MATH 231A: LECTURE 7
FEBRUARY 6, 2024
SCRIBE: BENJAMIN CHUNG

1. Representations of $\mathfrak{sl}_2(\mathbb{C})$

Recall that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of $\mathrm{SL}_2(\mathbb{C})$ consists of the 2×2 complex matrices with trace zero. Last time, we singled out the basis of $\mathfrak{sl}_2(\mathbb{C})$ given by the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy the commutation relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

We used this basis to classify the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ up to isomorphism. As for representations that are not necessarily irreducible, we have the following result.

Theorem 1.1. *Suppose $\pi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ is any finite-dimensional representation, not necessarily irreducible. With X, Y, and H as above, the following hold:*

(1) Every eigenvalue of $\pi(H)$ is an integer.
(2) $\pi(X)$ and $\pi(Y)$ are nilpotent.
(3) Define $S \in \mathrm{GL}(V)$ by

$$S = e^{\pi(X)} e^{-\pi(Y)} e^{\pi(X)}.$$

Then

$$S\pi(H)S^{-1} = -\pi(H).$$

(4) If k is an eigenvalue of $\pi(H)$, then so are $-|k|, -|k|+2, \cdots, |k|-2, |k|$.

Proof.

(1) This follows from the arguments given in the final theorem last lecture, but we shall repeat them for completeness. Let v be an eigenvector of $\pi(H)$ with eigenvalue α. By the lemma from last lecture, we have

$$\pi(H)\pi(X)v = (\alpha + 2)\pi(X)v,$$
$$\pi(H)\pi(Y)v = (\alpha - 2)\pi(Y)v,$$

so by induction, for all $k \in \mathbb{N}$,

$$\pi(H)\pi(X)^k v = (\alpha + 2k)\pi(X)^k v, \tag{1.1}$$
$$\pi(H)\pi(Y)^k v = (\alpha - 2k)\pi(Y)^k v. \tag{1.2}$$

Thus, $\pi(X)^k v$ is an eigenvector of $\pi(H)$ with eigenvalue $\alpha + 2k$; but $\pi(H)$ has only finitely many eigenvalues, so there must be some positive integer N such that $\pi(X)^N v \neq 0$, but $\pi(X)^{N+1} v = 0$. Let $v_0 = \pi(X)^N v$ and $\lambda = \alpha + 2N$, so that v_0 is an eigenvector of $\pi(H)$ with eigenvalue λ, and let

$$v_k = \pi(Y)^k v_0.$$

By (1.2), v_k is an eigenvalue of $\pi(H)$ with eigenvector $\lambda - 2k$, and by similar reasoning there is a nonnegative integer m such that $v_m \neq 0$ and $v_{m+1} = 0$.

Now, from last lecture, we have the relation

$$\pi(X)v_k = k(\lambda - (k-1))v_{k-1}$$

for all $k \in \mathbb{N}$. Therefore,

$$0 = \pi(X)v_{m+1} = (m+1)(\lambda - m)v_m,$$

but v_m and $m + 1$ are nonzero, so $\lambda = m$. This means that λ is an integer, so $\alpha = \lambda - 2N$ is also an integer.

(2) We first claim that

$$(\pi(H)-(a+2)I)^k\pi(X)=\pi(X)(\pi(H)-aI)^k \tag{1.3}$$

for all $a\in\mathbb{C}$ and all $k\in\mathbb{N}$. The proof will be by induction on k. Indeed, since $[H,X]=2X$, and Lie algebra homomorphisms preserve Lie brackets, we have $[\pi(H),\pi(X)]=2\pi(X)$, and therefore

$$\pi(H)\pi(X)=\pi(X)\pi(H)+2\pi(X). \tag{1.4}$$

Subtracting $(a+2)\pi(X)$ from both sides gives

$$(\pi(H)-(a+2)I)\pi(X)=\pi(X)(\pi(H)-aI),$$

which proves the base case $k=1$. Now suppose $m>1$, and that (1.3) has been established for $k=m-1$, so that

$$(\pi(H)-(a+2)I)^{m-1}\pi(X)=\pi(X)(\pi(H)-aI)^{m-1}.$$

Then multiplying both sides by $\pi(H)-aI$ on the right gives

$$(\pi(H)-(a+2)I)^{m-1}\pi(X)(\pi(H)-aI)=\pi(X)(\pi(H)-aI)^m,$$

and applying (1.4) on the left-hand side gives

$$(\pi(H)-(a+2)I)^{m-1}(\pi(H)\pi(X)-(a+2)\pi(X))=\pi(X)(\pi(H)-aI)^m,$$

which when simplified establishes (1.3) for $k=m$. Thus, by induction, (1.3) holds for all natural numbers k.

