MATH 231A: LECTURE 4
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SCRIBE: MILO MOSES

In this lecture, we discussed the basics of the Lie group/Lie algebra correspondence in the case of matrix groups. This involved working through the examples of $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{U}_n, \mathrm{O}_n$, and Sp_n over the fields of real and complex numbers. Subsequently, we proved that the Lie algebra of a matrix group forms a real vector space and that it is closed under the Lie bracket operation $[A, B] = AB - BA$. We concluded by proving that continuous homomorphisms of Lie groups descend uniquely to maps on the level of Lie algebras, and that these induced maps preserve the $\mathbb{R}$-linear structure and the bracket structure of the Lie algebras.

To contextualize these notes, we recall a few definitions and conventions. We take $\mathbb{K}$ to be a variable indicating either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. The set of all matrices of dimension n with entries in $\mathbb{K}$ is denoted $M_n(\mathbb{K})$. A matrix group over $\mathbb{K}$ is defined to be a closed subgroup of $\mathrm{GL}_n(\mathbb{K})$. Given a matrix group G, we define the Lie algebra of G to be

$$\mathfrak{g} := \left\{ X \in M_n(\mathbb{C}) \middle| \, e^{tX} \in G \; \forall t \in \mathbb{R} \right\}.$$

At the present moment, $\mathfrak{g}$ is a set with no additional algebraic structure. It will be shown in the course of these notes that $\mathfrak{g}$, in fact, forms a sub $\mathbb{R}$-vector space of $M_n(\mathbb{K})$. Throughout these notes, we will make repeated use of the following Proposition which was originally proved in an earlier lecture:

Proposition 1.1. *Every continuous group homomorphism $\phi\colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{K})$ is of the form $\phi(t) = e^{tX}$ for a unique matrix $X \in M_n(\mathbb{K})$.*

Proof. To begin, we show that ϕ must be smooth. For all $\epsilon > 0$, choose $f_\epsilon \colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $(-\epsilon, \epsilon)$ such that $\int_{-\infty}^{\infty} f_\epsilon(x)dx = 1$. Such "bump functions" are known to always exist[1]. Now, differentiation under the integral sign implies that the convolution

$$B_\epsilon(t) := \int_{-\infty}^{\infty} \phi(x) f_\epsilon(x-t)dx$$

must be smooth as well. Changing variables and using the hypothesis that ϕ is a homomorphism implies that

$$\begin{aligned} B_\epsilon(t) &= \int_{-\infty}^{\infty} \phi(x+t) f_\epsilon(x)dx \\ &= \phi(t)\left(\int_{-\infty}^{\infty} \phi(x) f_\epsilon(x)dx\right). \end{aligned}$$

Letting I denote the identity matrix, we compute

$$\begin{aligned} \limsup_{\epsilon\to 0}\left\|\int_{-\infty}^{\infty} \phi(x) f_\epsilon(x)dx - I\right\| &= \limsup_{\epsilon\to 0}\left\|\int_{-\infty}^{\infty} (\phi(x)-I) f_\epsilon(x)dx\right\| \\ &\leq \limsup_{\epsilon\to 0}\sup_{-\epsilon\leq x\leq\epsilon}\|\phi(x)-I\|. \end{aligned}$$

The hypothesis ϕ is a homomorphism implies $\phi(0) = I$, and thus the hypothesis that ϕ is continuous implies that $\limsup_{\epsilon\to 0}\sup_{-\epsilon\leq x\leq\epsilon}\|\phi(x)-I\|=0$. We conclude that

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \phi(x) f_\epsilon(x)dx = I.$$

[1]For instance, we can take take $f_\epsilon(x) = \lambda_\epsilon \cdot e^{1/(x^2-\epsilon^2)}$ on $-\epsilon \leq x \leq \epsilon$ and $f_\epsilon(x) = 0$ otherwise, for an appropriate choice of $\lambda_\epsilon \in \mathbb{R}$.

