MATH 115A SOLUTION SET I
JANUARY 13, 2005

1 (i) Suppose that $n > 1$ is a composite integer, with $n = rs$, say. Show that $2^n - 1$ is divisible by $2^r - 1$. (This shows that $2^n - 1$ is prime only if n is prime. Primes of the form $2^n - 1$ are called *Mersenne primes*.)

(ii) Show that if $2^k + 1$ is prime, then k must be a power of 2. (This explains why Fermat only had to consider numbers of the form $f_n = 2^{2^n} + 1$ (using the notation described in class) when he was hunting for primes, rather than more general numbers of the form $2^k + 1$ for an arbitrary positive integer k.)

Solution:

Suppose that $n = rs$. Then we have:

$$\begin{aligned} 2^{rs} - 1 &= (2^r)^s - 1 \\ &= [2^r - 1][(2^r)^{s-1} + ... + 2^r + 1]. \end{aligned}$$

Hence M_r divides M_n, as claimed.

To see why Fermat could restrict himself to the numbers f_n instead of considering the more general class of numbers $2^k + 1$, it suffices to show that if $2^k + 1$ is a prime, then k must be a power of two.

Suppose that p is an odd prime factor of k, so $k = mp$, say. Then

$$2^k + 1 = (2^m + 1)(1 - 2^m + 2^{2m} - ... + 2^{(p-1)m}).$$

Hence, if k is divisible by an odd prime, then $2^k + 1$ is composite, and this establishes the result we seek.

(2) The Fibonacci sequence $(F_n)_{n\geq 1}$ is defined recursively by the equations $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.

Let α be any real number larger than $\beta = \frac{1}{2}(1 + \sqrt{5})$. Prove by induction or otherwise that, for $n \geq 1$, $F_n < \alpha^n$. [Hint: You may find it helpful to note that β is a solution of the equation $x^2 = x + 1$.]

Solution:

We prove this result via induction.

First observe that the two roots of the quadratic equation $x^2 = x + 1$ are $\beta = (1 + \sqrt{5})/2$ and $\beta_1 = (1 - \sqrt{5})/2$, and we have that $\beta > \beta_1$. Hence it follows that if α is any real number strictly greater than β, then $\alpha^2 > \alpha + 1$.

Now suppose that α is a fixed real number strictly greater than β. Let $P(n)$ denote the statement '$F_n < \alpha^n$'. Since $F_1 = F_2 = 1$, it follows that $P(1)$ and $P(2)$ are true.

Let k be a positive integer such that $P(n)$ is true for all $n \leq k$. Then

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} && \text{(by definition)} \\ &< \alpha^k + \alpha^{k-1} && \text{(by our inductive hypothesis)} \\ &= \alpha^{k-1}(\alpha+1) \\ &< \alpha^{k+1} && (\text{since } \alpha^2 > \alpha + 1). \end{aligned}$$

Hence if $P(k)$ is true, then so is $P(k+1)$, and so it follows by induction that $P(n)$ is true for all positive integers n.

(3) Prove that, for $n \geq 1$,

$$\sum_1^n m = \frac{1}{2}n(n+1).$$

Solution:

We prove this result via induction.

Let $P(n)$ denote the statement '$\sum_1^n m = n(n+1)/2$'. Since $1 = 1(2)/2$, it follows that $P(1)$ is true.

Now let k be an integer for which $P(k)$ is true. Then

$$\begin{aligned} \sum_1^{k+1} m &= \sum_1^k m + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{(by our inductive hypothesis)} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Hence if $P(k)$ is true, then so is $P(k+1)$, and so it follows by induction that $P(n)$ is true for all positive integers n.

(4) Suppose that n pairs of gloves of different sizes are mixed together in a drawer. How many individual gloves must you take out if you are to be sure of having at least one complete pair. (Of course you must justify your answer!)

Solution:

There are n pairs of gloves. Imagine that each individual glove is labelled with an integer between 1 and n to indicate the pair to which it belongs. Then it follows from the Pigeonhole Principle that in any collection of $n+1$ individual gloves, at least two gloves must be labelled with the same integer, and these two gloves yield a complete pair. Hence if we remove $n+1$ gloves from the drawer, then we can be sure of having at least one complete pair. This is plainly the smallest number that will suffice, since it is certainly possible to have a collection of e.g. n individual right-hand gloves that do not form a complete pair.

(5) Prove that the integer $n = 2^{10}(2^{11} - 1)$ is not a perfect number by showing that $\sigma(n) \neq 2n$. [Hint: $2^{11} - 1 = 23 \times 89$.]