We continue with the proof of Point (2). Since the base field is $\mathbb{C}$, which is algebraically closed, V has a basis of generalized eigenvectors for $\pi(H)$. If v is a generalized eigenvector for $\pi(H)$ with eigenvalue λ, then

$$(\pi(H)-\lambda I)^m v=0$$

for some nonnegative integer m, so by (1.3),

$$(\pi(H)-(\lambda+2)I)^m\pi(X)v=0.$$

Thus, either $\pi(X)v = 0$, or $\pi(X)v$ is a generalized eigenvector of $\pi(H)$ with eigenvalue $\lambda + 2$. Then by induction, for each nonnegative integer k, either $\pi(X)^k v = 0$, or $\pi(X)^k v$ is a generalized eigenvector of $\pi(H)$ with eigenvector $\lambda + 2k$. As $\pi(H)$ has only finitely many eigenvalues, there must eventually be some $N_v \in \mathbb{N}$ such that $\pi(X)^{N_v} v = 0$.

Thus, for each generalized eigenvector u, there is an $N_u \in \mathbb{N}$ such that $\pi(X)^{N_u} u = 0$. Taking $\{u_1, \cdots, u_n\}$ to be a basis for V of generalized eigenvectors of $\pi(H)$, we then have

$$\pi(X)^{\max\{N_{u_1}, \cdots, N_{u_n}\}} = 0,$$

so $\pi(X)$ is nilpotent. A similar argument using the commutation relation $[H, Y] = -2Y$ shows that $\pi(Y)$ is also nilpotent.

(3) By the proposition from last time, we have

$$\begin{aligned} S\pi(H)S^{-1} &= e^{\pi(X)} e^{-\pi(Y)} e^{\pi(X)} \pi(H) e^{-\pi(X)} e^{\pi(Y)} e^{-\pi(X)} \\ &= \mathrm{Ad}_{e^{\pi(X)}} \circ \mathrm{Ad}_{e^{-\pi(Y)}} \circ \mathrm{Ad}_{e^{\pi(X)}}(\pi(H)) \\ &= e^{\mathrm{ad}_{\pi(X)}} \circ e^{\mathrm{ad}_{-\pi(Y)}} \circ e^{\mathrm{ad}_{\pi(X)}}(\pi(H)). \end{aligned} \tag{1.5}$$

To evaluate (1.5), we first compute

$$\begin{aligned} \mathrm{ad}_{\pi(X)}(\pi(X)) &= [\pi(X), \pi(X)] \\ &= 0, \\ \mathrm{ad}_{\pi(X)}(\pi(Y)) &= [\pi(X), \pi(Y)] \\ &= \pi(H), \\ \mathrm{ad}_{\pi(X)}(\pi(H)) &= [\pi(X), \pi(H)] \\ &= -2\pi(X). \end{aligned}$$

whence

$$\begin{aligned} e^{\mathrm{ad}_{\pi(X)}}(\pi(H)) &= \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{ad}_{\pi(X)}^k(\pi(H)) \\ &= \pi(H) - 2\pi(X). \end{aligned}$$

Continuing, we compute

$$\begin{aligned}
\mathrm{ad}_{-\pi(Y)}(\pi(X)) &= [-\pi(Y), \pi(X)] \\
&= \pi(H), \\
\mathrm{ad}_{-\pi(Y)}(\pi(Y)) &= [-\pi(Y), \pi(Y)] \\
&= 0, \\
\mathrm{ad}_{-\pi(Y)}(\pi(H)) &= [-\pi(Y), \pi(H)] \\
&= -2\pi(Y),
\end{aligned}$$

so

$$\begin{aligned}
e^{\mathrm{ad}_{-\pi(Y)}} \circ e^{\mathrm{ad}_{\pi(X)}}(\pi(H)) &= e^{\mathrm{ad}_{-\pi(Y)}}(\pi(H) - 2\pi(Y)) \\
&= (\pi(H) - 2\pi(Y)) - 2\left(\pi(X) + \pi(H) + \frac{1}{2}(-2\pi(Y))\right) \\
&= -\pi(H) - 2\pi(X).
\end{aligned}$$

Finally, then, (1.5) gives

$$\begin{aligned}
S\pi(H)S^{-1} &= e^{\mathrm{ad}_{\pi(X)}}(-\pi(H) - 2\pi(X)) \\
&= -(\pi(H) - 2\pi(X)) - 2(\pi(X)) \\
&= -\pi(H),
\end{aligned}$$

as desired.