Thus, for sufficiently small ϵ the integral $\int_{-\infty}^{\infty} \phi(x) f_\epsilon(x) dx$ is invertible and the function

$$\phi(t) = B_\epsilon(t) \left(\int_{-\infty}^{\infty} \phi(x) f_\epsilon(x) dx \right)^{-1}$$

is smooth.

We now show that ϕ must take the form $\phi(t) = e^{tX}$. Set

$$X := \lim_{\epsilon \to 0} \frac{d}{dt} \phi(t) \bigg|_{t=0}.$$

Since $\phi(0) = I$ and ϕ is continuous, there exists an open neighborhood $U \ni 0$ in $\mathbb{R}$ on which $\log \phi(t)$ is defined for all $t \in U$. Choosing $t \in U$, we compute using Taylor series expansions

$$\begin{aligned}
\log \phi(t) &= \log \phi(t/m)^m \\
&= m \cdot \log \phi(t/m) \\
&= m \cdot \log \left(I + \frac{t}{m} X + O\left((t/m)^2\right) \right) \\
&= m \cdot \left(\frac{t}{m} X + O\left((t/m)^2\right) \right) \\
&= tX + O(t^2/m).
\end{aligned}$$

Taking $m \to \infty$, we conclude $\log(\phi(t)) = tX$ and thus $\phi(t) = e^{tX}$. Seeing as for any $t \in \mathbb{R}$ we will have $t/m \in U$ for large enough m, we conclude that $\phi(t) = e^{tX}$ for all $t \in \mathbb{R}$.

Seeing as X was defined intrinsically in terms of ϕ, we conclude that the expression of ϕ in the form $\phi(t) = e^{tX}$ is unique. Since $\frac{d}{dt}\phi(t)$ is real whenever $\phi(t)$ is real, we conclude that $X \in M_n(\mathbb{R})$ whenever ϕ has codomain $\mathrm{GL}_n(\mathbb{R})$. This completes the proof. $\square$

We illustrate the utility of this proposition with a corollary, which we will use implicitly in all of our computations of Lie algebras over the field of real numbers:

Corollary 1.2. *If G is a matrix group over $\mathbb{R}$, then its Lie algebra $\mathfrak{g}$ is a subset of $M_n(\mathbb{R})$.*

Proof. If $e^{tX} \in G$ for all $t \in \mathbb{R}$, then $\phi(t) = e^{tX}$ defines a continuous group homomorphism $\phi\colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$. Thus Proposition 1.1 says that $\phi(t) = e^{tY}$ for some matrix $Y \in M_n(\mathbb{R})$, and uniqueness guarantees that $Y = X$. □

We can now move on to the computation of the Lie algebras associated with various Lie groups.

1.1. The group $\mathrm{GL}_n(\mathbb{K})$.

The most basic Lie group is $\mathrm{GL}_n(\mathbb{K})$, and hence its associated Lie algebra will be the most basic Lie algebra. The computation here is in a sense trivial, but it uses some key themes and ideas which will be repeatedly be used implicitely and explicitely throughout the rest of these notes. To begin, we recall a Lemma which was proved in an earlier lecture:

Lemma 1.3. *For any matrix $M \in M_n(\mathbb{K})$,*

$$\det\left(e^{X}\right) = e^{\mathrm{tr}(X)}.$$

Proof. Suppose first that X is a diagonal matrix with entries $\lambda_1 ... \lambda_n$. The trace of a matrix is the sum of its diagonal entries, and thus

$$e^{\mathrm{tr}(X)} = e^{\sum_{i=1}^{n} \lambda_i}.$$

The exponential of X will be a diagonal matrix with entries $e^{\lambda_1} ... e^{\lambda_n}$. The determinant of a diagonal matrix is the product of its entries, and thus

$$\det\left(e^X\right) = \prod_{i=1}^{n} e^{\lambda_i} = e^{\sum_i \lambda_i}.$$

Thus, $\det\left(e^X\right) = e^{\operatorname{tr}(X)}$ for diagonal matrices. Since the determinant and the trace are invariants of conjugacy classes in GL_n and conjugation passes through the exponential, we conclude that $\det\left(e^X\right) = e^{\operatorname{tr}(X)}$ for all diagonalizable matrices. Since diagonalizable matrices are dense in the space of all matrices, we conclude the result. □

