Math 108A - Home Work # 1 Solutions

1. For any $z \in \mathbb{C}$, prove that $z \in \mathbb{R}$ if and only if $\overline{z} = z$.

   Solution. Let $z = a + bi \in \mathbb{C}$ for $a, b \in \mathbb{R}$. If $z \in \mathbb{R}$, then $b = 0$ and $z = a$. Then $\bar{z} = \overline{a + 0i} = a - 0i = a = z$. Conversely, if $\bar{z} = z$, we have $a + bi = a - bi$, which implies $2bi = 0$, and hence $b = 0$. Thus $z = a \in \mathbb{R}$.

2. Is the set $\mathbb{Z}$ of integers (with the usual operations of addition and multiplication) a vector space? Why or why not?

   Solution. No. There is no operation of scalar multiplication by either the reals or the complex numbers on $\mathbb{Z}$.

3. Consider the set $V = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}$ consisting of all vectors in the first quadrant of $\mathbb{R}^2$ (considered with usual vector addition and scalar multiplication). Which vector space axioms (as listed on p. 9) hold for V and which fail? Justify your answers.

   Solution. As in the previous question, V fails to be a vector space since scalar multiplication (by negative real numbers) is not defined on V. Of the axioms listed on p. 9, the only one that fails to hold for V is the existence of additive inverses. For instance, there is no vector in the first quadrant that can be added to $(1, 1)$ to produce the 0-vector.

4. Let $\mathcal{P}(\mathbb{R})$ denote the set of all polynomials in the variable x with real coefficients. Show that $\mathcal{P}(\mathbb{R})$ is a vector space over $\mathbb{R}$. (You should *briefly* justify/check each of the axioms.)

   Solution. Let $p(x) = \sum_{i=0}^{k} a_i x^i$, $q(x) = \sum_{i=0}^{m} b_i x^i$ and $r(x) = \sum_{i=0}^{n} c_i x^i$ be polynomials with real coefficients a_i, b_i, c_i. We may assume $k = m = n$ by adding on extra terms with 0-coefficients to whichever of $p(x), q(x), r(x)$ does not have maximal degree.

   Commutativity: $p(x) + q(x) = \sum_{i=0}^{n} (a_i + b_i)x^i = \sum_{i=0}^{n} (b_i + a_i)x^i = q(x) + p(x)$.

   Associativity: $(p(x) + q(x)) + r(x) = \sum_{i=0}^{n} (a_i + b_i + c_i)x^i = p(x) + (q(x) + r(x))$.

   Additive identity: $0(x) = 0$ for all x. Then $p(x) + 0(x) = p(x)$.

   Additive Inverse: Let $-p(x) = \sum_{i=0}^{n} -a_i x^i$. Then $p(x) + -p(x) = 0(x)$.

   Multiplicative Identity: $1 \cdot p(x) = \sum_{i=0}^{n} 1 \cdot a_i x^i = p(x)$.

   Distributive Properties: $a(p(x) + q(x)) = \sum_{i=0}^{n} a(a_i + b_i)x^i = \sum_{i=0}^{n} aa_i x^i + \sum_{i=0}^{n} ab_i x^i = ap(x) + aq(x)$; and $(a + b)p(x) = \sum_{i=0}^{n} (a + b)a_i x^i = \sum_{i=0}^{n} aa_i x^i + \sum_{i=0}^{n} ba_i x^i = ap(x) + bp(x)$.

5. Let V be a vector space over F. In class we saw that any vector v has a unique additive inverse, denoted $-v$.

   (a) Using only the vector space axioms, show that for any $v \in V$, the additive inverse of v is given by $-1 \cdot v$. Mention which axiom you are using in each step of the proof. (Thus, we now know that $-v = -1 \cdot v$ for any vector $v \in V$.)

   Solution.

$$\begin{aligned} v + -1 \cdot v &= 1 \cdot v + -1 \cdot v \qquad (e) \\ &= (1 + -1)v \qquad (f) \\ &= 0v = 0 \end{aligned}$$

   where the last equality $0v = 0$ was proved in lecture. This shows that $-1 \cdot v$ is an additive inverse of v. Since $-v$ is the unique additive inverse of v, we must have $-1 \cdot v = -v$.

   (b) Let V be a vector space over F. Show that $-(-v) = v$ for any $v \in V$. Again, mention which axioms or previously proved results you are using in each step.

   Solution. By definition, $-(-v)$ is the additive inverse of $-v$, which is the additive inverse of v. We also showed in class that the additive inverse of any vector is unique. So, since $v + -v = 0$, by commutativity $-v + v = 0$, and thus v is the unique additive inverse of $-v$. Hence $v = -(-v)$.

   Alternatively, using the previous exercise, $-(-v) = -1 \cdot (-1 \cdot v) = (-1)^2 v = 1v = v$, by axioms (b) associativity of scalar product, and (e) multiplicative identity.