Solution:

We have

$$n = 2^{10}(2^{11} - 1) = 2^{10} \times 23 \times 89.$$

Hence,

$$\begin{aligned}\sigma(n) &= (1 + 2 + 2^2 + ... + 2^{10})(1 + 23)(1 + 89) \\ &= (2^{11} - 1) \times 24 \times 90,\end{aligned}$$

while

$$2n = 2 \times 2^{10}(2^{11} - 1) = 2^{11}(2^{11} - 1).$$

Therefore $\sigma(n) \neq 2n$, and so n is not perfect.

(6) Verify each of the following statements:
(a) No power of a prime can be a perfect number.
(b) The product of two odd primes is never a perfect number.

Solution:

(a) Suppose that p is a prime, and that a is a positive integer. Then p^a is perfect if and only if $\sigma(p^a) = p^a$. This in turn holds if and only if

$$1 + p + p^2 + ... + p^{a-1} = p^a.$$

However,

$$1 + p + p^2 + ... + p^{a-1} = \frac{p^a - 1}{p - 1} < p^a,$$

and so it follows that p^a cannot be perfect.

(b) Suppose that p and q are two distinct odd primes. Then pq is perfect if and only if $\sigma(pq) = 2pq$. Now

$$\sigma(pq) = (1 + p)(1 + q) = 1 + p + q + pq,$$

and so it follows that pq is perfect if and only if

$$1 + p + q = pq.$$

We now observe that, since p and q are odd primes, we have that $(p - 1)(q - 1) > 2$. This implies that

$$pq > 1 + p + q.$$

It therefore follows that pq cannot be perfect.

(7) (i) If n is a perfect number, prove that $\sum_{d|n}(1/d) = 2$.
(ii) Show that no proper divisor of a perfect number can be perfect.

Solution:

If n is perfect, then $\sigma(n) = 2n$, i.e.

$$\sum_{d|n} d = 2n.$$

Multiplying both sides of this equation by $1/n$ yields

$$\sum_{d|n} \frac{d}{n} = 2.$$

Since

$$\sum_{d|n} \frac{d}{n} = \sum_{d|n} \frac{1}{d},$$

the result follows.

(ii) Suppose that n is a perfect number, and let m be any proper divisor of n. Then, using (i) above, we have that

$$\begin{aligned} \sum_{d|m} \frac{1}{d} &< \sum_{d|n} \frac{1}{d} \\ &= 2. \end{aligned}$$

It therefore follows that m cannot be perfect.

(8) If $\sigma(n) = kn$, where $k \geq 3$, then the positive integer n is called a *k-perfect number*. Establish the following assertions concerning k-perfect numbers:

(a) If n is a 3-perfect number, and $3 \nmid n$, then $3n$ is 4-perfect.

(b) If n is a 5-perfect number, and $5 \nmid n$, then $5n$ is 6-perfect.

(c) If $3n$ is a $4k$-perfect number, and $3 \nmid n$, then n is $3k$-perfect.

Solution:

Let us first make the following preliminary observation. Suppose that m and n are relatively prime positive integers, and that their prime factorisations are given by

$$m = p_1^{a_1} ... p_r^{a_r}, \qquad n = q_1^{b_1} ... q_s^{b_s},$$

where $p_1, .., p_r, q_1, .., q_s$ are distict primes. Then

$$\begin{aligned} \sigma(m)\sigma(n) &= \left[\prod_{i=1}^{r} (1 + p_i + ... + p_i^{a_i})\right] \left[\prod_{j=1}^{s} (1 + q_j + ... + q_j^{b_j})\right] \\ &= \sigma(mn) \end{aligned}$$

(a) Since $3 \nmid n$, we have

$$\begin{aligned} \sigma(3n) &= \sigma(3)\sigma(n) \\ &= (1 + 3) \times 3n \\ &= 4 \times 3n. \end{aligned}$$

Hence $3n$ is 4-perfect.

(b) Since $5 \nmid n$, we have

$$\begin{aligned} \sigma(5n) &= \sigma(5)\sigma(n) \\ &= (1 + 5) \times 5n \\ &= 6 \times 5n. \end{aligned}$$

Hence $5n$ is 6-perfect.

(c) Since $3n$ is $4k$-perfect, we have that $\sigma(3n) = 4k \times 3n$. On the other hand, since $3 \nmid n$, we have

$$\sigma(3n) = (1 + 3)\sigma(n) = 4\sigma(n).$$

Hence it follows that
$$4\sigma(n) = 4k \times 3n.$$
This implies that $\sigma(n) = 3k \times n$, i.e that n is $3k$-perfect.