With this lemma in hand, our main computation is immediate:

Proposition 1.4. *The Lie algebra of* $\mathrm{GL}_n(\mathbb{K})$ *is* $M_n(\mathbb{K})$*.*

Proof. By the definition of the Lie algebra associated to a matrix group,

$$\mathfrak{g} = \left\{ X \in M_n(\mathbb{K}) \middle| \, e^{tX} \in \mathrm{GL}_n(\mathbb{K}) \ \forall t \in \mathbb{R} \right\}.$$

By Lemma 1.3 for all $X \in M_n(\mathbb{K})$ we have $\det\left(e^{tX}\right) = e^{t \operatorname{tr}(X)} \neq 0$, and thus e^{tX} is always in $\mathrm{GL}_n(\mathbb{K})$. Hence, $X \in \mathfrak{g}$ for all $X \in M_n(\mathbb{K})$ as desired. □

1.2. **The group** $\mathrm{SL}_n(\mathbb{K})$**.**

We now move on to the $\mathrm{SL}_n(\mathbb{K})$, the special linear group of matricies with determinant 1. The computation here is again quite simple, but demonstrates key themes will will be reccuring throughout these notes:

Proposition 1.5. *The Lie algebra of* $\mathrm{SL}_n(\mathbb{K})$ *is the set of trace zero matrices in* $M_n(\mathbb{K})$*.*

Proof. By the definition of the Lie algebra associated to a matrix group,

$$\mathfrak{g} = \left\{ X \in M_n(\mathbb{K}) \middle| \, e^{tX} \in \mathrm{SL}_n(\mathbb{K}) \ \forall t \in \mathbb{R} \right\}.$$

By Lemma 1.3 for all $X \in M_n(\mathbb{K})$ we have $\det\left(e^{tX}\right) = e^{t\,\mathrm{tr}(X)}$. Thus, $e^{tX} \in \mathrm{SL}_n(\mathbb{K})$ for all $t \in \mathbb{R}$ if and only if $e^{t\,\mathrm{tr}(X)} = 1$ for all $t \in \mathbb{R}$. From complex analysis we know that $e^{t\,\mathrm{tr}(X)} = 1$ if and only if $t\,\mathrm{tr}(X) \in 2\pi i\mathbb{Z}$. This cannot be true for all $t \in \mathbb{R}$ unless $\mathrm{tr}(X) = 0$. Thus, we conclude that $e^{tX} \in \mathrm{SL}_n(\mathbb{K})$ for all $t \in \mathbb{R}$ if and only if X has trace zero, as desired. □

1.3. **The groups U_n and SU_n.**

We now compute the Lie algebras associated to U_n and SU_n, the unitary and special unitary groups. The unitary group U_n is defined to be the set of matricies in $A \in M_n(\mathbb{C})$ such that $A^* = A$, where A^* denotes the conjugate transpose. The special unitary group is defined as $\mathrm{SU}_n = \mathrm{U}_n \cap \mathrm{SL}_n(\mathbb{C})$. For these computations we require a small lemma:

Lemma 1.6. *For all $X \in M_n(\mathbb{C})$, $\left(e^X\right)^* = e^{X^*}$.*

Proof. Since $(X+Y)^* = X^* + Y^*$ and $(X^n)^* = (X^*)^n$, this result follows immediately from the fact that the Taylor expansion of e^X has only real coefficients. □

Proposition 1.7. *The Lie algebra of U_n is the set of matrices $X \in M_n(\mathbb{C})$ with the property $X^* = -X$.*

Proof. By the definition of the Lie algebra associated to a matrix group,

$$\mathfrak{g} = \left\{ X \in M_n(\mathbb{C}) \middle|\, e^{tX} \in \mathrm{U}_n \ \ \forall t \in \mathbb{R} \right\}.$$

