}) \simeq \mathcal{X}_\mathfrak{q}(\mathfrak{K}_\infty)(\eta^{-1})/\mathcal{I}_{D_\infty}\mathcal{X}_\mathfrak{q}(\mathfrak{K}_\infty)(\eta^{-1}).$$

Set

$$L_\mathfrak{q}(\eta; \rho, s) := \mathcal{L}_\mathfrak{q}(\eta \cdot \rho^s),$$

and write

$$n_{\mathfrak{q},\rho}(\eta) := \mathrm{ord}_{s=0}\, L_\mathfrak{p}(\eta; \rho, s).$$

Definition 3.1. For any non-negative integer m, we define

$$\mathcal{L}_\mathfrak{q}^{(m)}(\eta; \rho) := \lim_{s\to 0} \frac{L_\mathfrak{p}(\eta; \rho, s)}{s^m}. \tag{3.5}$$

□

Proposition 3.2. *With the above notation, we have*

$$n_{\mathfrak{q},\rho}(\eta) = \mathrm{ord}_{X=0}\, H_{\mathfrak{q},\rho}^{(\eta)}(X), \tag{3.6}$$

and

$$\mathcal{L}_\mathfrak{q}^{(n_{\mathfrak{q},\rho}(\eta))}(\eta; \rho) = \lim_{s\to 0} \frac{L_\mathfrak{p}(\eta; \rho, s)}{s^{n_{\mathfrak{q},\rho}(\eta)}} \sim \left\{ \frac{\log_p(\rho(\gamma_\mathcal{H}))}{\rho(\gamma_\mathcal{H})} \right\}^{n_{\mathfrak{q},\rho}} \cdot \left. \frac{H_{\mathfrak{q},\rho}^{(\eta)}(X)}{X^{n_{\mathfrak{q},\rho}(\eta)}} \right|_{X=0}, \tag{3.7}$$

where '$\sim$' denotes equality up to multiplication by a p-adic unit (which, in this case, lies in $\mathcal{O}^\times$).

Proof. This follows from (3.1), (3.2), and (3.4). □

We therefore see that the order of vanishing $n_{\mathfrak{q},\rho}(\eta)$ of $\mathcal{L}_{\mathfrak{q}}$ at η in the direction of ρ, and the p-adic valuation of $\mathcal{L}_{\mathfrak{q}}^{(n_{\mathfrak{q},\rho}(\eta))}(\eta;\rho)$ may be determined by studying $H_{\mathfrak{q},\rho}^{(\eta)}$, and that this may be done algebraically.

4. Selmer groups

In this section we shall define various Selmer groups that we require, and we shall establish some of their properties.

Suppose that F is a finite extension of K.

Definition 4.1. For each finite place v of F, we write $H^1_f(F,V)$ for the Bloch-Kato cohomology group at v associated to V. Hence

$$H^1_f(F_v,V) = \begin{cases} \mathrm{Ker}[H^1(F_v,V) \to H^1(\mathrm{Gal}(\overline{F}_v/F_v^{nr},V))] & \text{if } v \nmid p; \\ \mathrm{Ker}[H^1(F_v,V) \to H^1(F_v, B_{\mathrm{crys}} \otimes_{\mathbf{Q}_p} V)] & \text{if } v \mid p, \end{cases}$$

where F_v^{nr} is the maximal unramified extension of F_v, and B_{crys} denotes Fontaine's ring of crystalline periods.

There is a tautological exact sequence

$$0 \to T \to V \to W \to 0, \tag{4.1}$$

and we define $H^1_f(F_v,T)$ and $H^1(F_v,V)$ to be the pre-image and image respectively of $H^1(F_v,V)$ under the maps on cohomology groups induced by the exact sequence (4.1).

For each positive integer n, there is an exact sequence

$$0 \to W_{p^n} \to W \xrightarrow{\times p^n} W \to 0, \tag{4.2}$$

and we define $H^1_f(F_v,W_{p^n})$ to be the inverse image of $H^1(F_v,W)$ under the map on cohomology induced by (4.2).

We define similar groups with V replaced by V^* in an entirely analogous manner. □

Example 4.2. Suppose that $\phi = \psi^i \psi^{*j}$. For each place v of F lying above p, we set

$$m_v(\phi) = \begin{cases} i & \text{if } v \mid \mathfrak{p}; \\ j & \text{if } v \mid \mathfrak{p}^*. \end{cases}$$

The following table lists the groups $H^1_f(F_v, -)$ for $v \mid p$:

	$m_v(\phi) < 0$	$m_v(\phi) = 0$	$m_v(\phi) > 0$
V	0	0	$H^1(F_v, V)$
T	$H^1(F, T)_{\mathrm{tors}}$	$H^1(F_v, T)_{\mathrm{tors}}$	$H^1(F_v, T)$
W	0	0	$H^1(F_v, W)_{\mathrm{div}}$

If $v \nmid p$, then we have

$$H^1_f(F_v, V) = H^1_f(F_v, W) = 0;$$
$$H^1_f(F_v, T) = H^1(F_v, T)_{\mathrm{tors}},$$

irrespective of the values of i and j. □

Definition 4.3. Suppose that $M \in \{W, W^*, W_{p^n}, W^*_{p^n}\}$ and that $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}^*\}$. If $c \in H^1(F, M)$, then we write $\mathrm{loc}_v(c)$ for the image of c in $H^1(F_v, M)$. We define

- the *true Selmer group* $\mathrm{Sel}(F, M)$ by

$$\mathrm{Sel}(F, M) = \left\{c \in H^1(F, M) \mid \mathrm{loc}_v(c) \in H^1_f(F_v, M) \, \text{for all } v\right\};$$

- the *relaxed Selmer group* $\mathrm{Sel}_{\mathrm{rel}}(F, M)$ by

$$\mathrm{Sel}_{\mathrm{rel}}(F, M) = \left\{c \in H^1(F, M) \mid \mathrm{loc}_v(c) \in H^1_f(F_v, M) \, \text{for all } v \text{ not dividing } p\right\};$$

- the *strict Selmer group* $\mathrm{Sel}_{\mathrm{str}}(L, M)$ by

$$\mathrm{Sel}_{\mathrm{str}}(F, M) = \{c \in \mathrm{Sel}(F, M) \mid \mathrm{loc}_v(c) = 0 \, \text{for all } v \text{ dividing } p\};$$

- the $\mathfrak{q}$-*strict Selmer group* $\mathrm{Sel}_{\mathrm{str}(\mathfrak{q})}(F, M)$ by

$$\mathrm{Sel}_{\mathrm{str}(\mathfrak{q})}(F, M) = \{c \in \mathrm{Sel}(F, M) \mid \mathrm{loc}_v(c) = 0 \, \text{for all } v \text{ dividing } \mathfrak{q}\};$$

- the $\mathfrak{q}$-*restricted Selmer group* (or simply *restricted Selmer group* for short when $\mathfrak{q}$ is understood) $\Sigma_{\mathfrak{q}}(F, M)$ by

$$\Sigma_{\mathfrak{q}}(F, M) = \{c \in \mathrm{Sel}_{\mathrm{rel}}(F, M) \mid \mathrm{loc}_v(c) = 0 \, \text{for all } v \nmid \mathfrak{q}\}.$$

We also define

$$\check{\mathrm{Sel}}_?(F,T) := \varprojlim_n \mathrm{Sel}_?(F,W_{p^n}), \quad \check{\mathrm{Sel}}_?(F,T^*) := \varprojlim_n \mathrm{Sel}_?(F,W^*_{p^n}),$$
$$\check{\Sigma}_{\mathfrak{q}}(F,T) := \varprojlim_n \Sigma_{\mathfrak{q}}(F,W_{p^n}), \quad \check{\Sigma}_{\mathfrak{q}}(F,T^*) := \varprojlim_n \Sigma_{\mathfrak{q}}(F,W^*_{p^n}).$$

If L/K is an infinite extension, we define

$$\mathrm{Sel}_?(L,M) = \varinjlim \mathrm{Sel}_?(L',M), \quad \Sigma_{\mathfrak{q}}(L,M) = \varinjlim \Sigma_{\mathfrak{q}}(L',M),$$
$$\check{\mathrm{Sel}}_?(L,T) = \varinjlim \check{\mathrm{Sel}}_?(L',T), \quad \check{\mathrm{Sel}}_?(L,T^*) = \varinjlim \check{\mathrm{Sel}}_?(L',T^*),$$

where the direct limits are taken with respect to restriction over all subfields $L' \subset L$ finite over K.

For any extension L/K, we set

$$\mathrm{Sel}_?(L,M)^\wedge = X_?(L,M), \qquad \Sigma_{\mathfrak{q}}(L,M)^\wedge = X_{\mathfrak{q}}(L,M).$$

□

Example 4.4. Suppose that $\phi = \psi^i \psi^{*j}$. By using the table given in Example 4.2, it is not hard to show that the $\mathbf{Z}_p$-coranks of the true Selmer groups $H^1_f(K,W)$ as i and j vary are equal to the $\mathbf{Z}_p$-coranks of the Selmer groups in the following table:

	$j<0$	$j=0$	$j>0$
$i<0$	$\mathrm{Sel}_{\mathrm{str}}(K,W)$	$\mathrm{Sel}_{\mathrm{str}}(K,W)$	$\Sigma_{\mathfrak{p}^*}(K,W)$
$i=0$	$\mathrm{Sel}_{\mathrm{str}}(K,W)$	$\mathrm{Sel}_{\mathrm{str}}(K,W)$	$\Sigma_{\mathfrak{p}^*}(K,W)$
$i>0$	$\Sigma_{\mathfrak{p}}(K,W)$	$\Sigma_{\mathfrak{p}}(K,W)$	$\mathrm{Sel}_{\mathrm{rel}}(K,W)$

The reader may find it helpful to draw a diagram of the $i-j$ plane to illustrate the table above. □

The following result is an analogue for restricted Selmer groups of a well known theorem of Coates about true Selmer groups associated to torsion points on CM elliptic curves [10, Theorem 12].

Theorem 4.5. *Let L be any field such that $\mathcal{F}_\infty \subseteq L \subseteq \mathfrak{F}_\infty$, and suppose that $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}^*\}$. Then there is an isomorphism*

$$X_{\mathfrak{q}}(L, W) \simeq \mathcal{X}^{(\mathfrak{q})}(L)(\phi^{-1})$$

of $\Lambda(L)$-modules. In particular, $X_{\mathfrak{q}}(L, W)$ is a torsion $\Lambda(L)$-module.

Proof. The proof of this result is very similar to that of [10, Theorem 12]. We begin by observing that , since $\mathcal{F}_\infty \subseteq L$, there are $\Lambda(L)$-module isomorphisms

$$\mathcal{X}^{(\mathfrak{q})}(L)(\phi^{-1}) \simeq \mathrm{Hom}(T, \mathcal{X}^{(\mathfrak{q})}(L)), \quad \mathcal{X}^{(\mathfrak{q})}(L)(\phi^{-1})^{\wedge} \simeq \mathrm{Hom}(\mathcal{X}^{(\mathfrak{q})}(L), W).$$

Therefore, in order to establish the desired result, it suffices to show that there is a natural isomorphism

$$\Sigma_{\mathfrak{q}}(L, W) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{X}^{(\mathfrak{q})}(L), W). \tag{4.3}$$

This follows via an argument entirely analogous to that used to establish [10, Theorem 12]. □

Let

$$\eta : G_K \to \mathbf{Z}_p^\times$$

be a character of infinite order, and write $\mathcal{K}_\infty^\eta$ for the extension of K cut out by η. Let K_∞^η denote the $\mathbf{Z}_p$-extension of K contained in $\mathcal{K}_\infty^\eta$. If F/K is any finite extension, we set

$$\mathcal{F}_\infty^\eta := F \cdot \mathcal{K}_\infty^\eta, \quad F_\infty^\eta := F \cdot K_\infty^\eta.$$

We now state a 'control theorem' for restricted Selmer groups.

Theorem 4.6. *Suppose that $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}^*\}$.*

(a) Let $I_{\mathcal{F}_\infty^\eta}$ denote the kernel of the quotient map

$$\Pi_{\mathcal{F}_\infty^\eta} : \Lambda(\mathfrak{F}_\infty) \to \Lambda(\mathcal{F}_\infty^\eta).$$

Then the kernel of the restriction map

$$\Sigma_{\mathfrak{q}}(\mathcal{F}_\infty^\eta, W) \to \Sigma_{\mathfrak{q}}(\mathfrak{F}_\infty, W)[I_{\mathcal{F}_\infty^\eta}]$$

is finite.

Let γ be a topological generator of $\mathrm{Gal}(\mathcal{F}^{\eta}_{\infty}/F)$*, and, for each place v of $\mathcal{F}^{\eta}_{\infty}$ lying above $\mathfrak{q}$, let γ_v be a topological generator of* $\mathrm{Gal}(\mathcal{F}^{\eta}_{\infty,v}/F_v) \leq \mathrm{Gal}(\mathcal{F}^{\eta}_{\infty}/F)$*. Let* $\widetilde{\gamma} \in \mathrm{Gal}(\mathfrak{F}_{\infty}/F)$ *and* $\widetilde{\gamma_v} \in \mathrm{Gal}(\mathfrak{F}_{\infty,v}/F_v)$ *be lifts of γ and γ_v respectively. Then acharacteristic power series in* $\Lambda(\mathcal{F}^{\eta}_{\infty})$ *of the Pontryagin dual of the cokernel of this restriction map is given by*

$$e_F = (\gamma - \phi^{-1}(\widetilde{\gamma}))^{-1} \prod_{v|\mathfrak{q}} (\gamma_v - \phi^{-1}(\widetilde{\gamma_v})),$$

(The action of e_F on the Pontryagin dual of the cokernel of the restiction map is independent of the choices of $\widetilde{\gamma}$ and $\widetilde{\gamma_v}$.)

Hence if $f \in \Lambda(\mathfrak{F}_{\infty})$ is a characteristic power series of $X_{\mathfrak{q}}(\mathfrak{F}_{\infty}, W)$, then $e_F^{-1}\Pi_{\mathcal{F}^{\eta}_{\infty}}(f) \in \Lambda(\mathcal{F}^{\eta}_{\infty})$ is a characteristic power series of $X_{\mathfrak{q}}(\mathcal{F}^{\eta}_{\infty}, W)$.

(b) Suppose that L is any field such that $F \subseteq L \subseteq \mathcal{F}^{\eta}_{\infty}$, and write I_L for the kernel of the quotient map $\Lambda(\mathcal{F}^{\eta}_{\infty}) \to \Lambda(L)$. Then the restriction map

$$\Sigma_{\mathfrak{q}}(L, W) \to \Sigma_{\mathfrak{q}}(\mathcal{F}^{\eta}_{\infty}, W)[I_L]$$

is an isomorphism.

Hence the dual of this restriction map is an isomorphism of $\Lambda(L)$-modules:

$$X_{\mathfrak{q}}(\mathcal{F}^{\eta}_{\infty}, W)/I_L X_{\mathfrak{q}}(\mathcal{F}^{\eta}_{\infty}, W) \xrightarrow{\sim} X_{\mathfrak{q}}(L, W).$$

Proof. Let $\mathcal{N}$ denote the maximal extension of $\mathfrak{F}_{\infty}$ that is unramified away from all places of $\mathfrak{F}_{\infty}$ lying above p. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \Sigma_{\mathfrak{q}}(\mathcal{F}^{\eta}_{\infty}, W) & \longrightarrow & H^1(\mathcal{N}/\mathcal{F}^{\eta}_{\infty}, W) & \xrightarrow{\mathrm{loc}_{\mathfrak{q}^*}} & \prod_{v|\mathfrak{q}^*} H^1(\mathcal{N}_v/\mathcal{F}^{\eta}_{\infty,v}, W) \\
& {\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \Sigma_{\mathfrak{q}}(\mathfrak{F}_{\infty}, W)[I_{\mathcal{F}^{\eta}_{\infty}}] & \longrightarrow & H^1(\mathcal{N}/\mathfrak{F}_{\infty}, W)[I_{\mathcal{F}^{\eta}_{\infty}}] & \xrightarrow{\mathrm{loc}_{\mathfrak{q}^*}} & \prod_{v|\mathfrak{q}^*} H^1(\mathcal{N}_v/\mathfrak{F}_{\infty,v}, W)
\end{array}$$

in which the vertical arrows are the obvious restriction maps.

