Math 115A Fall Quarter Midterm Examination Solutions
February 8, 2005

(1) Prove by mathematical induction that

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1).$$

Solution:

Let $P(n)$ denote the statement

$$`1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1).'$$

Then since $1 = \frac{1}{2} \cdot 1 \cdot 2$, it follows that $P(1)$ is true.
Suppose for our inductive hypothesis that $k \geq 1$ is an integer such that $P(k)$ is true.
Then

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{1}{2}(k+1)(k+2). \end{aligned}$$

Hence it follows that $P(k+1)$ is also true. It therefore follows by induction that $P(n)$ is true for all integers $n \geq 1$.

(2)(a) Prove that if $m \geq 1$ is a composite integer, then $2^m - 1$ is not prime.
(b) Prove that there are only finitely many prime numbers that may be written in the form $n^2 - 1$ for some positive integer n.

Solution:

(a) Suppose that $m = rs$, say, with $r, s > 1$. Then

$$\begin{aligned} 2^m - 1 &= 2^{rs} - 1 \\ &= (2^r)^s - 1 \\ &= (2^r - 1)(2^r + 2^{2r} + \cdots + 2^{r(s-1)} + 1), \end{aligned}$$

and so it follows that $2^m - 1$ is not prime.

(b) Observe that

$$n^2 - 1 = (n+1)(n-1).$$

Hence, if $n \geq 3$, then $n^2 - 1$ is composite.

(3) Let n be a positive integer.

(i) Define what it means to say that n is a *perfect number*.
(ii) Show that if n is a power of 2, then n cannot be a perfect number.

Solution:

(i) The integer n is perfect if it is equal to the sum of all of its divisors except n itself.
Equivalently, n is perfect if $\sigma(n) = 2n$.

(ii) Suppose that $n = 2^m$. Then

$$\begin{aligned}\sigma(n) &= 1 + 2 + 2^2 + \cdots + 2^m \\ &= \frac{2^{m+1} - 1}{2 - 1} \\ &= 2^{m+1} - 1 \\ &< 2^{m+1}.\end{aligned}$$

Hence $\sigma(n) \neq 2n$, and so it follows that n is not perfect.

(4)(i) Use the Euclidean algorithm to find the highest common factor of 143 and 227.
(ii) State whether or not the diophantine equation

$$143x + 227y = 42$$

can be solved. (You must justify your answer.)

Solution:

(i) Applying the Euclidean algorithm yields:

$$\begin{aligned}227 &= 1 \times 143 + 84 \\ 143 &= 1 \times 84 + 59 \\ 84 &= 1 \times 59 + 25 \\ 59 &= 2 \times 25 + 9 \\ 25 &= 2 \times 29 + 7 \\ 9 &= 1 \times 7 + 2 \\ 7 &= 3 \times 2 + 1 \\ 2 &= 2 \times 1.\end{aligned}$$

Hence the HCF of 143 and 227 is 1.

(ii) The equation can be solved if and only if $(143, 227) \mid 42$. Since (as shown above), $(143, 227) = 1$, it follows that the equation can be solved.

(5)(i) State Euler's Theorem
(ii) Use Euler's Theorem to find the last digit of 7^{101}.

Solution:

(i) Let m be a positive integer, and let x be an integer with $(x, m) = 1$. Euler's Theorem states that $x^{\phi(m)} \equiv 1 \pmod{m}$, where here ϕ denotes the Euler ϕ-function.

(ii) We first observe that $\phi(10) = 4$, and so (via Euler's Theorem), we have that $7^4 \equiv 1 \pmod{10}$. Hence

$$\begin{aligned}7^{101} &= (7^4)^{25} \cdot 7 \\ &\equiv 1 \cdot 7 \pmod{10} \\ &\equiv 7 \pmod{10}.\end{aligned}$$

This implies that the last digit of 7^{101} is equal to 7.