By Lemma 1.6, the condition that $\left(e^{tX}\right)^* = \left(e^{tX}\right)^{-1}$ is equivalent to the condition $e^{tX^*} = e^{-tX}$. By the uniqueness of matrix-exponential forms for continuous group homomorphisms from Proposition 1.1, we conclude that $e^{tX^*} = e^{-tX}$ for all $t \in \mathbb{R}$ if and only if $X^* = -X$. □

Corollary 1.8. *The Lie algebra of* SU_n *is the set of matrices* $X \in M_n(\mathbb{C})$ *with trace zero and the property* $X^* = -X$*.*

Proof. This follows immediately from combining the computations of the Lie algebras of SU_n in Proposition 1.5 and U_n in Proposition 1.7, since $\mathrm{SU}_n = \mathrm{U}_n \cap \mathrm{SL}_n(\mathbb{C})$. □

1.4. **The groups O_n and SO_n.**

In this section we compute the Lie algebras of O_n and SO_n, the orthogonal and special orthogonal groups. The orthogonal group O_n is defined to be the set of matricies $A \in M_n(\mathbb{R})$ for which $A^T = A$, where A^T denotes the transpose of A. The special orthogonal group is defined as $\mathrm{SO}_n = O_n \cap \mathrm{SL}_n(\mathbb{R})$. Since the conjugate transpose restricts to the transpose when applied to a real matrix, we may equivalently define $\mathrm{O}_n = \mathrm{U}_n \cap \mathrm{GL}_n(\mathbb{R})$. The computation of the Lie algebra of O_n follows directly from our earlier results:

Proposition 1.9. *The Lie algebra of* O_n *is the set of matrices* $X \in M_n(\mathbb{R})$ *with the property* $X^T = -X$*.*

Proof. Since O_n=$\mathrm{U}_n \cap \mathrm{GL}_n(\mathbb{R})$, the Lie algebra of O_n will be the set of real matrices with $X^* = -X$ by Proposition 1.7. Since the conjugate transpose is equal to the transpose for real matrices, we conclude the result. □

We now observe the following. Since the transpose fixes the diagonal entries of a matrix $\mathrm{tr}\left(X^T\right) = X$, and since the trace is linear $\mathrm{tr}(-X) = -\mathrm{tr}(X)$. Thus, any matrix in the Lie algebra of O_n must have $\mathrm{tr}(X) = \mathrm{tr}(X^T) = \mathrm{tr}(-X) = -\mathrm{tr}(X)$. In other words, any matrix Xin the Lie algebra of O_n must have trace zero. This is an especially curious observation given the fact that the Lie algebra of $\mathrm{SL}_n(\mathbb{R})$ consists of trace zero matrices by Proposition 1.5, and thus the Lie algebras of SO_n and O_n must be equal. The fact that SO_n

and O_n have the same Lie algebras is part of a more general phenominon. The Lie group O_n is disconnected, since there is no way to continuously deform between a matrix with determinant $+1$ and determinant -1. The Lie subgroup SO_n is the connected component of the identity[2]. Here is the general result:

Proposition 1.10. *Let G be a Lie group. Let G° be the connected component of the identity of G. We have that G° is a closed normal matrix subgroup of G. The Lie algebras of G and G° are equal.*

Proof. The fact that G° is topologically closed is immediate, since connected components are maximal closed sets. To show that G° is a subgroup we choose two matrices $X, Y \in G^\circ$, seeking to prove $XY \in G^\circ$. By the definition of G°, we know that there exists continuous paths $\gamma_X, \gamma_Y : [0,1] \to G$ such that $\gamma_X(0) = \gamma_Y(0) = I$ and $\gamma_X(1) = X$, $\gamma_Y(1) = Y$. The product path $\gamma_{XY} : [0,1] \to G$ defined by $\gamma_{XY}(t) = \gamma_X(t)\gamma_Y(t)$ is continuous as well. Since $\gamma_{XY}(0) = \gamma_X(0)\gamma_Y(0) = I$ and $\gamma_{XY}(1) = \gamma_X(1)\gamma_Y(1) = XY$ we have thus exhibited a path connecting I to XY, and hence $XY \in G^\circ$ as desired.