Applying the Snake Lemma (together with the inflation-restriction exact sequence) to this diagram yields the exact sequence

$$\begin{aligned}
0 \to \mathrm{Ker}(\alpha) \to H^1(\mathfrak{F}_{\infty}/\mathcal{F}^{\eta}_{\infty}, W^{G_{\mathcal{F}^{\eta}_{\infty}}}) &\xrightarrow{g_1} \prod_{v|\mathfrak{q}^*} H^1(\mathfrak{F}_{\infty,v}/\mathcal{F}^{\eta}_{\infty,v}, W^{G_{\mathcal{F}^{\eta}_{\infty,v}}}) \to \\
\to \mathrm{Coker}(\alpha) \to H^2(\mathfrak{F}_{\infty}/\mathcal{F}^{\eta}_{\infty}, W^{G_{\mathcal{F}^{\eta}_{\infty}}}) &\xrightarrow{g_2} \prod_{v|\mathfrak{q}^*} H^2(\mathfrak{F}_{\infty,v}/\mathcal{F}^{\eta}_{\infty,v}, W^{G_{\mathcal{F}^{\eta}_{\infty,v}}}) \to 0. \qquad (4.4)
\end{aligned}$$

Now,

$$\begin{aligned} &H^1(\mathfrak{F}_\infty/\mathcal{F}^\eta_\infty, W^{G_{\mathcal{F}^\eta_\infty}}) \simeq \mathrm{Hom}(\mathrm{Gal}(\mathfrak{F}_\infty/\mathcal{F}^\eta_\infty), W^{G_{\mathcal{F}^\eta_\infty}}); \\ &\prod_{v|\mathfrak{q}^*} H^1(\mathfrak{F}_{\infty,v}/\mathcal{F}^\eta_{\infty,v}, W^{G_{\mathcal{F}^\eta_{\infty,v}}}) \simeq \prod_{v|\mathfrak{q}^*} \mathrm{Hom}(\mathrm{Gal}(\mathfrak{F}_{\infty,v}/\mathcal{F}^\eta_{\infty,v}), W^{G_{\mathcal{F}^\eta_{\infty,v}}}), \end{aligned} \tag{4.5}$$

and, as $\mathrm{Gal}(\mathfrak{F}_\infty/\mathcal{F}^\eta_\infty) \simeq \Delta \times \mathbf{Z}_p$ with $p \nmid \Delta$, we have

$$\begin{aligned} &H^2(\mathfrak{F}_\infty/\mathcal{F}^\eta_\infty, W^{G_{\mathcal{F}^\eta_\infty}}) \simeq H^0(\mathfrak{F}_\infty/\mathcal{F}^\eta_\infty, W^{G_{\mathcal{F}^\eta_\infty}}) \simeq W^{G_{\mathcal{F}^\eta_\infty}}; \\ &\prod_{v|\mathfrak{q}^*} H^2(\mathfrak{F}_{\infty,v}/\mathcal{F}^\eta_{\infty,v}, W^{G_{\mathcal{F}^\eta_{\infty,v}}}) \simeq \prod_{v|\mathfrak{q}^*} H^0(\mathfrak{F}_{\infty,v}/\mathcal{F}^\eta_{\infty,v}, W^{G_{\mathcal{F}^\eta_{\infty,v}}}) \simeq \prod_{v|\mathfrak{q}^*} W^{G_{\mathcal{F}^\eta_{\infty,v}}}. \end{aligned}$$

We now deduce that g_2 is injective, and that g_1 is non-zero, and therefore has finite kernel (since $H^1(\mathfrak{F}_\infty/\mathcal{F}^\eta_\infty, W^{G_{\mathcal{F}^\eta_\infty}})$ is either finite or divisible). It follows from (4.4) that $\mathrm{Ker}(\alpha)$ is finite, and that there is an exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to H^1(\mathfrak{F}_\infty/\mathcal{F}^\eta_\infty, W^{G_{\mathcal{F}^\eta_\infty}}) \xrightarrow{g_1} \prod_{v|\mathfrak{q}^*} H^1(\mathfrak{F}_{\infty,v}/\mathcal{F}^\eta_{\infty,v}, W^{G_{\mathcal{F}^\eta_{\infty,v}}}) \to \mathrm{Coker}(\alpha) \to 0. \tag{4.6}$$

It follows from (4.5) that

$$\begin{aligned} &\mathrm{Char}_{\Lambda(\mathcal{F}^\eta_\infty)} \left(H^1(\mathfrak{F}_\infty/\mathcal{F}^\eta_\infty, W^{G_{\mathcal{F}^\eta_\infty}})\right)^\wedge = \gamma - \phi^{-1}(\widetilde{\gamma}); \\ &\mathrm{Char}_{\Lambda(\mathcal{F}^\eta_\infty)} \left(\prod_{v|\mathfrak{q}^*} H^1(\mathfrak{F}_{\infty,v}/\mathcal{F}^\eta_{\infty,v}, W^{G_{\mathcal{F}^\eta_{\infty,v}}})\right)^\wedge = \prod_{v|\mathfrak{q}^*} (\gamma_v - \phi^{-1}(\widetilde{\gamma}_v)). \end{aligned}$$

Hence we deduce from (4.6) that

$$\mathrm{Char}_{\Lambda(\mathcal{F}^\eta_\infty)}(\mathrm{Coker}(\alpha))^\wedge = e_F = (\gamma - \phi^{-1}(\widetilde{\gamma}))^{-1} \prod_{v|\mathfrak{q}^*} (\gamma_v - \phi^{-1}(\widetilde{\gamma}_v)),$$

as asserted.

(b) In this case we consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma_{\mathfrak{q}}(L, W) & \longrightarrow & H^1(\mathcal{N}/L, W) & \xrightarrow{\mathrm{loc}_{\mathfrak{q}^*}} & \prod_{v|\mathfrak{q}^*} H^1(\mathcal{N}_v/L_v, W) \\ & & \beta_1 \big\downarrow & & \beta_2 \big\downarrow & & \beta_3 \big\downarrow \\ 0 & \longrightarrow & \Sigma_{\mathfrak{q}}(\mathcal{F}^\eta_\infty, W)[I_L] & \longrightarrow & H^1(\mathcal{N}/\mathcal{F}^\eta_\infty, W) & \xrightarrow{\mathrm{loc}_{\mathfrak{q}^*}} & \prod_{v|\mathfrak{q}^*} H^1(\mathcal{N}/\mathcal{F}^\eta_{\infty,v}, W) \end{array}$$

We have that

$$\operatorname{Ker}(\beta_2) = H^1(\mathcal{F}^{\eta}_{\infty}/L, W^{G_L}) = 0,$$

$$\operatorname{Ker}(\beta_3) = \prod_{v \mid \mathfrak{q}^*} H^1(\mathcal{F}^{\eta}_{\infty,v}/L_v, W^{G_{L_v}}) = 0,$$

$$\operatorname{Coker}(\beta_2) = H^2(\mathcal{F}_{\infty}/L, W^{G_L}) = 0,$$

(cf. [26, p. 40], for example), and so the Snake Lemma implies that β_1 is an isomorphism, as claimed. □

Corollary 4.7. *For any field L with $F \subseteq L \subseteq \mathcal{F}_{\infty}$, we have an isomorphism*

$$X_{\mathfrak{q}}(L, T) \simeq \mathcal{X}_{\mathfrak{q}}(\mathcal{F}_{\infty})(\phi^{-1})/I_L(\mathcal{X}_{\mathfrak{q}}(\mathcal{F}_{\infty})(\phi^{-1})) \tag{4.7}$$

of $\Lambda(L)$-modules.

Proof. This follows directly from Proposition 4.6 and Theorem 4.5, where we take $\eta = \phi$. □

Remark 4.8. , If we take $F = K$ in Proposition 4.6, then

$$e_K = (\gamma - \phi^{-1}(\widetilde{\gamma}))^{-1} \cdot (\gamma_{\mathfrak{q}} - \phi^{-1}(\widetilde{\gamma}_{\mathfrak{q}})),$$

and so $e_K \in \Lambda(\mathcal{K}^{\eta}_{\infty})^{\times}$. We therefore see from Proposition 4.6(a) and Corollary 4.7 that the element $H^{(K)}_{\mathfrak{q},\eta} \in \Lambda(K^{\eta}_{\infty})$ fixed in Section 3 is a characteristic power series of $X_{\mathfrak{q}}(K^{\eta}_{\infty}, W)$. Let us also remark that as $\mathcal{L}_{\mathfrak{q}} \in \Lambda(\mathfrak{K}_{\infty})_{\mathcal{O}}$ is a characteristic power series of $\mathcal{X}_{\mathfrak{q}}(\mathfrak{K}_{\infty})$, it follows that $\operatorname{Tw}_{\phi}(\mathcal{L}_{\mathfrak{q}})$ is a characteristic power series of $X_{\mathfrak{q}}(\mathfrak{K}_{\infty}, W)$. □

5. The p-adic height pairing

Let F/K be a finite extension. Suppose that $\rho : G_F \to \mathbf{Z}_p$ is a non-trivial character (necessarily of infinite order), ans assume that $\{\phi, \phi^*\} \neq \{\mathbf{1}, \chi_{\mathrm{cyc}}\}$.. In this section we shall use the methods of [24], [26] and [3] to construct a p-adic height pairing

$$[-,-]^{(\rho)}_{F,\mathfrak{q}} : \Sigma_{\mathfrak{q}^*}(F, T^*) \times \Sigma_{\mathfrak{q}}(F, T) \to \mathbf{Z}_p,$$

and we shall describe some of its properties.

Let S be any finite set of finite places of F containing all places dividing p and all places of bad reduction for E, and write $\mathcal{N}$ for the maximal extension of F unramified away from S. Set $G_{F,S} := \mathrm{Gal}(\mathcal{N}/F)$.

We shall require the following result.

Proposition 5.1. *There is an isomorphism of* $\mathrm{Gal}(\mathcal{F}_n^*/F)$*-modules*

$$H^1(\mathcal{F}_n^*, W_{p^n}) \xrightarrow{\sim} \mathrm{Hom}(W_{p^n}^*, \mathcal{F}_n^{*\times}/\mathcal{F}_n^{*\times p^n}); \quad f \mapsto \tilde{f}. \tag{5.1}$$

For each place v of $\mathcal{F}_n^*$, there is also a corresponding local isomorphism

$$H^1(\mathcal{F}_{n,v}^*, W_{p^n}) \xrightarrow{\sim} \mathrm{Hom}(W_{p^n}^*, \mathcal{F}_{n,v}^{*\times}/\mathcal{F}_{n,v}^{*\times p^n}).$$

Proof. We first observe that for each integer $n \geq 1$, $\mathrm{Gal}(\overline{F}/\mathcal{F}_n)$ acts trivially on W_{p^n}. This implies that the map

$$\mu_{p^n} \to W_{p^n}; \quad \zeta_n \mapsto w_n$$

induces an isomorphism

$$\mathrm{Tw}_{\phi^{*-1}}^{(n)} : H^1(\mathcal{F}_n^*, \mu_{p^n}) \xrightarrow{\sim} H^1(\mathcal{F}_n^*, W_{p^n}).$$

The isomorphism (5.1) is now defined as follows. We first identify $\mathcal{F}_n^{*\times}/\mathcal{F}_n^{*\times p^n}$ with $H^1(\mathcal{F}^*, \mu_{p^n})$ via Kummer theory, and then we define $\tilde{f}$ by setting

$$\tilde{f}(w_n^*) = \mathrm{Tw}_{\phi^{*-1}}^{(n)}(f).$$

It is not hard to check that the map $f \mapsto \tilde{f}$ is a $\mathrm{Gal}(\mathcal{F}_n^*/F)$-isomorphism. □

We apply Proposition 5.1 in the following way. Suppose that $h \in \Sigma_{\mathfrak{q}^*}(\mathcal{F}_n, W_{p^n}^*)$. Then it follows from the local conditions defining $\Sigma_{\mathfrak{q}^*}(\mathcal{F}_n, W_{p^n}^*)$ that, for each finite place v of F, we have:

(a) $\tilde{h}(u) \in \mathcal{F}_{n,v}^{\times p^n}$ for all $v \mid \mathfrak{q}$;

(b) $p^n \mid v_{\mathcal{F}_n}(\tilde{h}(u))$ for all $v \nmid \mathfrak{q}^*$.

(There are no local conditions imposed at places lying above $\mathfrak{q}^*$.)

For any algebraic extension L/F, write $J(L)$ for the group of finite ideles of L. Let $V_{\mathfrak{q}}(L)$ denote the subgroup of $J(L)$ consisting of those elements of $J(L)$ whose components are equal to 1 at all places dividing $\mathfrak{q}$, and are units at all places not dividing $\mathfrak{q}^*$. We set

$$C_{n,\mathfrak{q}} := J(\mathcal{F}_n)/(V_{\mathfrak{q}}(\mathcal{F}_n)\cdot \mathcal{F}_n^{\times}),$$
$$\mathcal{C}_{\mathfrak{q}} := J(\mathcal{N})/(V_{\mathfrak{q}}(\mathcal{N})\cdot \mathcal{N}^{\times}),$$
$$\Omega_{n,\mathfrak{q}} := \prod_{v\mid \mathfrak{q}} \mu_{p^n}(\mathcal{F}_{n,v}),$$

and we note that the order of $\Omega_{n,\mathfrak{q}}$ remains bounded as n varies (because $\{\phi,\phi^*\}\neq\{\mathbf{1},\chi_{\mathrm{cyc}}\}$). Let p^c denote the exponent of $\cup_n \Omega_{n,\mathfrak{q}}$.

Proposition 5.2. *There is an exact sequence*

$$\mathrm{Hom}(W_{p^n},\Omega_{n,\mathfrak{q}})^{G_{F,S}} \to \mathrm{Hom}(W_{p^n},C_{n,\mathfrak{q}})^{G_{F,S}} \xrightarrow{\eta_n} \Sigma_{\mathfrak{q}^*}(F,W^*_{p^n}) \to 0. \tag{5.2}$$

Proof. The proof of this Proposition is identical, *mutatis mutandis*, to that of [24, Proposition 3.13]. □

We construct $[-,-]^{(\rho)}_{F,\mathfrak{q}}$ by first defining pairings

$$[-,-]^{(\rho)}_{F,\mathfrak{q},n} : \Sigma_{\mathfrak{q}^*}(F,W^*_{p^n})\times \Sigma_{\mathfrak{q}}(F,W_{p^n}) \to \mathbf{Z}/p^n\mathbf{Z}$$

'at level n', and then passing to appropriate limits.

Suppose that $x=(x_n)\in \Sigma_{\mathfrak{q}^*}(F,W^*_{p^n})$, $y=(y_n)\in\Sigma_{\mathfrak{q}}(F,W_{p^n})$, and write $\rho=(\rho_n)\in H^1(G_{F,S},\mathbf{Z}_p)$.

For each integer $n\geq 1$, (5.2) implies that we may choose an element

$$\widetilde{x}_n \in \mathrm{Hom}(W_{p^n},C_{n,\mathfrak{q}})^{G_{F,S}} = H^0(G_{F,S},\mathrm{Hom}(W_{p^n},C_{n,\mathfrak{q}})^{G_{F,S}})$$

such that $\eta_n(\widetilde{x}_n)=x_n$. It follows from (5.2) that

$$\overline{x}'_n := p^c\cdot \widetilde{x}_n$$

is well-defined. We write $\overline{x}_n$ for the image of $\overline{x}'_n$ in $H^0(G_{F,S},\mathrm{Hom}(W_{p^n},\mathcal{C}_{\mathfrak{q}})^{G_{F,S}})$ via the natural map $C_{n,\mathfrak{q}}\to\mathcal{C}_{\mathfrak{q}}$.

We now see that

$$z_n := \overline{x}_n\cup y_n\cup\rho_n \in H^2(G_{F,S},\mathcal{C}_{\mathfrak{q}})[p^n],$$

and we define

$$[x_n, y_n]^{(\rho)}_{F,\mathfrak{q},n} := \sum_v \mathrm{inv}_v(\mathrm{loc}_v(z_n)) \in \mathbf{Z}/p^n\mathbf{Z},$$

where inv_v denotes the local invariant map at v arising via class field theory.