(4) We continue to use the notation introduced in the proof of Point (1). Suppose α is an eigenvalue of $\pi(H)$ with corresponding eigenvector v, and first suppose that $\alpha \geq 0$. Since we showed at the end of the proof of Point (1) that $\lambda = m$, it follows that $v_0, v_1, \cdots, v_\lambda$ are nonzero, with corresponding eigenvalues $\alpha + 2N, \alpha, \cdots, -\alpha, -(\alpha + 2N)$. This establishes (4) in the case that $\alpha \geq 0$. On the other hand, suppose $\alpha \leq 0$. As $-\pi(H)S = S\pi(H)$ by Point (3), we have

$$-\pi(H)Sv = S\pi(H)v = \alpha Sv,$$

so Sv is an eigenvector of $\pi(H)$ with eigenvalue $-\alpha$. Thus, the preceding argument goes through with α replaced with $-\alpha$.

□

2. The Special Linear Lie Algebra

In general, we have

$$\mathfrak{sl}_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \mathrm{tr}(X) = 0\}.$$

This will serve as our prototypical example of a Lie algebra, and we shall use it to illustrate many of our definitions. If $X \in \mathfrak{sl}_n(\mathbb{C})$ and $A \in \mathfrak{gl}_n(\mathbb{C})$, then

$$\mathrm{tr}([X, A]) = \mathrm{tr}(XA - AX) = 0.$$

Hence, $[X, A] \in \mathfrak{sl}_n(\mathbb{C})$. This means that $\mathfrak{sl}_n(\mathbb{C})$ is a (proper, nontrivial) ideal in $\mathfrak{gl}_n(\mathbb{C})$, so $\mathfrak{gl}_n(\mathbb{C})$ is not simple. In contrast,

Proposition 2.1. $\mathfrak{sl}_n(\mathbb{C})$ *is simple for* $n \geq 2$.

Proof. Suppose that $\mathfrak{k}$ is a nonzero ideal of $\mathfrak{sl}_n(\mathbb{C})$. Write E_{ij} for the elementary matrix with 1 in position (i, j) and zeroes elsewhere.

For any matrix B, we compute

$$[B, E_{ij}] = \begin{pmatrix} & & & b_{1,i} & & & \\ & & & \vdots & & & \\ & & & b_{i-1,i} & & & \\ -b_{j,1} & \cdots & -b_{j,j-1} & b_{i,i} - b_{j,j} & b_{j,j+1} & \cdots & -b_{j,n} \\ & & & b_{i+1,i} & & & \\ & & & \vdots & & & \\ & & & b_{n,i} & & & \end{pmatrix},$$

where the shown column occupies column j, the shown row occupies row i, and any unoccupied entry is zero.

In terms of components,

$$[B, E_{ij}]_{k\ell} = (BE_{ij})_{k\ell} - (E_{ij}B)_{k\ell}$$

$$
\begin{aligned}
&= \sum_{m=1}^{n} \left(B_{km}(E_{ij})_{m\ell} - (E_{ij})_{km}B_{m\ell}\right) \\
&= \sum_{m=1}^{n} \left(B_{km}\delta_{im}\delta_{j\ell} - \delta_{ik}\delta_{jm}B_{m\ell}\right) \\
&= B_{ki}\delta_{j\ell} - B_{j\ell}\delta_{ik}. \qquad (2.1)
\end{aligned}
$$

With these in hand, we wish to show that $\mathfrak{k} = \mathfrak{sl}_n(\mathbb{C})$. We break the proof into three steps.

Step 1. In this step, we show that $\mathfrak{k}$ must contain at least one non-diagonal matrix (this is where we use the fact that $n \geq 2$). Indeed, if a nonzero $A \in \mathfrak{k}$ is diagonal, then its diagonal entries cannot all be the same, as the trace of A must be zero. Say $A_{ii} \neq A_{jj}$ for some $i \neq j$. Then by (2.2),

$$[A, E_{ij}]_{k\ell} = A_{ki}\delta_{j\ell} - A_{j\ell}\delta_{ik},$$

whose (i,j)-entry is $A_{ii} - A_{jj} \neq 0$, so $[A, E_{ij}]$ is not diagonal. Moreover, $[A, E_{ij}] \in \mathfrak{k}$, since $E_{ij} \in \mathfrak{sl}_n(\mathbb{C})$. This shows that $\mathfrak{k}$ has at least one non-diagonal matrix.