To see that G° is a normal subgroup, let $\gamma : [0,1] \to G$ be a continuous path of matrices connecting $\gamma(0) = I$ to $\gamma(1) = X$. For any $Y \in G$, the path $\widetilde{\gamma}(t) = Y\gamma(t)Y^{-1}$ connects $\widetilde{\gamma}(0) = Y\gamma(0)Y^{-1} = I$ to $\widetilde{\gamma(1)} = Y\gamma(1)Y^{-1} = YXY^{-1}$. Thus, $YXY^{-1} \in G^\circ$. Since $X \in G^\circ$ and $Y \in G$ were chosen generically, we conclude the result.

We now move on to the computation of Lie algebras. The key observation is that if $X \in M_n(\mathbb{K})$ is a matrix such that $e^X \in G$, then we must have $e^X \in G^\circ$. This is because the matrices e^{tX} for $0 \le t \le 1$ give a continuous

[2] The fact that SO_n is connected is not immediately obvious. It can be seen from the fact that the group SO_n consists of proper rotations on $\mathbb{R}^n$, and all rotations can be deformed continuously from one another by varying angles.

path connecting I and e^X. Now, the fact that the Lie algebras of G and G° are equal is immediate since the condition $e^{tX} \in G\ \forall t \in \mathbb{R}$ is equivalent to the condition $e^{tX} \in G^\circ\ \forall t \in \mathbb{R}$. $\square$

1.5. **The groups** $\mathrm{Sp}_{2n}(\mathbb{R})$**,** $\mathrm{Sp}_{2n}(\mathbb{C})$**, and** Sp_{2n}**.**

Another important family of matrix groups whose Lie algebras should be computed are the symplectic grups. We recall their definition here. Fix $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The complex symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ is defined to be the subset of matrices $X \in \mathrm{GL}_{2n}(\mathbb{C})$ which preserves the bilinear form induced by Ω. Or, in other words, the set of matrices with $X^T\Omega X = \Omega$. Similarly, we define $\mathrm{Sp}_{2n}(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{C}) \cap \mathrm{GL}_{2n}(\mathbb{R})$ and $\mathrm{Sp}_{2n} = \mathrm{Sp}_{2n}(\mathbb{C}) \cap U_{2n}$.

Proposition 1.11.

(1) *The Lie algebra of* $\mathrm{Sp}_{2n}(\mathbb{C})$ *is the set of matrices in* $M_{2n}(\mathbb{C})$ *such that* $\Omega X^T \Omega = X$*.*
(2) *The Lie algebra of* $\mathrm{Sp}_{2n}(\mathbb{R})$ *is the set of matrices in* $M_{2n}(\mathbb{R})$ *such that* $\Omega X^T \Omega = X$*.*
(3) *The Lie algebra of* Sp_{2n} *is the set of matrices in* $M_{2n}(\mathbb{C})$ *such that* $\Omega X^T \Omega = X$ *and* $X^* = -X$*.*

Proof.

(1) By the definition of the Lie algebra associated to a matrix group,

$$\mathfrak{g} = \left\{ X \in M_{2n}(\mathbb{C}) \middle|\, e^{tX} \in \mathrm{Sp}_{2n}(\mathbb{C})\ \forall t \in \mathbb{R} \right\}.$$

We find now that $e^{tX} \in \mathrm{Sp}_{2n}(\mathbb{C})$ if and only if $(e^{tX})^T \Omega e^{tX} = \Omega$. Applying Lemma 1.6 and manipulating the expression, we arrive at the equation $e^{tX} = e^{-t(\Omega^{-1}X^T\Omega)}$. By the uniqueness result in Proposition

1.1, we thus find that X is in the Lie algebra if and only if $X = -\Omega^{-1}X^T\Omega$. Since $\Omega^{-1} = -\Omega$, we conclude the desired result.

(2) This follows immediately from intersecting with $\mathrm{GL}_n(\mathbb{R})$.

(3) This follows immediately from intersecting with U_{2n} and applying Proposition 1.7.