Next, we observe that the pairing $[x_n, y_n]^{(\rho)}_{F,\mathfrak{q},n}$ induces a homomorphism

$$\Sigma_{\mathfrak{q}^*}(F, W_{p^n}) \to X_{\mathfrak{q}}(F, W_{p^n}).$$

Passing to inverse limits on the left and direct limits on the right yields a homomorphism

$$\xi^{(\rho)}_F : \Sigma_{\mathfrak{q}^*}(F, T^*) \to X_{\mathfrak{q}}(F, W).$$

Composing $\xi^{(\rho)}_F$ with the sequence of surjective maps

$$X_{\mathfrak{q}}(F, W) \to [\Sigma_{\mathfrak{q}}(F, W)_{\mathrm{div}}]^{\wedge} \to \mathrm{Hom}(\Sigma_{\mathfrak{q}}(F, T), \mathbf{Z}_p)$$

yields a homomorphism

$$\Xi^{(\rho)}_F : \Sigma_{\mathfrak{q}^*}(F, T^*) \to \mathrm{Hom}(\Sigma_{\mathfrak{q}}(F, T), \mathbf{Z}_p).$$

We define

$$[x, y]^{(\rho)}_{F,\mathfrak{q}} = \Xi^{(\rho)}_F(x)(y).$$

Finally, we set

$$[x, y]^{(\rho)}_{F,\mathfrak{q}} := \varinjlim_n [x_n, y_n]^{(\rho)}_{F,\mathfrak{q},n} \in \mathbf{Z}_p.$$

Remark 5.3. (a) We remark that it follows from the definitions that

$$[x, y]^{(\rho)}_{F,\mathfrak{q}} = \Xi^{(\rho)}_F(x)(y).$$

(b) It is shown in [3] that the pairing $[x, y]^{(\rho)}_{F,\mathfrak{q}}$ agrees with the pairing defined in [25] (and therefore also with the pairing defined in [22]) when both pairings are defined.

Let us expand upon this last comment. Write F^{ρ}_{∞} for the $\mathbf{Z}_p$-extension of F cut out by ρ. Set $\Gamma := \mathrm{Gal}(F^{\rho}_{\infty}/F)$, and $\Lambda := \mathbf{Z}_p[[\Gamma]]$. In [26], it is shown that if $X_{\mathfrak{q}}(F^{\rho}_{\infty}, W)$ is a Λ-torsion module, and if $H^2(\mathrm{Gal}(\mathcal{N}/F^{\rho}_{\infty}), W) = 0$ (this is a weak p-adic Leopoldt hypothesis), then there is a canonical isomorphism

$$\Phi^{(\rho)}_F : \Sigma_{\mathfrak{q}^*}(F, T^*) \to X_{\mathfrak{q}}(F^{\rho}_{\infty}, W)^{\Gamma}. \tag{5.3}$$

Composing $\Phi_F^{(\rho)}$ with the natural map

$$X_{\mathfrak{q}}(F_\infty^\rho, W)^\Gamma \to X_{\mathfrak{q}}(F_\infty^\rho, W)$$

yields a homomorphism

$$\overline{\xi}_F^{(\rho)} : \Sigma_{\mathfrak{q}^*}(F, T^*) \to X_{\mathfrak{q}}(F, W).$$

It is shown in [3] that

$$\xi_F^{(\rho)} = \overline{\xi}_F^{(\rho)}, \tag{5.4}$$

which in turn implies that the pairing $[x, y]_{F,\mathfrak{q}}^{(\rho)}$ agrees with the p-adic height pairing defined in [25]. □

Definition 5.4. If $x_1, \ldots, x_m$ and $x_1^*, \ldots, x_m^*$ are $\mathbf{Z}_p$-bases modulo torsion of $\Sigma_{\mathfrak{q}}(F, T)$ and $\Sigma_{\mathfrak{q}^*}(F, T^*)$ respectively, then we define the regulator $\mathcal{R}_{F,\mathfrak{q}}^{(\rho)}$ associated to $[-,-]_{F,\mathfrak{q}}^{(\rho)}$ by

$$\mathcal{R}_{F,\mathfrak{q}}^{(\rho)} := \det([x_i^*, x_j]_{F,\mathfrak{q}}^{(\rho)}).$$

□

Definition 5.5. We write

$$[-,-]_{F,\mathrm{str}}^{(\rho)} : \mathrm{Sel}_{\mathrm{str}}(F, T^*) \times \mathrm{Sel}_{\mathrm{str}}(F, T) \to \mathbf{Z}_p$$

for the restriction of the pairing $[-,-]_{F,\mathfrak{q}}^{\rho}$ to strict Selmer groups, and we let $\mathcal{R}_{F,\mathrm{str}}^{(\rho)}$ denote the regulator of this pairing with respect to some choice of $\mathbf{Z}_p$- bases of $\mathrm{Sel}_{\mathrm{str}}(F, T^*)$ and $\mathrm{Sel}_{\mathrm{str}}(F, T)$ modulo torsion. (So $\mathcal{R}_{F,\mathrm{str}}^{(\rho)}$ is well-defined up to multiplication by an element of $\mathbf{Z}_p^\times$.)

It is conjectured that, for some choice of ρ, the pairing $[-,-]_{F,\mathrm{str}}^{\rho}$ is non-degenerate modulo torsion (see [27, Section 3.1.4]), and so in particular $\mathrm{Sel}_{\mathrm{str}}(F, T^*)$ and $\mathrm{Sel}_{\mathrm{str}}(F, T)$ always have the same $\mathbf{Z}_p$-rank. □

We conclude this section by recalling the statement of a certain formula for the p-adic height due to K. Rubin.

Take $F = K$, and let γ be a topological generator of $\mathrm{Gal}(K_\infty^\rho/K)$. Suppose that $z = (z_n) \in H^1_{\mathrm{Iw}}(K_\infty^\rho, T^*)$ with $z_0 \in \Sigma_{\mathfrak{q}^*}(K, T^*)$. Then $\mathrm{loc}_{\mathfrak{q}}(z_0) = 0.$ and so

$$\mathrm{loc}_{\mathfrak{q}}(z) \in (\gamma - 1) \cdot H^1_{\mathrm{Iw}}(K_{\mathfrak{q},\infty}^\rho, T^*).$$

Choose $\alpha = (\alpha_n) \in H^1_{\mathrm{Iw}}(K^{\rho}_{\mathfrak{q},\infty}, T^*)$ such that

$$\log_p(\rho(\gamma)) \cdot \mathrm{loc}_{\mathfrak{q}}(z) = (\gamma - 1) \cdot \alpha.$$

(Note that our hypotheses imply that $\alpha_0 \in H^1(K_{\mathfrak{q}}, T^*)$ is independent of the choice of γ.)

Rubin's formula may be stated as follows.

Theorem 5.6. *(Rubin) Suppose that $y \in \Sigma_{\mathfrak{q}}(K, T)$. Then*

$$[z, y]^{(\rho)}_{K,\mathfrak{q}} = \mathrm{inv}_{\mathfrak{q}}(\alpha_0 \cup \mathrm{loc}_{\mathfrak{q}}(y)).$$

Proof. See [35, Theorem 3.2] for a proof of the formula for Tate modules of abelian varieties with good, ordinary reduction at p. (For a proof of the formula in a setting–viz. that of CM forms of heigher weight–closer to that considered here, see [5, Sections 4.2 and 4.3]. See also [4, Theorem 3.1.2 and Section 3.2].) □

6. A leading term formula

We retain the notation of the previous sections. Recall that F^{ρ}_{∞}/F denotes the $\mathbf{Z}_p$-extension cut out by a character $\rho : G_F \to \mathbf{Z}_p$. Set $\Gamma := \mathrm{Gal}(F^{\rho}_{\infty})/F$, and fix a topological generator γ of Γ. We may identify $\Lambda(F^{\rho}_{\infty})$ with the power series ring $\mathbf{Z}_p[[X]]$ via the map $\gamma \mapsto X + 1$, and we let $H^{(\rho)}_{\mathfrak{q},\phi} \in \Lambda(F^{\rho}_{\infty})$ be a characteristic power series of $X_{\mathfrak{q}}(F^{\rho}_{\infty}, W)$. Recall that S is a finite set of places of F containing all places above p, and that $\mathcal{N}$ is the maximal extension of F that is unramified outside S.

In this section, we shall, under certain conditions, calculate the p-adic valuation of the leading coefficient of $H^{(\rho)}_{\mathfrak{q},\phi}$.

Throughout this section, we make the following assumptions:

Assumption 6.1. (a) The $\Lambda(F^{\rho}_{\infty})$-module $X(F^{\rho}_{\infty}, W)$ is torsion.

(b) The $\Lambda(F^{\rho}_{\infty})$-module $X(F^{\rho}_{\infty}, W)$ has no finite submodules.

(c) $H^2(\mathrm{Gal}(\mathcal{N}/F^{\rho}_{\infty}), W) = 0$. (This is a weak p-adic Leopoldt hypothesis.)

(d) $\mathcal{R}^{(\rho)}_{F,\mathfrak{q}} \neq 0$, i.e. the p-adic height pairing

$$[-,-]^{(\rho)}_{F,\mathfrak{q}} : \Sigma_{\mathfrak{q}^*}(F, T^*) \times \Sigma_{\mathfrak{q}}(F, T) \to \mathbf{Z}_p$$

is non-degenerate. □

Remark 6.2. Assumption 6.1(b) is known in many cases due to work of R. Greenberg (see e.g. [14], [18]). □

Set

$$n_{\mathfrak{q}}(\phi) = n_{\mathfrak{q}}^{(F)}(\phi) := \mathrm{rk}_{\mathbf{Z}_p}(\Sigma_{\mathfrak{q}}(F,T)).$$

Theorem 6.3. *Under Assumptions 6.1 (a)–(d), we have that*

$$\mathrm{ord}_{X=0}\, H_{\mathfrak{q},\phi}^{(\rho)}(X) = n_{\mathfrak{q}}(\phi),$$

and

$$\left.\frac{H_{\mathfrak{q},\phi}^{(\rho)}(X)}{X_{\mathfrak{q}}^{n_{\mathfrak{q}}(\phi)}}\right|_{X=0} \sim |\Sigma_{\mathfrak{q}}(F,W)/\Sigma_{\mathfrak{q}}(F,W)_{\mathrm{div}}| \cdot |\check{\Sigma}_{\mathfrak{q}}(F,T)_{\mathrm{tors}}| \cdot \mathcal{R}_{F,\mathfrak{q}}^{(\rho)}.$$

Proof. We first observe that there is a surjective homomorphism

$$X_{\mathfrak{q}}(F_{\infty}^{\rho},W) \to [\Sigma_{\mathfrak{q}}(F,W)_{\mathrm{div}}]^{\wedge};$$

this implies that $H_{\mathfrak{q},\phi}^{(\rho)}$ is divisible by $X^{n_{\mathfrak{q}}(\phi)}$. Let Z_∞ denote the kernel of this map. Then the Snake Lemma yields the exact sequence

$$0 \to (Z_\infty)^{\Gamma} \to X_{\mathfrak{q}}(F_{\infty}^{\rho},W)^{\Gamma} \xrightarrow{f} [\Sigma_{\mathfrak{q}}(F,W)_{\mathrm{div}}]^{\wedge} \to$$
$$\to (Z_\infty)_{\Gamma} \to X_{\mathfrak{q}}(F_{\infty}^{\rho},W)_{\Gamma} \to [\Sigma_{\mathfrak{q}}(F,W)_{\mathrm{div}}]^{\wedge} \to 0.$$

We now observe that the kernel of the last map

$$X_{\mathfrak{q}}(F_{\infty}^{\rho},W)_{\Gamma} \to [\Sigma_{\mathfrak{q}}(F,W)_{\mathrm{div}}]^{\wedge}$$

is dual to the cokernel of the map

$$\Sigma_{\mathfrak{q}}(F,W)_{\mathrm{div}} \to \Sigma_{\mathfrak{q}}(F_{\infty}^{\rho},W)^{\Gamma}.$$

Since $\Sigma_{\mathfrak{q}}(F,W) \simeq \Sigma_{\mathfrak{q}}(F_{\infty}^{\rho},W)^{\Gamma}$ (via Theorem 4.6(b)), it follows that this cokernel is isomorphic to $\Sigma_{\mathfrak{q}}(F,W)/\Sigma_{\mathfrak{q}}(F,W)_{\mathrm{div}}$, which is finite because $\Sigma_{\mathfrak{q}}(F,W)$ is cofinitely generated.

We therefore deduce that the multiplicity of X in $H_{\mathfrak{q},\phi}^{(\rho)}$ is equal to $n_{\mathfrak{q}}(\phi)$ if and only if $(Z_\infty)_{\Gamma}$ is finite, which in turn is the case if and only if the cokernel of f is finite.

Consider the following diagram in which the left-hand square commutes (see Remark 5.3):

$$\begin{array}{ccccc} X_{\mathfrak{q}}(F_\infty^\rho, W)^\Gamma & \xrightarrow{f_1} & X_{\mathfrak{q}}(F,W) & \xrightarrow{f_2} & [\Sigma_{\mathfrak{q}}(F,W)_{\mathrm{div}}]^\wedge \\ \uparrow \Phi_F^{(\rho)} & & \uparrow \xi_F^{(\rho)} & & \downarrow \wr \\ \Sigma_{\mathfrak{q}^*}(F,T^*) & \xrightarrow{=} & \Sigma_{\mathfrak{q}^*}(F,T^*) & & \mathrm{Hom}(\Sigma_{\mathfrak{q}}(F,T), \mathbf{Z}_p) \end{array}$$

By definition, $f = f_2 \circ f_1$. Assumptions 6.1(a) and 6.1(c) imply that the map $\Phi_F^{(\rho)}$ is an isomorphism, and so $\xi_F^{(\rho)}$ is surjective with finite kernel (see Remark 5.3). We now deduce that the cokernel of f is finite if and only if the p-adic height pairing $[-,-]_{F,\mathfrak{q}}^{(\rho)}$ is non-degenerate.