Step 2. We claim that $\mathfrak{k}$ must contain at least one of the E_{ij}s. To show this, let $A \in \mathfrak{k}$ be the non-diagonal matrix guaranteed from Step 1. Say $A_{ij} \neq 0$ for some $i \neq j$. From (2.1), we compute

$$
\begin{aligned}
[[A, E_{ji}], E_{ji}]_{k\ell} &= [A, E_{ji}]_{kj}\delta_{i\ell} - [A, E_{ji}]_{i\ell}\delta_{jk} \\
&= (A_{kj}\delta_{ij} - A_{ij}\delta_{jk})\delta_{i\ell} - (A_{ij}\delta_{i\ell} - A_{i\ell}\delta_{ji})\delta_{jk} \\
&= -2A_{ij}\delta_{jk}\delta_{i\ell} \\
&= -2A_{ij}E_{ji},
\end{aligned}
$$

and therefore

$$[[A, E_{ji}], E_{ji}] = -2A_{ij}E_{ji},$$

so $E_{ji} \in \mathfrak{k}$.

Step 3. In this step, we finally show that $\mathfrak{k}$ must be all of $\mathfrak{sl}_n(\mathbb{C})$. We start by noting two special cases of (2.1). Firstly, if $i \neq k$, then

$$\begin{aligned}[E_{ij}, E_{jk}]_{ab} &= (E_{ij})_{aj}\delta_{kb} - (E_{ij})_{kb}\delta_{ja} \\ &= \delta_{ia}\delta_{kb} - \delta_{ik}\delta_{jb}\delta_{ja} \\ &= \delta_{ia}\delta_{kb} \\ &= (E_{ik})_{ab},\end{aligned}$$

so

$$[E_{ij}, E_{jk}] = E_{ik} \text{ when } i \neq k, \tag{2.2}$$

and secondly,

$$\begin{aligned}[E_{ij}, E_{ji}]_{ab} &= (E_{ij})_{aj}\delta_{ib} - (E_{ij})_{ib}\delta_{ja} \\ &= \delta_{ia}\delta_{ib} - \delta_{jb}\delta_{ja} \\ &= (E_{ii} - E_{jj})_{ab},\end{aligned}$$

so

$$[E_{ij}, E_{ji}] = E_{ii} - E_{jj}. \tag{2.3}$$

From Step 2, we know there is some elementary matrix $E_{uv} \in \mathfrak{k}$. Using this, we will show that $E_{k\ell} \in \mathfrak{k}$ for all $k \neq \ell$.

Indeed, suppose $k \neq \ell$, and that $k \neq u$. Then by repeated applications of (2.2),

$$\begin{aligned}E_{k\ell} &= [E_{ku}, E_{u\ell}] \\ &= -[E_{u\ell}, E_{ku}] \\ &= -[[E_{uv}, E_{v\ell}], E_{ku}],\end{aligned}$$

so $E_{k\ell} \in \mathfrak{k}$. If, on the other hand, $k \neq v$, then we similarly have

$$\begin{aligned}E_{k\ell} &= [E_{kv}, E_{v\ell}] \\ &= [[E_{ku}, E_{uv}], E_{v\ell}],\end{aligned}$$

which is also in $\mathfrak{k}$. This covers all cases, as we assumed $u \neq v$. Thus, $E_{k\ell} \in \mathfrak{k}$ for all $k \neq \ell$, as claimed.

Moreover, (2.3) then gives that $E_{kk} - E_{\ell\ell} \in \mathfrak{k}$ for all $k \neq \ell$. This, in particular, tells us that

$$\{E_{k\ell} : k \neq \ell\} \cup \{E_{kk} - E_{k+1,k+1} : k = 1, \cdots, n-1\} \subseteq \mathfrak{k}.$$

But $\mathfrak{sl}_n(\mathbb{C})$ has complex dimension $n^2 - 1$, and the left-hand side above consists of $n^2 - n + (n-1) = n^2 - 1$ linearly independent elements, so it must be a complex basis for $\mathfrak{sl}_n(\mathbb{C})$. We conclude that $\mathfrak{k} = \mathfrak{sl}_n(\mathbb{C})$.