□

1.6. **Algebraic structure in Lie algebras.**

In our discussion so far, we have only defined the Lie algebras of Lie groups as sets. However, these Lie algebras happen to be closed under several arithmetic operations which endow them with powerful algebraic structure. To prove these key results, we recall the Lie product formula:

Proposition 1.12. *For all $X, Y \in M_n(\mathbb{K})$,*

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{X/m}e^{Y/m}\right)^m.$$

Proof. We first prove this formula in the case that $\|X\|, \|Y\| \leq 1/4$. In this case, X, Y, and $X+Y$ will all be in the domain of the logarithm. We compute using Taylor expansions that

$$\begin{aligned}
&\log\left(\left(e^{X/m}e^{Y/m}\right)^m\right) \\
&= m\log\left(e^{X/m}e^{Y/m}\right) \\
&= m\log\left(\left(I + \frac{X}{m} + O((\|X\|/m)^2)\right)\left(I + \frac{Y}{m} + O((\|Y\|/m)^2)\right)\right) \\
&= m\log\left(I + \frac{X+Y}{m} + O\left(((\|X\| + \|Y\|)/m)^2\right)\right) \\
&= m\left(\frac{X+Y}{m} + O\left(((\|X\| + \|Y\|)/m)^2\right)\right).
\end{aligned}$$

Taking $m \to \infty$ we thus find $\log\left(\left(e^{X/m}e^{Y/m}\right)^m\right) = X + Y$. Exponentiating gives the desired result. Now, let $X, Y \in M_n(\mathbb{R})$ be arbitrary. Choose $n \geq 4(\|X\| + \|Y\|)$. By construction $\|X/m\|, \|Y/m\| \leq 1/4$. Hence, we can apply to previously derived case to find

$$\begin{aligned} e^{X+Y} &= \left(e^{(X+Y)/n}\right)^n \\ &= \lim_{m\to\infty} \left(e^{X/mn}e^{Y/mn}\right)^{mn}. \end{aligned}$$

By power series expansion arguments the sequence $\left(e^{X/m}e^{Y/m}\right)^m$ has a limit. Hence, showing that a subsequence converges to e^{X+Y} is enough to conclude the desired result. □

These key closure results are stated below:

Proposition 1.13. *Let G be a matrix Lie group over $\mathbb{K}$, with associated Lie algebra $\mathfrak{g}$. Choose $X, Y \in \mathfrak{g}$.*

(1) *$\mathfrak{g}$ is a closed subset of $M_n(\mathbb{K})$,*
(2) *$AXA^{-1} \in \mathfrak{g}$ for all $A \in G$,*
(3) *$tX \in \mathfrak{g}$ for all $t \in \mathbb{R}$,*
(4) *$X + Y \in \mathfrak{g}$,*
(5) *$XY - YX \in \mathfrak{g}$*

Proof.

(1) Let $(X_n)_{n=1}^{\infty}$ be a sequence of matrices in $\mathfrak{g}$ with limit X_∞. Since the exponential is continuous,

$$e^{tX_\infty} = \lim_{n\to\infty} e^{tX_n}$$

for all $t \in \mathbb{R}$. Since $e^{tX_\infty} \in \mathrm{GL}_n(\mathbb{K})$ and $e^{tX_n} \in G$, the fact that G is a closed subset of $\mathrm{GL}_n(\mathbb{K})$ implies that $e^{tX_\infty} \in G$.

(2) For all $t \in \mathbb{R}$,

$$e^{tZXZ^{-1}} = Ze^{tX}Z^{-1} \in G.$$

Hence, $ZXZ^{-1} \in \mathfrak{g}$ as desired.

(3) If $X \in \mathfrak{g}$ then $e^{s(tX)} \in G$ for all $s \in \mathbb{R}$. Hence, $tX \in \mathfrak{g}$ as well.