We now see that if $[-,-]_{F,\mathfrak{q}}^{(\rho)}$ is non-degenerate, then $(Z_\infty)_\Gamma$ is finite. This implies that $(Z_\infty)^\Gamma$ is also finite, whence it follows from Assumption 6.1(b) that $(Z_\infty)^\Gamma = 0$. Hence we have

$$\left.\frac{H_{\mathfrak{q},\phi}^{(\rho)}(X)}{X^{n_{\mathfrak{q}}(\phi)}}\right|_{X=0} \sim |(Z_\infty)_\Gamma| \sim |\Sigma_{\mathfrak{q}}(F,W)/\Sigma_{\mathfrak{q}}(F,W)_{\mathrm{div}}| \cdot |\operatorname{Coker}(f)|.$$

Now

$$\begin{aligned} |\operatorname{Coker}(f)| &= [(\Sigma_{\mathfrak{q}}(F,W)_{\mathrm{div}})^\wedge : f(X_{\mathfrak{q}}(F_\infty^\rho, W)^{\Gamma_F})] \\ &= [T_p(\Sigma_{\mathfrak{q}}(F,W)) : \xi_F^{(\rho)}(\check{\Sigma}_{\mathfrak{q}^*}(F,T^*))] \\ &= \mathcal{R}_{F,\mathfrak{q}}^{(\rho)} \cdot \left| \left[\operatorname{Ker}(\check{\Sigma}_{\mathfrak{q}}(F,T) \to T_p(\Sigma_{\mathfrak{q}}(F,W)))\right] \right| \\ &= \mathcal{R}_{F,\mathfrak{q}}^{(\rho)} \cdot |\check{\Sigma}_{\mathfrak{q}}(F,T)_{\mathrm{tors}}|. \end{aligned}$$

Hence

$$\left.\frac{H_{\mathfrak{q},\phi}^{(\rho)}(X)}{X^{n_{\mathfrak{q}}(\phi)}}\right|_{X=0} \sim |\Sigma_{\mathfrak{q}}(F,W)/\Sigma_{\mathfrak{q}}(F,W)_{/\,\mathrm{div}}| \cdot |\check{\Sigma}_{\mathfrak{q}}(F,T)_{\mathrm{tors}}| \cdot \mathcal{R}_{F,\mathfrak{q}^*}^{(\rho)},$$

as claimed. □

Corollary 6.4. *Suppose further that $F = K$. We have that*

$$\operatorname{ord}_{s=1} L_{\mathfrak{q}}(\phi;\rho,s) \geq n_{\mathfrak{q}}(\phi),$$

with equality if and only if $\mathcal{R}_{K,\mathfrak{q}}^{(\rho)} \neq 0$. If $\mathcal{R}_{K,\mathfrak{q}^*}^{(\rho)} \neq 0$, then

$$\mathcal{L}_{\mathfrak{q}}^{(n_{\mathfrak{q}}(\phi))}(\phi;\rho) \sim |\Sigma_{\mathfrak{q}}(K,W)_{/\,\mathrm{div}}| \cdot |\check{\Sigma}_{\mathfrak{q}}(K,T)_{\mathrm{tors}}| \cdot \mathcal{R}_{K,\mathfrak{q}}^{(\rho)},$$

Proof. This is a direct consequence of Theorem 6.3 and its proof (see also Remark 4.8). □

7. Restricted Selmer groups over K

In this section we shall use Poitou-Tate duality to study the relationships between the ranks of different Selmer groups over K. Throughout this section, we take $F = K$, and we assume that $\phi \notin \{\chi_{\mathrm{cyc}}, \mathbf{1}\}$. In accord with our earlier notation, we write

$$n_{\mathfrak{q}}(\phi) = n_{\mathfrak{q}}^{(K)}(\phi) := \mathrm{rk}_{\mathbf{Z}_p}(\Sigma_{\mathfrak{q}}(K,T)). \tag{7.1}$$

We set

$$n_{\mathrm{str}}(\phi) := \mathrm{rk}_{\mathbf{Z}_p}(\mathrm{Sel}_{\mathrm{str}}(K,T)), \quad n_{\mathrm{rel}}(\phi) := \mathrm{rk}_{\mathbf{Z}_p}(\mathrm{Sel}_{\mathrm{rel}}(K,T)),$$

and we let $r_{\mathfrak{q}}(\phi)$ denote the $\mathbf{Z}_p$-rank of the image of the localisation map

$$\mathrm{loc}_{\mathfrak{q}} : \Sigma_{\mathfrak{q}}(K,T) \to H^1(K_{\mathfrak{q}}, T).$$

Equivalently, $r_{\mathfrak{q}}(\phi)$ is equal to the $\mathbf{Z}_p$-rank of the Pontrygin dual of the image of the localisation map

$$\mathrm{loc}_{\mathfrak{q}} : \Sigma_{\mathfrak{q}}(K,W) \to H^1(K_{\mathfrak{q}}, W).$$

Lemma 7.1. *We have that* $\mathrm{rk}_{\mathbf{Z}_p}(H^1(K_{\mathfrak{q}},T)) = 1$. *Hence* $r_{\mathfrak{q}}(\phi)$ *is equal to* 0 *or* 1.

Proof. The first assertion follows from [16, Proposition 1, p.109], while the second assertion is an immediate consequence of the first. □

Remark 7.2. Note that Lemma 7.1 is false if $\phi \in \{\chi_{\mathrm{cyc}}, \mathbf{1}\}$. □

The following result will play a critical role in much of our subsequent work.

Theorem 7.3. *(a) Suppose that* $r_{\mathfrak{q}}(\phi) = 1$. *Then* $r_{\mathfrak{q}}(\phi^*) = 0$, *and we have*

$$\Sigma_{\mathfrak{q}}(K,W^*)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{str}}(K,W^*)_{\mathrm{div}}.$$

(b) Suppose that

$$r_{\mathfrak{q}}(\phi) = r_{\mathfrak{q}^*}(\phi^*) = 1.$$

Then we have

$$\Sigma_{\mathfrak{q}}(K,W)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{rel}}(K,W)_{\mathrm{div}};$$

$$\Sigma_{\mathfrak{q}^*}(K,W)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{str}}(K,W)_{\mathrm{div}}.$$

In particular

$$n_{\mathfrak{q}}(\phi) = n_{\mathfrak{q}^*}(\phi) + 1.$$

(c) Suppose that $n_{\mathfrak{q}}(\phi) = n_{\mathfrak{q}^*}(\phi^*)$. Suppose also that $n_{\mathrm{str}}(\phi) = n_{\mathrm{str}}(\phi^*)$ (recall that this is conjecturally always true—see Definition 5.5). Then

$$r_{\mathfrak{q}}(\phi) = r_{\mathfrak{q}^*}(\phi^*).$$

Proof. (a) We first observe that the Poitou-Tate exact sequence yields

$$0 \to \Sigma_{\mathfrak{q}^*}(K,T) \to \mathrm{Sel}_{\mathrm{rel}}(K,T) \xrightarrow{\alpha} H^1(K_{\mathfrak{q}},T) \to \Sigma_{\mathfrak{q}}(K,W^*)^{\wedge}. \tag{7.2}$$

The cokernel of α is equal to the Pontryagin dual of the image of the localisation map

$$\mathrm{loc}_{\mathfrak{q}} : \Sigma_{\mathfrak{q}}(K,W^*) \to H^1(K_{\mathfrak{q}},W^*),$$

and so has $\mathbf{Z}_p$-rank $r_{\mathfrak{q}}(\phi^*)$. Hence

$$\mathrm{rk}_{\mathbf{Z}_p}(\mathrm{Im}(\alpha)) = 1 - r_{\mathfrak{q}}(\phi^*).$$

Since $r_{\mathfrak{q}}(\phi) = 1$, it follows that $\mathrm{rk}_{\mathbf{Z}_p}(\mathrm{Im}(\alpha)) = 1$, and so $r_{\mathfrak{q}}(\phi^*) = 0$, as claimed.

As $\mathrm{rk}_{\mathbf{Z}_p}(\mathrm{Im}(\alpha)) = 1$, we now see from the sequence

$$0 \to \mathrm{Sel}_{\mathrm{str}}(K,T^*) \to \Sigma_{\mathfrak{q}^*}(K,T^*) \xrightarrow{\mathrm{loc}_{\mathfrak{q}^*}} H^1(K_{\mathfrak{q}^*},T^*)$$

that

$$\mathrm{rk}_{\mathbf{Z}_p}(\mathrm{Sel}_{\mathrm{str}}(K,T^*)) = \mathrm{rk}_{\mathbf{Z}_p}(\Sigma_{\mathfrak{q}^*}(K,T^*)),$$

and this implies that

$$\Sigma_{\mathfrak{q}}(K,W^*)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{str}}(K,W^*)_{\mathrm{div}},$$

as claimed.

(b) Since $r_{\mathfrak{q}^*}(\phi^*) = 1$, part (a) above (with ϕ replaced by ϕ^* and $\mathfrak{q}$ replaced by $\mathfrak{q}^*$) implies that $r_{\mathfrak{q}^*}(\phi) = 0$ and $\Sigma_{\mathfrak{q}^*}(K, W)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{str}}(K, W)_{\mathrm{div}}$.

It now follows from the following diagram (in which the first two vertical arrows are inclusion maps, and the third is the identity):

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma_{\mathfrak{q}^*}(K,T) & \longrightarrow & \mathrm{Sel}_{\mathrm{rel}}(K,T) & \xrightarrow{\mathrm{loc}_{\mathfrak{q}}} & H^1(K_{\mathfrak{q}},T) \\
 & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow \\
0 & \longrightarrow & \mathrm{Sel}_{\mathrm{str}}(K,T) & \longrightarrow & \Sigma_{\mathfrak{q}}(K,T) & \xrightarrow{\mathrm{loc}_{\mathfrak{q}}} & H^1(K_{\mathfrak{q}},T)
\end{array}$$

that

$$\mathrm{rk}_{\mathbf{Z}_p}(\mathrm{Sel}_{\mathrm{rel}}(K,T)) = \mathrm{rk}_{\mathbf{Z}_p}(\Sigma_{\mathfrak{q}}(K,T).$$

This implies that $\Sigma_{\mathfrak{q}}(K, W)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{rel}}(K, W)_{\mathrm{div}}$, as claimed.

(c) We observe that it follows from the definitions of $r_{\mathfrak{q}}(\phi)$ and $r_{\mathfrak{q}^*}(\phi^*)$ that we have

$$n_{\mathfrak{q}}(\phi) = n_{\mathrm{str}}(\phi) + r_{\mathfrak{q}}(\phi), \quad n_{\mathfrak{q}^*}(\phi^*) = n_{\mathrm{str}}(\phi^*) + r_{\mathfrak{q}^*}(\phi^*).$$

This implies the desired result.

□

Proposition 7.4. *Suppose that*

$$n_{\mathfrak{q}}(\phi) = n_{\mathfrak{q}^*}(\phi^*), \quad n_{\mathrm{str}}(\phi) = n_{\mathrm{str}}(\phi^*).$$

Then:

$$\Sigma_{\mathfrak{q}}(K, W)_{\mathrm{div}} = \begin{cases} \mathrm{Sel}_{\mathrm{str}}(K, W)_{\mathrm{div}} & \text{if } r_{\mathfrak{q}} = 0; \\ \mathrm{Sel}_{\mathrm{rel}}(K, W)_{\mathrm{div}} & \text{if } r_{\mathfrak{q}} = 1. \end{cases}$$

Proof. This follows directly from Theorem 7.3. □

To continue our analysis, let us suppose that

$$n_{\mathfrak{q}}(\phi) = n_{\mathfrak{q}^*}(\phi^*)$$

and that

$$r_{\mathfrak{q}}(\phi) = r_{\mathfrak{q}^*}(\phi^*) = 1.$$

Then we have $r_{\mathfrak{q}^*}(\phi) = 0$ (see Theorem 7.3, and so (see Propostition 7.4,

$$\Sigma_{\mathfrak{q}}(K,W)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{rel}}(K,W)_{\mathrm{div}}, \quad \Sigma_{\mathfrak{q}^*}(K,W)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{str}}(K,W)_{\mathrm{div}}.$$

Proposition 7.5. *Suppose that $n_{\mathfrak{q}}(\phi) = n_{\mathfrak{q}^*} = 1$ and that $r_{\mathfrak{q}}(\phi) = r_{\mathfrak{q}^*}(\phi^*) = 1$.*

(a) We have that

$$\frac{|\,\mathrm{Sel}_{\mathrm{rel}}(K,W)/_{\mathrm{div}}|}{|\Sigma_{\mathfrak{q}}(K,W)/_{\mathrm{div}}|} = [H^1(K_{\mathfrak{q}^*},T^*) : \mathrm{loc}_{\mathfrak{q}^*}(\Sigma_{\mathfrak{q}^*}(K,T^*))]. \tag{7.3}$$

(b) We have that

$$\frac{|\,\mathrm{Sel}_{\mathrm{rel}}(K,W)/_{\mathrm{div}}|}{|\Sigma_{\mathfrak{q}^*}(K,W)/_{\mathrm{div}}|} = \frac{[H^1(K_{\mathfrak{q}},T) : \mathrm{loc}_{\mathfrak{q}}(\mathrm{Sel}_{\mathrm{rel}}(K,T))]}{|H^1(K_{\mathfrak{q}},T)_{\mathrm{tors}}|\,|\,\mathrm{loc}_{\mathfrak{q}}(\mathrm{Sel}_{\mathrm{rel}}(K,W))/_{\mathrm{div}}|}. \tag{7.4}$$

Proof. (a) Since $r_{\mathfrak{q}}(\phi) = 1$, we have (as remarked above) that

$$\Sigma_{\mathfrak{q}}(K,W)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{rel}}(K,W)_{\mathrm{div}}.$$

The Poitou-Tate exact sequence implies that there is an exact sequence

$$0 \to \Sigma_{\mathfrak{q}}(K,W) \to \mathrm{Sel}_{\mathrm{rel}}(K,W) \xrightarrow{\mathrm{loc}_{\mathfrak{q}^*}} H^1(K_{\mathfrak{q}^*},W) \xrightarrow{\alpha} \Sigma_{\mathfrak{q}^*}(K,T^*)^{\wedge}.$$

The kernel of α is equal to the Pontryagin dual of the cokernel of the map

$$\Sigma_{\mathfrak{q}^*}(K,T^*) \to H^1(K_{\mathfrak{q}^*},W)^{\wedge} \simeq H^1(K_{\mathfrak{q}^*},T^*),$$

and so (7.3) follows.

(b) There is an exact sequence

$$0 \to \Sigma_{\mathfrak{q}^*}(K,W) \to \mathrm{Sel}_{\mathrm{rel}}(K,W) \xrightarrow{\mathrm{loc}_{\mathfrak{q}}} \mathrm{loc}_{\mathfrak{q}}(\mathrm{Sel}_{\mathrm{rel}}(K,W)) \to 0. \tag{7.5}$$

If M is any cofinitely generated, torsion $\mathbf{Z}_p$-module, then we have

$$\mathrm{Ext}^1(\mathbf{Q}_p/\mathbf{Z}_p, M) \simeq M/_{\mathrm{div}}, \quad \mathrm{Ext}^1(\mathbf{Q}_p/\mathbf{Z}_p, M_{\mathrm{div}}) = \mathrm{Ext}^2(\mathbf{Q}_p/\mathbf{Z}_p, M) = 0.$$

Since $r_{\mathfrak{q}}(\phi) = 1$, and $H^1(K_{\mathfrak{q}},W)$ is of $\mathbf{Z}_p$-corank one, we have that

$$\begin{aligned}\mathrm{Hom}(\mathbf{Q}_p/\mathbf{Z}_p, \mathrm{loc}_{\mathfrak{q}}(\mathrm{Sel}_{\mathrm{rel}}(K,W))) &= \mathrm{Hom}(\mathbf{Q}_p/\mathbf{Z}_p, \mathrm{loc}_{\mathfrak{q}}(\mathrm{Sel}_{\mathrm{rel}}(K,W))_{\mathrm{div}}) \\ &= \mathrm{Hom}(\mathbf{Q}_p/\mathbf{Z}_p, H^1(K_{\mathfrak{q}},W)_{\mathrm{div}}) \\ &= H^1(K_{\mathfrak{q}},T)/H^1(K_{\mathfrak{q}},T)_{\mathrm{tors}}.\end{aligned}$$

Hence, applying the functor $\mathrm{Hom}(\mathbf{Q}_p/\mathbf{Z}_p, -)$ to the exact sequence (7.5) yields

$$0 \to \Sigma_{\mathfrak{q}^*}(K,T) \to \mathrm{Sel}_{\mathrm{rel}}(K,T) \to H^1(K_{\mathfrak{q}},T)/H^1(K_{\mathfrak{q}},T)_{\mathrm{tors}} \to$$
$$\to \Sigma_{\mathfrak{q}^*}(K,W)/_{\mathrm{div}} \to \mathrm{Sel}_{\mathrm{rel}}(K,W)/_{\mathrm{div}} \to \mathrm{loc}_{\mathfrak{q}}(\mathrm{Sel}_{\mathrm{rel}}(K,W))/_{\mathrm{div}} \to 0,$$

and this immediately implies (7.4). □

Let $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathbf{Z}_p$ be a non-trivial character, and write K^ρ/K for the $\mathbf{Z}_p$-extension cut out by ρ. Recall that $H^{(\rho)}_{\mathfrak{q},\phi} \in \Lambda(K^\rho_\infty)$ denotes a characteristic power series of $X_{\mathfrak{q}}(K^\rho_\infty, W)$. Let us set

$$H^{(\rho)}_{\mathfrak{q},\phi}(0)^* := \left.\frac{H^{(\rho)}_{\mathfrak{q},\phi}(X)}{X^{n_{\mathfrak{q}}(\phi)}}\right|_{X=0}.$$

Theorem 7.6. *Suppose that $r_{\mathfrak{q}}(\phi) = r_{\mathfrak{q}^*}(\phi^*) = 1$, and that the p-adic height pairings $[-,-]^{(\rho)}_{K,\mathfrak{q}}$ and $[-,-]^{(\rho)}_{K,\mathfrak{q}^*}$ are non-degenerate. Suppose also that $n_{\mathrm{str}}(\phi) = n_{\mathrm{str}}(\phi^*)$. Then we have*

$$|\,\mathrm{Sel}_{\mathrm{rel}}(K,W)/_{\mathrm{div}}| \sim$$
$$\frac{H^{(\rho)}_{\mathfrak{q},\phi}(0)^*}{|\Sigma_{\mathfrak{q}}(K,T)_{\mathrm{tors}}|\mathcal{R}^{(\rho)}_{K,\mathfrak{q}}} \cdot [H^1(K_{\mathfrak{q}^*},T^*) : \mathrm{loc}_{\mathfrak{q}^*}(\Sigma_{\mathfrak{q}^*}(K,T^*))] \sim$$
$$\frac{H^{(\rho)}_{\mathfrak{q}^*,\phi}(0)^*}{|\Sigma_{\mathfrak{q}^*}(K,T)_{\mathrm{tors}}|\mathcal{R}^{(\rho)}_{K,\mathfrak{q}^*}} \cdot \frac{[H^1(K_{\mathfrak{q}},T) : \mathrm{loc}_{\mathfrak{q}}(\mathrm{Sel}_{\mathrm{rel}}(K,T))]}{|H^1(K_{\mathfrak{q}},T)_{\mathrm{tors}}||\,\mathrm{loc}_{\mathfrak{q}}(\mathrm{Sel}_{\mathrm{rel}}(K,W))/_{\mathrm{div}}|}.$$

Proof. This follows directly from Theorem 6.3 and Proposition 7.5. □

8. Formal groups and explicit reciprocity

In this section we shall recall various results that we need concerning explicit reciprocity laws.

