Math 108A - Home Work # 6

Due: May 20, 2009

1. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be the standard basis for F^3, and let $\mathcal{B}$ be the basis $\{(1,4,2),\ (2,0,1),\ (1,-1,0)\}$.
   (a) Find the coordinates of $(9,4,9)$ in the basis $\mathcal{B}$.
   (b) Find the coordintates of each of the standard basis vectors in the basis $\mathcal{B}$.
   (c) Compute the change of basis matrices between $\mathcal{E}$ and $\mathcal{B}$ – recall, these are the matrices $Mat(I, \mathcal{E}, \mathcal{B})$ and $Mat(I, \mathcal{B}, \mathcal{E})$. (One is easy, and all the work for the other is in (b).)
   (d) Let $T : F^3 \to F^3$ be the linear map defined by

   $$T(x,y,z) = (2z - x - y, 2x - y - z, 2y - x - y), \quad \forall\ x, y, z \in F.$$

   Find the standard matrix for T (i.e., $Mat(T, \mathcal{E}, \mathcal{E})$) and the matrix for T relative to the basis $\mathcal{B}$ (i.e., $Mat(T, \mathcal{B}, \mathcal{B})$.

2. Let $V = \mathcal{P}_3(F)$ and let $D = \frac{d}{dx} : V \to V$ be the differentiation map. Consider the two bases for V:

   $$\mathcal{B}_1 = \{1, x, x^2, x^3\} \quad \text{and} \quad \mathcal{B}_2 = \{1, x, x(x-1), x(x-1)(x-2)\}.$$

   (a) Compute the change of basis matrices $Mat(I, \mathcal{B}_1, \mathcal{B}_2)$ and $Mat(I, \mathcal{B}_2, \mathcal{B}_1)$.
   Now Compute the following matrices for D:
   (b) $Mat(D; \mathcal{B}_1, \mathcal{B}_2)$.
   (c) $Mat(D; \mathcal{B}_2, \mathcal{B}_1)$.
   (d) $Mat(D; \mathcal{B}_2, \mathcal{B}_2)$.

3. Let A be an $n \times n$ matrix with entries in F. Show that A is invertible if and only if its columns are linearly independent (column) vectors in F^n. (Since A has n columns and $n = \dim F^n$, this problem is equivalent to showing that A is invertible if and only if its columns are a basis of F^n.)

   Hint: This is a consequence of the following result proved in class. Let $T : V \to W$ be a linear map, and let $\{v_1, \ldots, v_n\}$ be a basis for V. Then T is invertible (i.e., bijective) if and only if $\{Tv_1, \ldots, Tv_n\}$ is a basis for W.

4. LADR p. 62: Exercise 26. (Hint: Write the system as a matrix equation, and think of the matrix as a linear map $F^n \to F^n$.)