(4) By Proposition 1.12,

$$e^{t(X+Y)} = \lim_{n\to\infty} \left(e^{tX/n}e^{tY/n}\right)^n.$$

Since $X, Y \in \mathfrak{g}$, the right hand side consists of a sequence of elements of G. Now, we recall that a matrix group G is defined to be a closed subgroup of $\mathrm{GL}_n(\mathbb{K})$. This means that if a sequence of elements of G has a limit in $\mathrm{GL}_n(\mathbb{K})$, then that limit must be in G. Thus, $e^{t(X+Y)} \in G$ for all $t \in \mathbb{R}$ so $X + Y \in \mathfrak{g}$.

(5) By part (2) of this Proposition, $e^{tX}Ye^{-tX} \in \mathfrak{g}$ for all $t \in \mathbb{R}$. Now, by the Leibnitz product rule we observe

$$\frac{d}{dt}e^{tX}Ye^{-tX}\bigg|_{t=0} = \left(Xe^{0\cdot X}\right)Ye^{-0\cdot X} - e^{0\cdot X}Y\left(Xe^{-0\cdot X}\right) = XY - YX.$$

Since $\mathfrak{g}$ is a closed subgroup of $M_n(\mathbb{K})$ by part (1) of this Proposition, the derivative of a parameterized family of elements of $\mathfrak{g}$ must lie in $\mathfrak{g}$. Hence, we conclude that $XY - YX \in \mathfrak{g}$ as desired.

□

We will make repeated use of part (5) of Proposition 1.13, as the quantities $XY - YX$ for $X, Y \in \mathfrak{g}$ will be of utmost importance. Hence, we introduce the notation $[X, Y] := XY - YX$. We call this operation the *Lie bracket.*

The important algebraic identity that the Lie bracket satisfies is the so-called Jacobi identity[3]:

Proposition 1.14. *Let G be a Lie group with associated Lie algebra $\mathfrak{g}$. For all $X, Y, Z \in \mathfrak{g}$, we have that*

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0.$$

Proof. Expanding, we find that

$$\begin{aligned}
&[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] \\
&= [XY - YX, Z] + [ZX - XZ, Y] + [YZ - ZY, X] \\
&= (XY - YX)Z - Z(XY - YX) \\
&+ (ZX - XZ)Y - Y(ZX - XZ) \\
&+ (YZ - ZY)X - X(YZ - ZY).
\end{aligned}$$

Every term in this expansion appears once with a plus sign, and once with a minus sign. Hence, we conclude the desired result. □

1.7. **The Lie group/Lie algebra correspondence.** Every Lie group has an associated Lie algebra. A very powerful tool in Lie theory is to take data on the level of Lie groups and push it down to the level of Lie algebras. In particular, every morphism of Lie groups descends to a morphism of Lie algebras[4], as we will show in this section.

[3]In fact, the term "Lie algebra" is used to abstractly refer to any vector space with a bilinear bracket operation $[\cdot,\cdot]$ satisfying the Jacobi identity.

[4]In category theory, we would say that the Lie group/Lie algebra correspondence is *functorial*.

Proposition 1.15. *Let G, H be matrix Lie groups with associated Lie algebras $\mathfrak{g}, \mathfrak{h}$ respectively. Let $\phi\colon G \to H$ be a continuous homomorphism of Lie groups. For all $X \in \mathfrak{g}$ let*

$$\widetilde{\phi}(X) = \frac{d}{dt}\phi(e^{tX})\bigg|_{t=0}$$

This induces a well-defined map $\widetilde{\phi}\colon \mathfrak{g} \to \mathfrak{h}$, uniquely characterized by the property that for all $t \in \mathbb{R}$

$$\phi\left(e^{tX}\right) = e^{t\widetilde{\phi}(X)}.$$

For all $X, Y \in \mathfrak{g}$, the following properties hold:

(1) *$\widetilde{\phi}$ is continuous,*
(2) *$\widetilde{\phi}(AXA^{-1}) = \phi(A)\widetilde{\phi}(X)\phi(A)^{-1}$ for all $A \in G$,*
(3) *$\widetilde{\phi}(tX) = t\widetilde{\phi}(X)$ for all $t \in \mathbb{R}$,*
(4) *$\widetilde{\phi}(X+Y) = \widetilde{\phi}(X) + \widetilde{\phi}(Y)$,*
(5) *$\widetilde{\phi}([X,Y]) = [\widetilde{\phi}(X), \widetilde{\phi}(Y)]$,*
(6) *For all Lie groups K and $\psi : H \to K$, we have that $\widetilde{(\psi \circ \phi)} = \widetilde{\psi} \circ \widetilde{\phi}$.*