8.1. **Formal groups.** Suppose that $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}^*\}$, and let $\mathbf{F}$ be a height one Lubin-Tate formal group defined over $O_{K,\mathfrak{q}}$. Recall that $\zeta = [\zeta_n]$ denotes the generator of $\mathbf{Z}_p(1)$ that was fixed at the begining of this paper in Section 2. For each n, let $\hat{\zeta}_n$ denote the parameter of ζ_n on the multiplicative formal group $\hat{\mathbf{G}}_m$. We choose an isomorphism

$$\eta_{\mathbf{F}} : \hat{\mathbf{G}}_m \xrightarrow{\sim} \mathbf{F}; \quad \eta_{\mathbf{F}}(X) \in \mathcal{O}[[X]] \tag{8.1}$$

over $\mathcal{O}$, and we set

$$\nu_n = \nu_{\mathbf{F},n} := \eta_{\mathbf{F}}(\hat{\zeta}_n).$$

We write

$$\mathbf{F}[p^n] := O_{K,\mathfrak{q}} \cdot \nu_n$$

(where here $O_{K,\mathfrak{q}}$ acts on ν_n via $\mathbf{F}$) for the group of p^n-torsion points on $\mathbf{F}$. Then

$$\nu = \nu_{\mathbf{F}} := [\nu_n]$$

is a generator of the p-adic Tate module $T_{\mathbf{F}}$ of $\mathbf{F}$. We write

$$\kappa_{\mathbf{F}} : \mathrm{Gal}(\overline{K}_{\mathfrak{q}}/K_{\mathfrak{q}}) \to \mathbf{Z}_p^{\times}$$

for the formal group character afforded by Galois action on $T_{\mathbf{F}}$, and

$$\Omega_{\eta_{\mathbf{F}}} := \eta'_{\mathbf{F}}(0) \tag{8.2}$$

for the p-adic period associated to our choice of isomorphism $\eta_{\mathbf{F}}$. Then, for each $\sigma \in \mathrm{Gal}(\overline{K}_{\mathfrak{q}}/K_{\mathfrak{q}})$, we have

$$\Omega_{\eta_{\mathbf{F}}}^{\sigma} = \kappa_{\mathbf{F}}^{*}(\sigma) \cdot \Omega_{\eta_{\mathbf{F}}}, \tag{8.3}$$

where $\kappa_{\mathbf{F}}^{*} := \kappa^{-1} \cdot \chi_{\mathrm{cyc}}$.

For each integer $n \geq 1$, we set $L_n := K_{\mathfrak{q}}(\nu_n)$, and we write H_n for the unique unramified extension of $K_{\mathfrak{q}}$ of degree $p^{n-1}(p-1)$. We put $M_n := L_n H_n$, and we set

$$L_\infty := \cup_{n\geq 1} L_n, \quad H_\infty := \cup_{n\geq 1} H_n, \quad M_\infty := \cup_{n\geq 1} M_n. \tag{8.4}$$

We define $L_0 = H_0 = K_{\mathfrak{q}}$.

We write R for the field of fractions of $\mathcal{O}$.

8.2. **Explicit reciprocity laws.** We retain the notation established in the previous subsection.

Let

$$\phi_{\mathfrak{q}} : \mathrm{Gal}(\overline{K}_{\mathfrak{q}}/K_{\mathfrak{q}}) \to \mathbf{Z}_p^{\times}$$

be a character, and set ${\phi_{\mathfrak{q}}}^* := {\phi_{\mathfrak{q}}}^{-1} \cdot \chi_{\mathrm{cyc}}$. Assume that $\phi_{\mathfrak{q}}$ (and therefore also ${\phi_{\mathfrak{q}}}^*$) is crystalline. Set

$$T_{\mathfrak{q}} := \mathbf{Z}_p(\phi_{\mathfrak{q}}), \quad V_{\mathfrak{q}} := \mathbf{Q}_p(\phi_{\mathfrak{q}}),$$

and define $T_{\mathfrak{q}}^*$, $V_{\mathfrak{q}}^*$ in an analogous way.

We write

$$\mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) := (B_{dR} \otimes_{K_{\mathfrak{q}}} V_{\mathfrak{q}})^{\mathrm{Gal}(\overline{K}_{\mathfrak{q}}/K_{\mathfrak{q}})},$$

where B_{dR} denotes the de Rham period ring of Fontaine. Recall that $\mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}})$ carries an action of a Frobenius endomorphism φ and has a decreasing filtration which in the present situation may be described explicitly as follows.

Let $m(V_{\mathfrak{q}})$ denote the (unique) Hodge-Tate weight of $V_{\mathfrak{q}}$, so

$$\phi_{\mathfrak{q}} \mid_{\mathrm{Gal}(\overline{K}_{\mathfrak{q}}/K_{\mathfrak{q}}^{\mathrm{nr}})} = \chi_{\mathrm{cyc}}^{m(V_{\mathfrak{q}})} \mid_{\mathrm{Gal}(\overline{K}_{\mathfrak{q}}/K_{\mathfrak{q}}^{\mathrm{nr}})} .$$

Then

$$\mathrm{Fil}^i(\mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}})) = \begin{cases} \mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) & \text{if } i \leq -m(V_{\mathfrak{q}}); \\ 0 & \text{if } i > -m(V_{\mathfrak{q}}). \end{cases}$$

Let

$$\exp_{V_{\mathfrak{q}}} : \frac{\mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}})}{\mathrm{Fil}^0(\mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}}))} \to H^1_f(K_{\mathfrak{q}}, V_{\mathfrak{q}}) \subseteq H^1(K_{\mathfrak{q}}, V_{\mathfrak{q}})$$

and

$$\exp^*_{V_{\mathfrak{q}}^*} : H^1(K_{\mathfrak{q}}, V_{\mathfrak{q}}) \to \frac{H^1(K_{\mathfrak{q}}, V_{\mathfrak{q}})}{H^1_f(K_{\mathfrak{q}}, V_{\mathfrak{q}})} \to \mathrm{Fil}^0(\mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}}))$$

denote the Bloch-Kato exponential and dual exponential maps respectively.

If $\mathrm{Fil}^0(\mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}})) = \mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}})$ (i.e. if $m(V_{\mathfrak{q}}) \leq -1$), then $\exp^*_{V_{\mathfrak{q}}^*}$ is an isomorphism, while $\exp_{V_{\mathfrak{q}}}$ is the zero map; if $\mathrm{Fil}^0(\mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}})) = 0$ (i.e. if $m(V_{\mathfrak{q}}) \geq 0$), then the reverse is true. We write

$$\log_{V_{\mathfrak{q}}} : H^1_f(K_{\mathfrak{q}}, V_{\mathfrak{q}}) \to \mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}})$$

for the inverse of $\exp_{V_\mathfrak{q}}$ when $\mathrm{Fil}^0(\mathrm{D}_{\mathrm{dR}}(V_\mathfrak{q})) = 0$, and we call this map the *Bloch-Kato logarithm* associated to $V_\mathfrak{q}$.

Let R denote the completion of $K_\mathfrak{q}^{\mathrm{nr}}$. Set $\Gamma_{\mathbf{F}} := \mathrm{Gal}(L_\infty/K_\mathfrak{q})$, and let $\mathcal{H}(\Gamma_{\mathbf{F}})$ denote the algebra of $\mathbf{Q}_p$-valued distributions on $\Gamma_{\mathbf{F}}$. Then there is a 'large exponential map'

$$\mathrm{Exp}_{V_\mathfrak{q}} : \mathcal{H}(\Gamma_{\mathbf{F}})_R \otimes_{K_\mathfrak{q}} \mathrm{D}_{\mathrm{dR}}(V_\mathfrak{q}) \to H^1_{\mathrm{Iw}}(L_\infty, V_\mathfrak{q})_R$$

(first constructed by Perrin-Riou and subsequently generalised to our present setting by Zhang (see [37]), following work of Colmez (see [12])) satisfying the following result.

Theorem 8.1. *Suppose that* $\mu \in \mathcal{H}(\Gamma_{\mathbf{F}})_R \otimes_{K_\mathfrak{q}} \mathrm{D}_{\mathrm{dR}}(V_\mathfrak{q})$ *and set*

$$z = (z_n) = \mathrm{Exp}_{V_\mathfrak{q}}(\mu) \in H^1_{\mathrm{Iw}}(L_\infty, V_\mathfrak{q})_R.$$

(a) Suppose that $m(V_\mathfrak{q}) \geq 1$. *Then*

$$z_0 = \exp_{V_\mathfrak{q}}\left[\left(\frac{(1-(p\cdot\varphi)^{-1})}{(1-\varphi)}\right)\int_{\Gamma_{K_\mathfrak{q}}} \mu\right].$$

(b) Suppose that $m(V_\mathfrak{q}) \leq -1$. *Then*

$$\exp^*_{V_\mathfrak{q}^*}(z_0) = \left(\frac{(1-(p\cdot\varphi)^{-1})}{(1-\varphi)}\right)\int_{\Gamma_{K_\mathfrak{q}}} \mu.$$

Proof. Part (a) is simply a statement of a special case of [37, Theorem 3.3] (with $r = 0$ and $h = 1$), in our present setting, while while part (b) is a special case of [37, Theorem 6.2] (recall that we assume that $V_\mathfrak{q}$ is crystalline).

□

8.3. **Periods.** Let us now describe a slightly different way of writing Theorem 8.1 by making certain periods explicit.

We put

$$t_{\mathbf{F}} := \Omega_{\eta_{\mathbf{F}}} \cdot t,$$

where t denotes the canonical element of B_{dR} upon which $\mathrm{Gal}(\overline{K}_\mathfrak{q}/K_\mathfrak{q})$ acts via the p-adic cyclotomic character χ_{cyc}. Then for each $\sigma \in \mathrm{Gal}(\overline{K}_\mathfrak{q}/K_\mathfrak{q})$, we have

$$t_{\mathbf{F}}^\sigma = \kappa(\sigma) \cdot t_{\mathbf{F}}.$$

We may write

$$\phi_{\mathfrak{q}} = \kappa^{m(V_{\mathfrak{q}})} \cdot \kappa^{*n(V_{\mathfrak{q}})},$$

with $n(V_{\mathfrak{q}}) \in \mathbf{Z}$. Set

$$\Omega(V_{\mathfrak{q}}) := t_{\mathbf{F}}^{m(V_{\mathfrak{q}})} \cdot \Omega_{\eta_{\mathbf{F}}}^{-n(V_{\mathfrak{q}})}. \tag{8.5}$$

(Note that $\Omega(V_{\mathfrak{q}})$ depends upon our choice of isomorphism $\eta_{\mathbf{F}}$; we omit this dependence from our notation.)

Let $e(V_{\mathfrak{q}})$ denote a fixed $K_{\mathfrak{q}}$ basis of $V_{\mathfrak{q}}$. Then $\Omega(V_{\mathfrak{q}})^{-1} \otimes e(V_{\mathfrak{q}})$ is a $K_{\mathfrak{q}}$-basis of $\mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}})$, and we make the identifications

$$\begin{aligned} K_{\mathfrak{q}} \xrightarrow{\sim} K_{\mathfrak{q}} \cdot \Omega(V_{\mathfrak{q}})^{-1} (\subseteq B_{dR}) \xrightarrow{\sim} \mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) \\ x \mapsto x \cdot \Omega(V_{\mathfrak{q}})^{-1} \mapsto x \cdot \Omega(V_{\mathfrak{q}})^{-1} \otimes e(V_{\mathfrak{q}}) \end{aligned} \tag{8.6}$$

of $K_{\mathfrak{q}}$-vector spaces.

Lemma 8.2. *Let* $\mathrm{Fr}_{\mathfrak{q}}$ *denote the absolute Frobenius element associated to* $\mathfrak{q}$*, then we have the following equalities in* B_{dR}*:*

$$\begin{aligned} (1 - (p \cdot \varphi)^{-1}) \cdot \Omega(V_{\mathfrak{q}})^{-1} = (1 - (p^{-1} \cdot \phi_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}))) \cdot \Omega(V_{\mathfrak{q}})^{-1}; \\ (1 - \varphi)^{-1} \cdot \Omega(V_{\mathfrak{q}})^{-1} = (1 - \phi_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}))^{-1} \cdot \Omega(V_{\mathfrak{q}})^{-1}. \end{aligned} \tag{8.7}$$

Proof. This follows directly from the definition of $\Omega(V_{\mathfrak{q}})$ via a straightforward calculation. □

If $\mu \in \mathcal{H}(\Gamma_{\mathbf{F}})_{K_{\mathfrak{q}}}$, then for any character χ say of $\Gamma_{K_{\mathfrak{q}}}$, we set

$$L_{\mu}(\chi) := \int_{\Gamma_{K_{\mathfrak{q}}}} \chi \cdot \mu.$$

If we make the identifications above, then Theorem 8.1 may be stated as follows.

Theorem 8.3. *Suppose that* $\mu \in \mathcal{H}(\Gamma_{\mathbf{F}})_{K_{\mathfrak{q}}} \otimes_{K_{\mathfrak{q}}} \mathrm{D}_{\mathrm{dR}}(V_{\mathfrak{q}})$ *and set*

$$z = (z_n) = \mathrm{Exp}_{V_{\mathfrak{q}}}(\mu) \in H^1_{\mathrm{Iw}}(L_{\infty}, V_{\mathfrak{q}})_R.$$

(a) If $m(V_{\mathfrak{q}}) \geq 1$*, then*

$$\log_{V_{\mathfrak{q}}}(z_0) = (1 - (p^{-1} \cdot \phi_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}))) \cdot (1 - \phi_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}))^{-1} \cdot \Omega(V_{\mathfrak{q}})^{-1} \cdot L_{\mu}(\mathbf{1}).$$

(b) If $m(V_{\mathfrak{q}}) \leq -1$, then

$$\exp^*_{V^*_{\mathfrak{q}}}(z_0) = (1 - (p^{-1} \cdot \phi_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}))) \cdot (1 - \phi_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}))^{-1} \cdot \Omega(V_{\mathfrak{q}})^{-1} \cdot L_{\mu}(\mathbf{1}).$$

9. The Katz two-variable p-adic L-function

If k, j are integers, recall that a Grossencharacter of type (k, j) is defined to be a $\overline{K}$-valued function ϵ which is defined on integral ideals of O_K coprime to a fixed ideal $\mathfrak{m}$ such that if $\mathfrak{a} = \alpha O_K$ with $\alpha \equiv 1 \pmod{\mathfrak{m}}$, then $\epsilon(\mathfrak{a}) = \alpha^k \overline{\alpha}^j$. For any ideal $\mathfrak{m}$ of O_K, we write $L_{\infty,\mathfrak{m}}(\epsilon, s)$ for the $\mathbf{C}$-valued completed Hecke L-function attached to ϵ with the Euler factors dividing $\mathfrak{m}$ removed. If ϵ is a Grossencharacter of conductor dividing $\mathfrak{m}$, then it has an associated p-adic Galois character

$$\epsilon_{\mathfrak{q}} : \mathrm{Gal}(K(\mathfrak{m}p^{\infty})/K) \to \mathbf{C}_p^{\times}; \quad \sigma_{\mathfrak{a}} \to i_{\mathfrak{q}}(\epsilon(\mathfrak{a})),$$

where $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}^*\}$ and $i_{\mathfrak{q}} : \overline{K} \hookrightarrow \overline{K}_{\mathfrak{q}}$ is the natural embedding afforded by $\mathfrak{q}$.