□

Now, let

$$\mathfrak{h} := \{h \in \mathfrak{sl}_n(\mathbb{C}) \mid h \text{ is diagonal}\}$$

(we shall later see that this is an example of a Cartan subalgebra). Then $\mathfrak{h}$ is a subalgebra of $\mathfrak{sl}_n(\mathbb{C})$, since products and sums (and therefore commutators) of diagonal matrices are diagonal. It has codimension 1 in the n-dimensional space of diagonal $n \times n$ complex matrices, since it is the kernel of the trace map, so $\dim_{\mathbb{C}} \mathfrak{h} = n - 1$. Moreover, since diagonal matrices commute, $\mathfrak{h}$ is abelian.

Proposition 2.2. *$\mathbb{C}E_{ij}$ is a left $\mathfrak{h}$-submodule of $\mathfrak{sl}_n(\mathbb{C})$ of complex dimension* 1*.*

Proof. If $h = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ is an arbitrary element of $\mathfrak{h}$, then

$$\begin{aligned}[h, E_{ij}] &= hE_{ij} - E_{ij}h \\ &= \lambda_i E_{ij} - \lambda_j E_{ij} \\ &= (\lambda_i - \lambda_j)E_{ij}.\end{aligned}$$

This shows that $\mathbb{C}E_{ij}$ is an $\mathfrak{h}$-submodule of $\mathfrak{sl}_n(\mathbb{C})$; its complex dimension is 1 because it is spanned by E_{ij}. □

It follows from Proposition 2.2, and from counting complex dimensions, that we have a decomposition

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{h} \oplus \sum_{i \neq j} \mathbb{C}E_{ij}$$

into left $\mathfrak{h}$-submodules. Each $\mathfrak{h}$-module $\mathbb{C}E_{ij}$ in this decomposition then yields a complex 1-dimensional representation of $\mathfrak{h}$ in the usual way, which by the proof of Proposition 2.2 is given by

$$\mathcal{E}_{ij} \colon \mathfrak{h} \longrightarrow \mathbb{C}$$
$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \longmapsto \lambda_i - \lambda_j.$$

Definition 2.3. The $n(n-1)$ representations of $\mathfrak{h}$ that arise in the above way are called the **roots** of $\mathfrak{sl}_n(\mathbb{C})$ with respect to $\mathfrak{h}$. The set of roots is denoted by Φ.

Here are some properties of Φ:

(1) If $\alpha \in \Phi$, then $-\alpha \in \Phi$, for if $\alpha = \mathcal{E}_{ij}$, then $-\alpha = \mathcal{E}_{ji}$.

(2) Define $\alpha_i \in \Phi$ by $\alpha_i := \mathcal{E}_{i,i+1}$. Then the set

$$\Pi := \{\alpha_1, \cdots, \alpha_{n-1}\}$$

of so-called **fundamental roots** is a basis for $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$. Indeed, suppose there is a dependence relation

$$\sum_{i=1}^{n-1} a_i \alpha_i = 0$$

amongst the α_is, for some complex a_is. Then for any $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ with $\lambda_1 + \cdots + \lambda_n = 0$, we would have

$$\sum_{i=1}^{n-1} a_i(\lambda_i - \lambda_{i+1}) = 0.$$

Then for $k = 1, 2, \cdots, n-1$, taking

$$\lambda_1, \cdots, \lambda_k = \frac{n-k}{n}$$

and

$$\lambda_{k+1}, \cdots, \lambda_n = -\frac{k}{n}$$

in the dependence relation yields $a_k = 0$. Hence, our dependence relation was actually trivial, so the α_is are linearly independent, and they span $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ because

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C}) = \dim_{\mathbb{C}}(\mathfrak{h}) = n - 1.$$

(3) Observe that we have

$$\mathcal{E}_{ij} = \begin{cases} \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} & \text{if } i < j, \\ -(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{i-1}) & \text{if } i > j, \end{cases}$$

so we may write $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ consists only of positive integer combinations of elements of Π, and Φ^- consists only of negative integer combinations of elements of Π.

3. Semisimple Lie Algebras

Definition 3.1. Suppose that $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$ and that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Define the **idealizer** $I(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ by

$$I(\mathfrak{h}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{h}] \subseteq \mathfrak{h}\}.$$

Say that $\mathfrak{h}$ satisfies the **idealizer condition** if $I(\mathfrak{h}) = \mathfrak{h}$.