Proof. Since ϕ is a homomorphism, the assignment $t \mapsto \phi(e^{tX})$ gives a continuous group homomorphism from $\mathbb{R}$ to H. Hence, it must be of the form $\phi(e^{tX}) = e^{t\widetilde{\phi}(X)}$ for a unique $\widetilde{\phi}(X)$ by Proposition 1.1. Thus, this defines $\widetilde{\phi}$ and shows that it is uniquely characterized by the property that $\phi\left(e^{X}\right) = e^{\widetilde{\phi}(X)}$. We now move on to demonstrating the desired properties.

(1) Choose $t > 0$ sufficiently small so that $\left\|t\widetilde{\phi}(X)\right\| \leq 1/4$. Taking a logarithm of the expression $e^{t\widetilde{\phi}(X)} = \phi(e^{tX})$ we have $\widetilde{\phi}(X) = \frac{1}{t}\log\phi(e^{tX})$. Since e^X, log, and ϕ are all continuous in a neighborhood of X, we find that $\widetilde{\phi}$ is continuous in a neighborhood of X as well. Since $X \in \mathfrak{g}$ was chosen arbitrarily, we conclude the result.

(2) We compute that

$$\begin{aligned}
e^{t\widetilde{\phi}(AXA^{-1})} &= \phi\left(e^{t(AXA^{-1})}\right) \\
&= \phi\left(Ae^{tX}A^{-1}\right) \\
&= \phi(A)\phi\left(e^{tX}\right)\phi(A)^{-1} \\
&= \phi(A)e^{t\widetilde{\phi}(X)}\phi(A)^{-1} \\
&= e^{t\phi(A)\widetilde{\phi}(X)\phi(A)^{-1}}.
\end{aligned}$$

Thus, by the uniqueness property in Proposition 1.1 we conclude the desired result.

(3) Since $e^{s(t\widetilde{\phi}(X))} = \phi(e^{stX}) = e^{s\widetilde{\phi}(tX)}$ for all $s \in \mathbb{R}$, we conclude the result by the uniqueness property in Proposition 1.1.

(4) Expanding using the Lie product formula, we find

$$\begin{aligned}
e^{t\widetilde{\phi}(X+Y)} &= \phi\left(e^{t(X+Y)}\right) \\
&= \phi\left(\lim_{m\to\infty}\left(e^{tX/m}e^{tY/m}\right)^m\right) \\
&= \lim_{m\to\infty}\left(\phi\left(e^{tX/m}\right)\left(e^{tY/m}\right)\right)^m \\
&= \lim_{m\to\infty}\left(e^{t\widetilde{\phi}(X)/m}e^{t\widetilde{\phi}(Y)/m}\right)^m \\
&= e^{t(\widetilde{\phi}(X)+\widetilde{\phi}(Y))}.
\end{aligned}$$

Thus, $\widetilde{\phi}(X+Y) = \widetilde{\phi}(X) + \widetilde{\phi}(Y)$ as desired.

(5) This result follows immediately from applying ϕ to the formula

$$\frac{d}{dt}e^{tX}Ye^{-tX}\bigg|_{t=0} = [X, Y].$$

(6) We observe that

$$\begin{aligned}(\psi \circ \phi)\left(e^{tX}\right) &= \psi\left(e^{\widetilde{\phi}(X)}\right) \\ &= e^{t(\widetilde{\psi} \circ \widetilde{\phi})(X)}.\end{aligned}$$

Thus, since $\widetilde{(\psi \circ \phi)}$ is uniquely characterized by the formula

$$(\psi \circ \phi)\left(e^{tX}\right) = e^{t\widetilde{(\psi \circ \phi)}(X)}$$

we conclude the desired result.

$\square$