Let $\mathfrak{f}$ denote the conductor of the elliptic curve E/K. A result of Katz (see [13, Theorem II.4.14]) asserts that there exists a p-adic measure $\mu_{\mathfrak{q}} \in \Lambda(K(\mathfrak{f}p^{\infty}))_{\mathcal{O}}$ such that if ϵ is any Grossencharacter of type (k, j) with $0 \leq -j < k$ and of conductor dividing $\mathfrak{f}p^{\infty}$, then there is an interpolation formula:

$$\alpha_{\mathfrak{q}}(\epsilon) \int \epsilon_{\mathfrak{q}} d\mu_{\mathfrak{q}} = \left(1 - \frac{\epsilon(\mathfrak{q})}{p}\right) \cdot L_{\infty,\mathfrak{f}\mathfrak{q}^*}(\epsilon^{-1}, 0).$$

Here $\alpha_{\mathfrak{q}}(\epsilon)$ is an explicit, non-zero constant (whose precise description we shall not need), the integral is over $\mathrm{Gal}(K(\mathfrak{f}p^{\infty})/K)$, and we view the right-hand side of the equality as lying in $\mathbf{C}_p$ via the embedding $i_{\mathfrak{q}}$.

Definition 9.1. We define the Katz two-variable p-adic L-function $\mathcal{L}_{\mathfrak{q}}$ by the interpolation formula

$$\mathcal{L}_{\mathfrak{q}}(\epsilon) = \int_{\mathrm{Gal}(K(\mathfrak{f}p^{\infty})/K)} \epsilon_{\mathfrak{q}} d\mu_{\mathfrak{q}}$$

for all Grossencharacters ϵ of conductor dividing $\mathfrak{f}p^{\infty}$, and we view $\mathcal{L}_{\mathfrak{q}}$ as lying in $\Lambda(K(\mathfrak{f}p^{\infty}))_{\mathcal{O}}$. Hence, if ϵ is of type (k, j) with $0 \leq -j < k$, then

$$\alpha_{\mathfrak{q}}(\epsilon) \cdot \mathcal{L}_{\mathfrak{q}}(\epsilon) = \left(1 - \frac{\epsilon(\mathfrak{q})}{p}\right) \cdot L_{\infty,\mathfrak{f}\mathfrak{q}^*}(\epsilon^{-1}, 0).$$

(Note that this agrees with the definition used in [30, Section 7], but is different from the definition given in [13, II.4.16].)

If $\epsilon_{\mathfrak{q}}$ factors through $\mathrm{Gal}(F/K)$ for some subextension F/K of $K(\mathfrak{f}p^\infty)/K$, then

$$\int_{\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)} \epsilon_{\mathfrak{q}} d\mu_{\mathfrak{q}} = \int_{\mathrm{Gal}(F/K)} \epsilon_{\mathfrak{q}} d\mu_{\mathfrak{q}},$$

and so we have that

$$\mathcal{L}_{\mathfrak{q}}(\epsilon) = (\mathcal{L}_{\mathfrak{q}}|_F)(\epsilon), \tag{9.1}$$

where $\mathcal{L}_{\mathfrak{q}}|_F$ denotes the image of $\mathcal{L}_{\mathfrak{q}}$ under the natural projection map

$$\Lambda(K(\mathfrak{f}p^\infty))_{\mathcal{O}} \to \Lambda(F)_{\mathcal{O}}.$$

We write $\mu_{\mathfrak{q}}(F) \in \Lambda(F)_{\mathcal{O}}$ for the image of $\mu_{\mathfrak{q}}$ under this last projection map. $\square$

Suppose now that

$$\chi : \mathrm{Gal}(K(\mathfrak{f}p^\infty)/K) \to \mathbf{Z}_p^\times$$

is surjective and totally ramified at $\mathfrak{q}$. Let K_∞^χ/K denote the extension cut out by χ. Set $L_\infty := K_{\infty,\mathfrak{q}}^\chi$. We identify $\mathrm{Gal}(K_\infty^\chi/K)$ with $\mathrm{Gal}(L_\infty/K_{\mathfrak{q}})$, and we write $\mu_{\mathfrak{q}}(L_\infty)$ for the image of $\mu_{\mathfrak{q}}$ in $\Lambda(L_\infty)$. We set $\rho := \langle \chi \rangle$, and we note that, by definition, ρ is locally Lubin-Tate at $\mathfrak{q}$ (see Definition C).

For an integer $r \geq 0$ and any Grossencharacter ϵ of conductor dividing $\mathfrak{f}p^\infty$, we set

$$\mathcal{L}_{\mathfrak{q}}^{(r)}(\epsilon;\rho) := \frac{1}{r!}\left(\frac{d}{ds}\right)^r \mathcal{L}_{\mathfrak{q}}(\epsilon \cdot \rho^s)\bigg|_{s=0}.$$

Definition 9.2. Let r be a non-negative integer. We define a measure $\mu_{\mathfrak{q}}^{(r;\rho)}$ in $\Lambda(K(\mathfrak{f}p^\infty))_{\mathcal{O}}$ by the equality

$$\mathcal{L}_{\mathfrak{q}}^{(r)}(\epsilon;\rho) = \int_{\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)} \epsilon \cdot \mu_{\mathfrak{q}}^{(r;\rho)}$$

for all Grossencharacters ϵ of conductor dividing $\mathfrak{f}p^\infty$. Note that with this definition, we have

$$\mu_{\mathfrak{q}}^{(0;\rho)} = \mu_{\mathfrak{q}}. \tag{9.2}$$

We also define measures $\mu_{\mathfrak{q}}^{(r;\rho)}(\phi)$ and $\mu_{\mathfrak{q}}^{(r;\rho)}(\phi^*)$ (which should be thought of as twists of $\mu^{(r;\rho)}$) in $\Lambda(K(\mathfrak{f}p^\infty))_{\mathcal{O}}$) by the equalities

$$\mathcal{L}_{\mathfrak{q}}^{(r)}(\phi \cdot \epsilon; \rho) = \int_{\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)} \epsilon \cdot \mu_{\mathfrak{q}}^{(r;\rho)}(\phi); \tag{9.3}$$

$$\mathcal{L}_{\mathfrak{q}}^{(r)}(\phi^* \cdot \epsilon; \rho) = \int_{\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)} \epsilon \cdot \mu_{\mathfrak{q}}^{(r;\rho)}(\phi^*). \tag{9.4}$$

If F/K is a subextension of $K(\mathfrak{f}p^\infty)/K$, we write $\mu_{\mathfrak{q}}^{(r;\rho)}(F)$, $\mu^{(r;\rho)}(F;\phi)$, and $\mu^{(r;\rho)}(F;\phi^*)$ for the images of $\mu_{\mathfrak{q}}^{(r;\rho)}$, $\mu_{\mathfrak{q}}^{(r;\rho)}(\phi)$ and $\mu^{(r;\rho)}(\phi^*)$ respectively under the projection map $\Lambda(K(\mathfrak{f}p^\infty))_{\mathcal{O}} \to \Lambda(F)_{\mathcal{O}}$.

We remark that it follows from the definitions that we have e.g.

$$\mathcal{L}_{\mathfrak{q}}^{(r)}(\phi^*; \rho) = \int_{\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)} \mu_{\mathfrak{q}}^{(r;\rho)}(\phi^*) = \int_{\mathrm{Gal}(L_\infty/K_{\mathfrak{q}})} \mu_{\mathfrak{q}}^{(r;\rho)}(L_\infty; \phi^*).$$

□

Proposition 9.3. *(a) Suppose that*

$$\mathrm{ord}_{s=0}\, \mathcal{L}_{\mathfrak{q}}(\phi^* \cdot \rho^s) = m,$$

and that γ is a topological generator of $\mathrm{Gal}(L_\infty/K_{\mathfrak{q}})$. Suppose also that $\widetilde{\mu}_{\mathfrak{q}}^{(m;\rho)} \in \Lambda(L_\infty)$ satisfies

$$(\log_p \rho(\gamma))^m \cdot \mu_{\mathfrak{q}}(L_\infty; \phi^*) = (\gamma - 1)^m \cdot \widetilde{\mu}_{\mathfrak{q}}^{(m;\rho)}.$$

Then

$$\mathcal{L}_{\mathfrak{q}}^{(m)}(\phi^*; \rho) = \int_{\mathrm{Gal}(L_\infty/K_{\mathfrak{q}})} \mu_{\mathfrak{q}}^{(m;\rho)}(L_\infty; \phi^*) = \int_{\mathrm{Gal}(L_\infty/K_{\mathfrak{q}})} \widetilde{\mu}_{\mathfrak{q}}^{(m;\rho)}$$

(b) Assume that V is crystalline at $\mathfrak{q}$. Let $d(V)$ be a basis of $\mathrm{D}_{\mathrm{dR}}(V)$, and suppose that $\beta_{\mathfrak{q}}^{(m;\rho)}(\phi) = (\beta_{\mathfrak{q}}^{(m;\rho)}(\phi)_n) \in H^1_{\mathrm{Iw}}(L_\infty, V)$ satisfies

$$(\log_p \rho(\gamma))^m \cdot \mathrm{Exp}_{V,\mathfrak{q}}(\mu_{\mathfrak{q}}(L_\infty; \phi^*) \otimes d(V)) = (\gamma - 1)^m \cdot \beta_{\mathfrak{q}}^{(m;\rho)}(\phi).$$

Then

$$\beta_{\mathfrak{q}}^{(m;\rho)}(\phi)_0 = (\mathrm{Exp}_{V,\mathfrak{q}}(\widetilde{\mu}_{\mathfrak{q}}^{(m;\rho)} \otimes d(V)))_0.$$

Proof. If we identify $\Lambda(L_\infty)$ with the power series ring $\mathbf{Z}_p[[(\gamma-1)]]$ in the standard manner, then we may write

$$\mu_{\mathfrak{q}}(L_\infty;\phi^*) = a_m\cdot(\gamma-1)^m + (\text{higher order terms in } (\gamma-1)),$$

and

$$\widetilde{\mu}_{\mathfrak{q}}^{(m;\rho)} = a_m\cdot(\log_p(\rho(\gamma)))^m + (\text{higher order terms in } (\gamma-1)).$$

A routine computation shows that

$$\frac{d^m}{ds^m}\left[(\rho(\gamma)^s-1)^m\right]\Bigg|_{s=0} = m!\cdot(\log_p\rho(\gamma))^m.$$

Hence

$$\begin{aligned}\mathcal{L}_{\mathfrak{q}}^{(m)}(\phi^*;\rho) &= \int_{\mathrm{Gal}(L_\infty/K_{\mathfrak{q}})}\mu_{\mathfrak{q}}^{(m;\rho)}(L_\infty;\phi^*)\cdot\mathbf{1}\\ &= a_m\cdot(\log_p(\rho(\gamma)))^m\\ &= \int_{\mathrm{Gal}(L_\infty/K_{\mathfrak{q}})}\widetilde{\mu}_{\mathfrak{q}}^{(m;\rho)},\end{aligned}$$

as claimed.

(b) As $\mathrm{Exp}_{V,\mathfrak{q}}$ is $\Lambda(L_\infty)$-equivariant, we have

$$(\log_p\rho(\gamma))^m\cdot\mathrm{Exp}_{V,\mathfrak{q}}(\mu_{\mathfrak{q}}(L_\infty;\phi^*)\otimes d(V)) = (\gamma-1)^m\cdot\mathrm{Exp}_{V.\mathfrak{q}}(\widetilde{\mu}_{\mathfrak{q}}^{(m;\rho)}\otimes d(V)).$$

This implies that

$$(\gamma-1)^m\cdot\beta_{\mathfrak{q}}^{(m;\rho)}(\phi) = (\gamma-1)^m\cdot\mathrm{Exp}_{V.\mathfrak{q}}(\widetilde{\mu}_{\mathfrak{q}}^{(m;\rho)}\otimes d(V)),$$

whence it follows that

$$\beta_{\mathfrak{q}}^{(m;\rho)}(\phi)_0 = \mathrm{Exp}_{V.\mathfrak{q}}(\widetilde{\mu}_{\mathfrak{q}}^{(m;\rho)}\otimes d(V))_0,$$

because $H^1_{\mathrm{Iw}}(L_\infty,V)$ is a torsion-free $\Lambda(L_\infty)$-module. □

The following result will allow us to apply the results of Section 8 in the present setting.

Proposition 9.4. *Let N_∞ be any totally ramified $\mathbf{Z}_p^\times$ extension of $K_{\mathfrak{q}}$, and let $\pi(N_\infty)$ be a uniformiser of $O_{K,\mathfrak{q}}$ that generates the group of universal norms of this extension. Then there exists a height one Lubin-Tate formal group $\mathbf{F}(N_\infty)$ associated to $\pi(N_\infty)$, such that $L_\infty = N_\infty$ (using the notation of* (8.4)*).*

Proof. This is a standard result which follows via local class field theory; see [23, Chapter V, §5], for example. ☐

We now fix a height one Lubin-Tate formal group $\mathbf{F}(L_\infty)$ over $K_\mathfrak{q}$ associated to the extension $L_\infty/K_\mathfrak{q}$ via Proposition 9.4. We assume that V is crystalline at $\mathfrak{q}$, and we let $\Omega_\mathfrak{q}(V)$ be a corresponging period associted to V (see (8.5)). Recall that if $e(V)$ is a fixed K-basis of V, then $d(V) := \Omega_\mathfrak{q}(V)^{-1} \otimes e(V)$ is a basis of $\mathrm{D}_{\mathrm{dR}}(V)$ (see §8.3).

It follows from the definition of $\mu_\mathfrak{q}^{(r;\rho)}(L_\infty;\phi^*)$ that we have

$$\mathcal{L}_\mathfrak{q}^{(r)}(\phi^*;\rho) = \int_{\mathrm{Gal}(L_\infty/K_\mathfrak{q})} \mu_\mathfrak{q}^{(r,\rho)}(L_\infty;\phi^*).$$

and so in $\mathrm{D}_{\mathrm{dR}}(V)_R$, we have

$$\mathcal{L}_\mathfrak{q}^{(r)}(\phi^*;\rho) \otimes d(V) = \int_{\mathrm{Gal}(L_\infty/K_\mathfrak{q})} \mu_\mathfrak{q}^{(r;\rho)}(L_\infty;\phi^*) \otimes d(V).$$

We define $z_\mathfrak{q}(\phi)^{(r;\rho)} = (z_\mathfrak{q}(\phi)_n^{(r;\rho)}) \in H^1_{\mathrm{Iw}}(L_\infty, V)_R$ by setting

$$z_\mathfrak{q}(\phi)^{(r;\rho)} := \mathrm{Exp}_{V.\mathfrak{q}}(\mu_\mathfrak{q}^{(r,\rho)}(L_\infty;\phi^*) \otimes d(V)).$$

We also write

$$z_\mathfrak{q}(\phi) := z_\mathfrak{q}(\phi)^{(0;\rho)} = \mathrm{Exp}_{V.\mathfrak{q}}(\mu_\mathfrak{q}(L_\infty;\phi^*) \otimes d(V)).$$

In order to be able to state our results in a uniform manner, it is helpful to introduce the following notation.