Proposition 3.2. *If $\mathfrak{h}$ is a subalgebra of a complex Lie algebra $\mathfrak{g}$, then $I(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ is an ideal of $I(\mathfrak{h})$, and $I(\mathfrak{h})$ is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{h}$ is an ideal.*

Proof. It is clear that $I(\mathfrak{h})$ is a vector subspace of $\mathfrak{g}$, and if $X_1, X_2 \in I(\mathfrak{h})$, and $H \in \mathfrak{h}$ is arbitrary, then by the Jacobi identity,

$$[[X_1, X_2], H] = [X_1, [X_2, H]] + [X_2, [X_1, H]].$$

Both terms on the right-hand side are in $\mathfrak{h}$ by the assumption that $X_1, X_2 \in I(\mathfrak{h})$, so $[[X_1, X_2], H]$ is also in $\mathfrak{h}$. This shows that $I(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$.

Now, the definition of $I(\mathfrak{h})$ immediately implies that $\mathfrak{h}$ is an ideal of $I(\mathfrak{h})$. To show maximality, suppose $\mathfrak{k} \subseteq \mathfrak{g}$ is another subalgebra such that $\mathfrak{h}$ is an ideal of $\mathfrak{k}$. Then, of course, every $X \in \mathfrak{k}$ satisfies $[X, \mathfrak{h}] \subseteq \mathfrak{h}$, so $\mathfrak{k}$ must be a subset of $I(\mathfrak{h})$. □

Definition 3.3. A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a **Cartan subalgebra** if $\mathfrak{h}$ is nilpotent and $I(\mathfrak{h}) = \mathfrak{h}$.

Theorem 3.4. *Every finite-dimensional Lie algebra $\mathfrak{g}$ has a Cartan subalgebra, and any two Cartan subalgebras are conjugate via an automorphism in $\mathfrak{g}$.*

Proof. We shall not have time to cover the proof, but the curious reader is referred to Sections 15 and 16 of Humphreys. □

For each $X \in \mathfrak{g}$ and $\lambda \in \mathbb{C}$, define

$$L_{\lambda,X} := \{Y \in \mathfrak{g} \mid (\mathrm{ad}_X - \lambda I)^i Y = 0 \text{ for some } i \in \mathbb{N}\}.$$

This is just the λ-generalized eigenspace in $\mathfrak{g}$ of ad_X. We know from linear algebra that $\mathfrak{g}$ splits as

$$\mathfrak{g} = \bigoplus_{\lambda} L_{\lambda,X}.$$

Definition 3.5. In the notation above, we say that an element $X \in \mathfrak{g}$ is **regular** if $\dim(L_{0,X})$ is minimal amongst all $X \in \mathfrak{g}$.

It turns out that if X is regular, $L_{0,X}$ is a Cartan subalgebra of $\mathfrak{g}$. The proof, however, is outside the scope of this course.

Definition 3.6. We say that a Lie algebra $\mathfrak{g}$ is **semisimple** if it has no nonzero soluble ideals. A Lie algebra $\mathfrak{g}$ is called **reductive** if there exists a compact matrix group K, with Lie algebra $\mathfrak{k}$, such that

$$\mathfrak{g} \cong \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{k}_{\mathbb{C}},$$

the complexification of $\mathfrak{k}$.

Example 3.7. The Lie algebra of U_n, which we shall denote $\mathfrak{u}_n$, is the set of $n \times n$ skew-Hermitian complex matrices. Then $\mathfrak{u}_n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{gl}_n(\mathbb{C})$, which follows from the fact that every $X \in M_n(\mathbb{C}) = \mathfrak{gl}_n(\mathbb{C})$ can be written as

$$X = \frac{X - X^*}{2} + i\frac{X + X^*}{2i} \in \mathfrak{u}_n + i\mathfrak{u}_n.$$

As U_n is compact, this shows that $\mathfrak{gl}_n(\mathbb{C})$ is reductive. But also note that

$$\mathfrak{gl}_n(\mathbb{C}) \cong \mathfrak{gl}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C},$$

so we have found two ways of writing $\mathfrak{gl}_n(\mathbb{C})$ as the complexification of the Lie algebra of a matrix group, but only the first revealed the reductivity of $\mathfrak{gl}_n(\mathbb{C})$.

The definition of a reductive Lie algebra may seem contrived at first, but we shall see in the next lecture that reductivity guarantees the existence of an inner product with certain nice properties.