Definition 9.5. Set

$$\mathrm{Eul}_\mathfrak{q}(\phi) := (1 - (p^{-1} \cdot \phi(\mathrm{Fr}_\mathfrak{q}))) \cdot (1 - \phi(\mathrm{Fr}_\mathfrak{q}))^{-1}$$

and

$$A_\mathfrak{q}(\phi;\rho) := \mathrm{Eul}_\mathfrak{q}(\phi) \cdot \Omega_\mathfrak{q}(V)^{-1}.$$

☐

Definition 9.6. We define

$$\mathfrak{Log}_V = \mathfrak{Log}_{V,\mathfrak{q}} : H^1(K_{\mathfrak{q}}, V) \to \mathrm{D}_{\mathrm{dR}}(V)$$

by

$$\mathfrak{Log}_V(x) = \mathfrak{Log}_{V,\mathfrak{q}}(x) = \begin{cases} \log_V(x) & \text{if } m(V) \geq 0; \\ \exp_V^*(x) & \text{if } m(V) \leq -1. \end{cases}$$

We remark that if we identify $\mathrm{D}_{\mathrm{dR}}(V)$ with $\mathbf{Q}_p$ (see (8.6)), then the cup product pairing

$$\cup : H^1(K_{\mathfrak{q}}, V) \times H^1(K_{\mathfrak{q}}, V^*) \to H^2(K_{\mathfrak{q}}, \mathbf{Z}_p(1)) \xrightarrow{\mathrm{inv}_{\mathfrak{q}}} \mathbf{Q}_p$$

is given by

$$x \cup x^* = \mathfrak{Log}_{V,\mathfrak{q}}(x) \cdot \mathfrak{Log}_{V^*,\mathfrak{q}}(x^*). \tag{9.5}$$

□

The following result is now a direct consequence of Theorems 8.1 and 8.3.

Theorem 9.7. *Assume that V is crystalline at $\mathfrak{q}$. Suppose that $r \geq 0$ is an integer, and that $m(V) \neq 0$. Then*

$$\mathfrak{Log}_{V,\mathfrak{q}}(z_{\mathfrak{q}}(\phi)_0^{(r;\rho)}) = A_{\mathfrak{q}}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(r)}(\phi^*;\rho).$$

10. Euler systems and Selmer conditions

We retain the notation established in the previous section. Throughout this section, we assume that V (and therefore also V^*) is crystalline at both places of K lying above p.

In what follows throughout this section,

$$\chi : \mathrm{Gal}(\overline{K}/K) \to \mathbf{Z}_p^{\times}$$

is surjective and totally ramified at $\mathfrak{q}$, $\rho := \langle \chi \rangle$, and K_∞^χ denotes the extension of K cut out by χ.

We assume that the regulators $\mathcal{R}_{K,\mathfrak{q}}^{(\rho)}$ and $\mathcal{R}_{K,\mathfrak{q}^*}^{(\rho)}$ are non-zero (so $n_{\mathfrak{q}}(\phi) = n_{\mathfrak{q}^*}(\phi^*)$) and that

$$r_{\mathfrak{q}}(\phi) = r_{\mathfrak{q}^*}(\phi^*) = 1;$$

(see Theorem 7.3). Then

$$n_{\mathfrak{q}}(\phi^*) = n_{\mathrm{str}}(\phi^*) = n_{\mathrm{str}}(\phi) = \min\{n_{\mathfrak{q}}(\phi^*), n_{\mathfrak{q}}(\phi)\},$$

and

$$n_{\mathfrak{q}}(\phi^*) = n_{\mathfrak{q}}(\phi) - 1.$$

To ease notation, we set

$$n_{\mathrm{str}} := n_{\mathfrak{q}}(\phi^*) = n_{\mathrm{str}}(\phi^*) = n_{\mathrm{str}}(\phi). \tag{10.1}$$

As $\mathcal{R}^{(\rho)}_{K,\mathfrak{q}}$ and $\mathcal{R}^{(\rho)}_{K,\mathfrak{q}^*}$ are non-zero, we thus also have that

$$n_{\mathrm{str}} = \min\{\mathrm{ord}_{s=0}\, \mathcal{L}_{\mathfrak{q}}(\phi^* \cdot \rho^s), \mathrm{ord}_{s=0}\, \mathcal{L}_{\mathfrak{q}^*}(\phi^* \cdot \rho^s)\}. \tag{10.2}$$

Under our hypotheses, we have (see Proposition 7.4):

$$\Sigma_{\mathfrak{q}}(K,V)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{rel}}(K,V)_{\mathrm{div}}, \quad \Sigma_{\mathfrak{q}^*}(K,V^*)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{rel}}(K,V^*)_{\mathrm{div}};$$

$$\Sigma_{\mathfrak{q}^*}(K,V)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{str}}(K,V)_{\mathrm{div}}, \quad \Sigma_{\mathfrak{q}}(K,V^*)_{\mathrm{div}} = \mathrm{Sel}_{\mathrm{str}}(K,V^*)_{\mathrm{div}}.$$

Remark 10.1. We remark that if the restriction

$$[-,-]^{(\rho)}_{K,\mathrm{str}} : \mathrm{Sel}_{\mathrm{str}}(K,V^*) \times \mathrm{Sel}_{\mathrm{str}}(K,V) \to \mathbf{Q}_p \tag{10.3}$$

of the p-adic height pairing $[-,-]^{(\rho)}_{K,\mathfrak{q}}$ to strict Selmer groups is non-degenerate (as is conjectured to always be the case), and if $\alpha = (\alpha_n) \in H^1_{\mathrm{Iw}}(K^{(\chi)}_{\infty}, V)$ is any element, then $\alpha_0 \in \mathrm{Sel}_{\mathrm{str}}(K,V)$ if and only if $\alpha_0 = 0$ (see e.g. [27, §3.1.4]). □

10.1. **Euler systems.** We shall require the following result.

Theorem 10.2. *There exists a canonical element $c(\phi) = (c_n(\phi)) \in H^1_{\mathrm{Iw}}(K^{\chi}_{\infty}, V)$ such that if $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}^*\}$. then*

$$\mathrm{loc}_{\mathfrak{q}}(c(\phi)) = \mathrm{Exp}_V(\mu_{\mathfrak{q}}(L_\infty; \phi^*) \otimes d(V)) = z_{\mathfrak{q}}(\phi).$$

Proof. This is a standard result from the theory of Euler systems. The essential ideas involved in the proof are as follows.

There is an Euler system of elliptic units (whose precise definition we shall not need) in $K(\mathfrak{f}p^\infty)/K$ attached to $\mathbf{Z}_p(1)$ which is associated to the measure $\mu_\mathfrak{q}$. This in turn implies that there exists an element $u = (u_n) \in H^1_{\mathrm{Iw}}(K^\chi_\infty, \mathbf{Q}_p(1))$ such that

$$\mathrm{Exp}_{\mathbf{Q}_p(1)}(\mu_\mathfrak{q}(L_\infty) \otimes d(\mathbf{Q}_p(1))) = \mathrm{loc}_\mathfrak{q}(u),$$

where $d(\mathbf{Q}_p(1))$ denotes a basis of $\mathrm{D}_{\mathrm{dR}}(\mathbf{Q}_p(1)$. (See [20], for example.)

The $\mathrm{Gal}(\overline{K}/K)$-module T is equal to the twist of $\mathbf{Z}_p(1)$ by the character $\phi^{*-1} = \chi_{\mathrm{cyc}}^{-1} \cdot \phi$. This induces an isomorphism

$$f : H^1_{\mathrm{Iw}}(K(\mathfrak{f}p^\infty, \mathbf{Z}_p(1)) \to H^1_{\mathrm{Iw}}(K(\mathfrak{f}p^\infty, T)$$

of $\Lambda(K(\mathfrak{f}p^\infty))$-modules satisfying

$$f \circ \mathrm{Tw}^{-1}_{\phi^*}(\lambda) = \lambda \circ f$$

for any $\lambda \in \Lambda(K(\mathfrak{f}p^\infty))$ (see [33, Chapter 6]). It follows that there is an Euler system of twisted elliptic units in $H^1_{\mathrm{Iw}}(K(\mathfrak{f}p^\infty), T)$ which is associated to the measure $\mathrm{Tw}_{\phi^*}(\mu_\mathfrak{q})$. Hence, writing $\mu_\mathfrak{q}(L_\infty; \phi^*)$ for the image of $\mathrm{Tw}_{\phi^*}(\mu_\mathfrak{q})$ in $\Lambda(L_\infty)$ (see Definition 9.2), we see that there exists $c(\phi) = (c_n(\phi)) \in H^1_{\mathrm{Iw}}(K^{(\chi)}, V)$ such that

$$\mathrm{loc}_\mathfrak{q}(c(\phi)) = \mathrm{Exp}_V(\mu_\mathfrak{q}(L_\infty; \phi^*) \otimes d(V)) = z_\mathfrak{q}(\phi).$$

□

This is essentially the only property of the Euler system that we shall require.

10.2. **Selmer conditions.** In this subsection we shall examine Selmer conditions associated to the elements $c(\phi)$.

We first make the following definition.

Definition 10.3. Let γ be a topological generator of $\mathrm{Gal}(K^\chi_\infty/K)$,and suppose that $c(\phi) \in (\gamma - 1)^b \cdot H^1_{\mathrm{Iw}}(K^\chi_\infty, V)$. Recall that $\rho = \langle \chi \rangle$. Choose $\widetilde{c}(\phi)^{(b;\rho)} \in H^1_{\mathrm{Iw}}(K^\chi_\infty, V)$ such that

$$(\log_p \rho(\gamma))^b \cdot c(\phi) = (\gamma - 1)^b \cdot \widetilde{c}(\phi)^{(b;\rho)}.$$

We set

$$c(\phi)^{(b;\rho)}_0 := \widetilde{c}(\phi)^{(b;\rho)}_0.$$

This is independent of the choice of γ.

We shall require the following result.

Proposition 10.4. *Let γ be a topological generator of $\mathrm{Gal}(K_\infty^\chi/K)$, and let $d \geq 0$ be the largest integer such that $c(\phi) \in (\gamma-1)^d \cdot H^1_{\mathrm{Iw}}(K_\infty^\chi, V)$. Then $d = n_{\mathrm{str}}$.*

Proof. We see from (10.2) that $d \leq n_{\mathrm{str}}$, because precisely one of $\mathrm{loc}_{\mathfrak{q}}(c(\phi))$ and $\mathrm{loc}_{\mathfrak{q}^*}(c(\phi))$ is exactly divisible by $(\gamma-1)^{n_{\mathrm{str}}}$.

We must then have $d = n_{\mathrm{str}}$, for else $c(\phi)_0^{(d;\rho)} \neq 0$ while

$$\mathrm{loc}_{\mathfrak{q}}(c(\phi)_0^{(d;\rho)}) = \mathrm{loc}_{\mathfrak{q}^*}(c(\phi)_0^{(d;\rho)}) = 0$$

which implies that the pairing (10.3) is degenerate (see Remark 10.1). This contradicts our assumption that $\mathcal{R}^{(\rho)}_{K,\mathfrak{q}} \neq 0$. □

Corollary 10.5. *The element $c(\phi)_0^{(n_{\mathfrak{q}}(\phi^*);\rho)}$ lies in $\Sigma_{\mathfrak{q}}(K,V)$, and*

$$\mathfrak{Log}_{V,\mathfrak{q}}(\mathrm{loc}_{\mathfrak{q}}(c(\phi)_0^{(n_{\mathfrak{q}}(\phi^*);\rho)}))) = A_{\mathfrak{q}}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(n_{\mathfrak{q}}(\phi^*))}(\phi^*;\rho),$$

i.e.

$$\mathfrak{Log}_{V,\mathfrak{q}}(\mathrm{loc}_{\mathfrak{q}}(c(\phi)_0^{(n_{\mathrm{str}};\rho)}))) = A_{\mathfrak{q}}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(n_{\mathrm{str}})}(\phi^*;\rho),$$

Proof. As $r_{\mathfrak{q}}(\phi) = r_{\mathfrak{q}^*}(\phi^*) = 1$, Theorem 7.3 implies that

$$n_{\mathfrak{q}}(\phi^*) = n_{\mathrm{str}}(\phi^*) = n_{\mathfrak{q}^*}(\phi^*) - 1 = n_{\mathrm{str}}.$$

It therefore follows from Proposition 10.4 that

$$\mathrm{loc}_{\mathfrak{q}^*}(\widetilde{c}(\phi^*)^{(n_{\mathfrak{q}}(\phi^*);\rho)}) \in (\gamma-1)\cdot H^1_{\mathrm{Iw}}(K_{\mathfrak{q}^*}^{(\chi)}, V),$$

and so

$$\mathrm{loc}_{\mathfrak{q}^*}(c(\phi^*)_0^{(n_{\mathfrak{q}}(\phi^*);\rho)}) = 0,$$

Hence $c(\phi)_0^{(n_{\mathfrak{q}}(\phi^*);\rho)} \in \Sigma_{\mathfrak{q}}(K,V)$, as claimed.

The second assertion follows from Proposition 9.3 and Theorem 9.7. □

10.3. **Height formulae.**

Theorem 10.6. *Suppose that $x \in \Sigma_{\mathfrak{q}}(K,V)$. Then*

$$[c(\phi^*)_0^{(n_{\rm str};\rho)}, x]_{K,\mathfrak{q}}^{(\rho)} = A_{\mathfrak{q}}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(n_{\rm str}+1)}(\phi;\rho) \cdot \mathfrak{Log}_{\mathfrak{q}}(\mathrm{loc}_{\mathfrak{q}}(x)).$$

Proof. Applying Theorem 5.6 and (9.5) gives:

$$\begin{aligned}[c(\phi^*)_0^{(n_{\rm str};\rho)}, x]_{K,\mathfrak{q}}^{(\rho)} &= z_{\mathfrak{q}}(\phi^*)^{(n_{\rm str}+1;\rho)} \cup \mathrm{loc}_{\mathfrak{q}}(x) \\ &= A_{\mathfrak{q}}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(n_{\rm str}+1)}(\phi;\rho) \cdot \mathfrak{Log}_{\mathfrak{q}}(\mathrm{loc}_{\mathfrak{q}}(x)).\end{aligned}$$

□

The following result is an immediate consequence of Theorem 10.6.

Corollary 10.7. *Let $\mathrm{Sel}_{\rm str}(K,V)^{\perp} \subseteq \Sigma_{\mathfrak{q}^*}(K,V^*)$ denote the orthogonal complement of $\mathrm{Sel}_{\rm str}(K,V)$ with respect to the p-adic height pairing*

$$[-,-]_{\mathfrak{q}}^{(\rho)} : \Sigma_{\mathfrak{q}^*}(K,V^*) \times \Sigma_{\mathfrak{q}}(K,V) \to \mathbf{Q}_p.$$

Then $c(\phi^*)_0^{(n_{\rm str};\rho)} \in \mathrm{Sel}_{\rm str}(K,V)^{\perp}$. □

Corollary 10.8. *We have*

$$[c(\phi^*)_0^{(n_{\rm str};\rho)}, c(\phi)_0^{(n_{\rm str};\rho)}]_{K,\mathfrak{q}}^{(\rho)} = A_{\mathfrak{q}}(\rho;\chi) \cdot A_{\mathfrak{q}}(\phi^*;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(n_{\rm str}+1)}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(n_{\rm str})}(\phi^*;\rho).$$

Proof. This follows from Theorem 10.6 and Corollary 10.5. □

Theorem 10.9. *(a) We have*

$$\mathcal{R}_{K,\mathfrak{q}}^{(\rho)} = \mathcal{R}_{K,\rm str}^{(\rho)} \cdot A_{\mathfrak{q}}(\phi;\rho) \cdot A_{\mathfrak{q}}(\phi^*;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(n_{\rm str}+1)}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(n_{\rm str})}(\phi^*;\rho).$$

(b) Suppose that $n_{\rm str} = 0$. Let $y \in \Sigma_{\mathfrak{q}}(K,V)$ and $y^* \in \Sigma_{\mathfrak{q}^*}(K,V^*)$ be elements of infinite order. Then

$$\frac{[y^*, y]_{K,\mathfrak{q}}^{(\rho)}}{\mathfrak{Log}_{V^*,\mathfrak{q}^*}(y^*) \cdot \mathfrak{Log}_{V,\mathfrak{q}}(y)} = \frac{A_{\mathfrak{q}}(\phi^*;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(1)}(\phi;\rho)}{A_{\mathfrak{q}^*}(\phi^*;\rho) \cdot \mathcal{L}_{\mathfrak{q}^*}(\phi)}.$$

Proof. (a) If $\{x_1, \ldots, x_{n_{\rm str}}\}$ and $\{x_1^*, \ldots, x_{n_{\rm str}}^*\}$ are bases of $\mathrm{Sel}_{\rm str}(K, V)$ and $\mathrm{Sel}_{\rm str}(K, V^*)$ respectively, then we may write

$$\Sigma_{\mathfrak{q}}(K, V) = \langle x_1, \ldots, x_{n_{\rm str}}, c(\phi)_0^{(n_{\rm str};\rho)}\rangle$$

$$\Sigma_{\mathfrak{q}}(K, V^*) = \langle x_1^*, \ldots, x_{n_{\rm str}}^*, c(\phi^*)_0^{(n_{\rm str};\rho)}\rangle$$

Hence

$$\begin{aligned}\mathcal{R}_{K,\mathfrak{q}}^{(\rho)} &= \mathcal{R}_{K,\rm str}^{(\rho)} \cdot [c(\phi^*)_0^{(n_{\rm str};\rho)}, c(\phi)_0^{(n_{\rm str};\rho)}]_{K,\mathfrak{q}}^{(\rho)} \\ &= \mathcal{R}_{K,\rm str}^{(\rho)} \cdot A_{\mathfrak{q}}(\phi;\rho) \cdot A_{\mathfrak{q}}(\phi^*;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(n_{\rm str}+1)}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(n_{\rm str})}(\phi^*;\rho),\end{aligned}$$

wnere the last equality follows from Corollary 10.8.

(b) If $m = 0$ then $n_{\mathfrak{q}}(\phi) = n_{\mathfrak{q}^*}(\phi^*) = 1$, and so the quotient

$$\frac{[y^*, y]_{K,\mathfrak{q}}^{(\chi)}}{\mathfrak{Log}_{V^*,\mathfrak{q}^*}(y^*) \cdot \mathfrak{Log}_{V,\mathfrak{q}}(y)} \tag{10.4}$$

is independent of the choices of y and y^*. (Here we have identified $K_{\mathfrak{q}}$ and $K_{\mathfrak{q}^*}$ (as well as $\mathrm{D}_{\rm dR}(V)$ and $\mathrm{D}_{\rm dR}(V^*)$) with $\mathbf{Q}_p$ (cf. (8.6)).)

We evaluate (10.4) for $y = c(\phi)_0$ and $y^* = c(\phi^*)_0$.

We first observe that part (a) implies

$$[y^*, y]_{K,\mathfrak{q}}^{(\rho)} = A_{\mathfrak{q}}(\phi;\rho) \cdot A_{\mathfrak{q}}(\phi^*;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(1)}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}(\phi^*).$$

Next, we see from Corollary 10.5 that

$$\mathfrak{Log}_{V,\mathfrak{q}}(\mathrm{loc}_{\mathfrak{q}}(c(\phi)_0)) = A_{\mathfrak{q}}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}(\phi^*);$$

$$\mathfrak{Log}_{V^*,\mathfrak{q}^*}(\mathrm{loc}_{\mathfrak{q}^*}(c(\phi^*)_0)) = A_{\mathfrak{q}^*}(\phi^*;\rho) \cdot \mathcal{L}_{\mathfrak{q}^*}(\phi).$$

Hence

$$\begin{aligned}\frac{[y^*, y]_{K,\mathfrak{q}}^{(\rho)}}{\mathfrak{Log}_{V^*,\mathfrak{q}^*}(y^*) \cdot \mathfrak{Log}_{V,\mathfrak{q}}(y)} &= \frac{A_{\mathfrak{q}}(\phi;\rho) \cdot A_{\mathfrak{q}}(\phi^*;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(1)}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}(\phi^*)}{A_{\mathfrak{q}}(\phi;\rho) \cdot \mathcal{L}_{\mathfrak{q}}(\phi^*) \cdot A_{\mathfrak{q}^*}(\phi^*;\rho) \cdot \mathcal{L}_{\mathfrak{q}^*}(\phi)} \\ &= \frac{A_{\mathfrak{q}}(\phi^*;\rho) \cdot \mathcal{L}_{\mathfrak{q}}^{(1)}(\phi;\rho)}{A_{\mathfrak{q}^*}(\phi^*;\rho) \cdot \mathcal{L}_{\mathfrak{q}^*}(\phi)}.\end{aligned}$$

∎

11. Special value formulae

In this section we shall explain how the results of Section 10 yield a common generalisation of the results of [30] and [2] to CM modular forms of higher weight.

Let us now consider the characters

$$\phi_k := \psi^{k+1}\psi^{*-k}, \quad \phi_k^* := \psi^{-k}\psi^{*(k+1)}, \quad (k \geq 0).$$

The character ϕ_k is naturally associated to the CM modular form of weight $2k+2$ attached to the Grossencharacter ψ^{2k+1} and lies within the range of interpolation of $\mathcal{L}_{\mathfrak{p}}$, with

$$\mathcal{L}_{\mathfrak{p}}(\phi_k) = \alpha_k \cdot L(\psi^{2k+1}, k+1), \tag{11.1}$$

where α_k is an explicit, non-zero constant whose precise description need not concern us. On the other hand, the character ϕ_k^* lies outside the range of interpolation of $\mathcal{L}_{\mathfrak{p}}$, and the behaviour of $\mathcal{L}_{\mathfrak{p}}$ at ϕ_k^* is less well-understood.

The following result is a generalisation of [2, Theorem 2] to the case of CM modular forms of higher weight. Note that ϕ_k and ϕ_k^* are crystalline at $\mathfrak{p}$ and $\mathfrak{p}^*$.

Theorem 11.1. *Suppose that* $\mathcal{L}_{\mathfrak{p}}(\phi_k) \neq 0$.

(a) We have $n_{\mathfrak{p}}(\phi_k) = n_{\mathfrak{p}^*}(\phi_k^*) = 0$*, and so in particular* $n_{\mathrm{str}}(\phi_k) = n_{\mathrm{str}}(\phi_k^*) = 0$ *and* $r_{\mathfrak{p}}(\phi_k) = r_{\mathfrak{p}^*}(\phi_k^* = 0$ *also.*

(b) We have $r_{\mathfrak{p}^*}(\phi_k) = r_{\mathfrak{p}}(\phi_k^*) = 1$*, and so* $n_{\mathfrak{p}^*}(\phi_k) = n_{\mathfrak{p}}(\phi_k^*) = 1$ *also.*

(c) Let $y \in \Sigma_{\mathfrak{p}^*}(K, V_k)$ *and* $y^* \in \Sigma_{\mathfrak{p}}(K, V_k^*)$ *be elements of infinite order. Then*

$$\frac{[y^*, y]^{(\rho)}_{K,\mathfrak{p}^*}}{\exp^*_{V_k^*,\mathfrak{p}}(y^*) \cdot \exp^*_{V_k,\mathfrak{p}^*}(y)} = \frac{A_{\mathfrak{p}^*}(\phi_k^*;\rho) \cdot \mathcal{L}^{(1)}_{\mathfrak{p}^*}(\phi_k;\rho)}{A_{\mathfrak{p}}(\phi_k^*;\rho) \cdot \mathcal{L}_{\mathfrak{p}}(\phi_k)}.$$

Hence we have that $\mathcal{L}^{(1)}_{\mathfrak{p}^*}(\phi_k;\rho) \neq 0$ *if and only if* $[y^*, y]^{(\rho)}_{K,\mathfrak{p}^*} \neq 0$.

Remark 11.2. When $k = 0$, it may be proved directly using elliptic units that $\mathcal{R}^{(\rho)}_{K,\mathfrak{p}^*} \neq 0$ (see [2, Theorems 8.3 and 8.4]). □

Proof. (a) The action of complex conjugation interchanges the measures $\mu_{\mathfrak{p}}$ and $\mu_{\mathfrak{p}^*}$ (see Definition 9.1) as well as the characters ϕ_k and ϕ_k^*. Hence if $\mathcal{L}_{\mathfrak{p}}(\phi_k^*) \neq 0$, then $\mathcal{L}_{\mathfrak{p}^*}(\phi_k) \neq 0$ also.

We now conclude from the proof of Theorem 6.3 together with the two-variable main conjecture that $n_{\mathfrak{q}}(\phi_k) = n_{\mathfrak{q}^*}(\phi_k^*) = 0$, as claimed

(b) Let $W(\phi_k)$ denote the Artin root number associated to ϕ_k. Then $W(\phi_k) = 1$ since $\mathcal{L}_{\mathfrak{p}}(\phi_k) \neq 0$. The p-adic functional equation satisfied by $\mathcal{L}_{\mathfrak{p}}$ (see [13, Section 6.4]) now implies that $\mathcal{L}_{\mathfrak{p}}(\phi_k^*) = 0$. This in turn implies, via the action of complex conjugation, that $\mathcal{L}_{\mathfrak{p}^*}(\phi_k) = 0$ also.

Recall that by hypothesis ρ is LLT with $\rho = \langle \chi \rangle$. Let γ be a topological generator of $\mathrm{Gal}(K_\infty^\chi/K)$. As $\mathcal{L}_{\mathfrak{p}}(\phi_k)$ and $\mathcal{L}_{\mathfrak{p}^*}(\phi_k^*)$ are both non-zero (because $n_{\mathrm{str}}(\phi_k) = n_{\mathrm{str}}(\phi_k^*) = 0$), we see that

$$c(\phi_k) \notin (\gamma - 1) \cdot H^1_{\mathrm{Iw}}(K_\infty^\chi, T_k), \quad c(\phi_k^*) \notin (\gamma - 1) \cdot H^1_{\mathrm{Iw}}(K_\infty^\chi, T_k^*)$$

(see Theorem 10.2 and Proposition 10.4).

We now see from Corollary 10.5 (with $\phi = \phi_k$ and $\mathfrak{q} = \mathfrak{p}^*$, so $n_{\mathfrak{q}}(\phi^*; \rho) = 0$) that $c(\phi_k)_0 \in \Sigma_{\mathfrak{p}^*}(K, V)$, and that

$$\mathrm{Exp}^*_{V_k, \mathfrak{p}^*}(\mathrm{loc}_{\mathfrak{p}^*}(c(\phi_k)_0)) \doteq \mathcal{L}_{\mathfrak{p}^*}(\phi_k^*) \neq 0,$$

where the symbol $\doteq$ denotes equality up to multiplication by an explicit, non-zero factor.

This implies that $\mathrm{loc}_{\mathfrak{p}^*}(c(\phi_k)_0)$ is of infinite order, and so $r_{\mathfrak{q}^*}(\phi_k) = 1$. Hence

$$n_{\mathfrak{q}^*}(\phi_k) = n_{\mathrm{str}}(\phi_k) + r_{\mathfrak{q}^*}(\phi_k) = 1$$

also.

A similar argument shows that $\mathrm{loc}_{\mathfrak{q}}(c(\phi^*)_0)$ is of infinite order, and that

$$n_{\mathfrak{q}}(\phi_k^*) = r_{\mathfrak{q}}(\phi_k^*) = 1.$$

(c) Follows directly from Theorem 10.9 with $\phi = \phi_k$ and $\mathfrak{q} = \mathfrak{p}^*$. □

Before stating our next result, let us make some preliminary remarks in order to help to orient the reader.

Suppose that $\mathcal{L}_{\mathfrak{p}}(\phi_k) = 0$, and that $r_{\mathfrak{p}}(\phi_k) = 1$. Then, just as in the proof of Theorem 11.1(a), we see that $\mathcal{L}_{\mathfrak{p}^*}(\phi_k^*) = 0$ also.

Next, observe that $c(\phi_k)_0 \in \mathrm{Sel}_{\mathrm{rel}}(K, V_k)$ is of infinite order if and only if $c(\phi_k) \notin (\gamma - 1) \cdot H^1_{\mathrm{Iw}}(K^{\chi}_{\infty}, V_k)$, which in turn happens if and only if $n_{\mathrm{str}}(\phi_k) = 0$ (see Proposition 10.4). As $r_{\mathfrak{p}}(\phi_k) = 1$ (by assumption) and $n_{\mathfrak{p}}(\phi_k) = n_{\mathrm{str}}(\phi_k) + r_{\mathfrak{p}}(\phi_k)$, we conclude that $c(\phi_k)_0$ is of infinite order if and only if $n_{\mathfrak{p}}(\phi_k) = 1$.

Corollary 10.5 now implies that

$$\mathrm{Log}_{V_k,\mathfrak{p}}(\mathrm{loc}_{\mathfrak{p}}(c(\phi_k)_0)) \doteq \mathcal{L}_{\mathfrak{p}}(\phi_k^*),$$

and

$$\mathrm{Log}_{V_k,\mathfrak{p}^*}(\mathrm{loc}_{\mathfrak{p}^*}(c(\phi_k)_0)) \doteq \mathcal{L}_{\mathfrak{p}^*}(\phi_k^*) = 0.$$

Hence we see that $c(\phi_k)_0 \in \Sigma_{\mathfrak{p}}(K, V_k)$ (see Corollary 10.5). Furthemore, if $c(\phi_k)_0$ is of infinite order (so $n_{\mathfrak{p}}(\phi_k) = 1$), then (since $r_{\mathfrak{p}}(\phi_k) = 1$), we see that $\mathcal{L}_{\mathfrak{p}}(\phi_k^*) \neq 0$.

The following result is a generalisation of [30, Theorem 10.1] to the case of CM modular forms of higher weight.

Theorem 11.3. *Suppose that*

$$n_{\mathfrak{p}}(\phi_k) = 1, \quad n_{\mathfrak{p}^*}(\phi_k) = 0,$$

and that $\mathcal{R}^{(\rho)}_{K,\mathfrak{p}} \neq 0$. *Suppose also that*

$$r_{\mathfrak{q}}(\phi_k) = r_{\mathfrak{q}^*}(\phi_k^*) = 1. \tag{11.2}$$

Let $y \in \Sigma_{\mathfrak{p}}(K, V_k)$ *and* $y^* \in \Sigma_{\mathfrak{p}^*}(K, V_k^*)$ *be elements of infinite order. Then*

$$\frac{[y^*, y]^{(\rho)}_{K,\mathfrak{p}}}{\log_{V_k^*,\mathfrak{p}^*}(y^*) \cdot \log_{V_k,\mathfrak{p}}(y)} = \frac{A_{\mathfrak{p}}(\phi_k^*; \rho) \cdot \mathcal{L}^{(1)}_{\mathfrak{p}}(\phi_k; \rho)}{A_{\mathfrak{p}^*}(\phi_k^*; \rho) \cdot \mathcal{L}_{\mathfrak{p}^*}(\phi_k)}.$$

Proof. Follows directly from Theorem 10.9 with $\phi = \phi_k$ and $\mathfrak{q} = \mathfrak{p}$. □

Remark 11.4. Let $W(\phi_k)$ denote the Artin root number of ϕ_k, so $W(\phi_k) = \pm 1$. A theorem of Greenberg and Rohrlich implies that if $W(\phi_k) = 1$ for some $k \geq 0$, then $\mathcal{L}_{\mathfrak{p}}(\phi_{k'}) = 0$ for only finitely many $k' \geq 0$ with $k' \equiv k \pmod{p-1}$ (see [15, Theorem 4] or [28, p.184]). Rohrlich has also shown that if $W(\phi_k) = -1$, then $L^{(1)}(\psi^{2k'+1}, k'+1) = 0$ for only finitely many $k' \geq 0$ with $k' \equiv k \pmod{p-1}$. It seems reasonable to expect that an analogous result holds for $\mathcal{L}_{\mathfrak{p}}$, namely that if $W(\phi_k) = -1$, then e.g. $\mathcal{L}^{(1)}_{\mathfrak{p}}(\phi_{k'}; \chi_{\mathrm{cyc}}) = 0$ for only finitely many $k' \geq 0$ with $k' \equiv k \pmod{p-1}$. Greenberg (unpublished) has shown that this would follow from a suitable generalisation of the results of [19]. $\square$

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