Math 108A - Home Work # 2 Solutions

Spring 2009

1. From LADR:

   5: Soluton.

   (a) Subspace. Closed under addition: If $x_1+2x_2+3x_3=0$ and $y_1+2y_2+3y_3=0$ then $(x_1+y_1)+2(x_2+y_2)+3(x_3+y_3)=0+0=0$. Closed under scalar multiplication: If $x_1+2x_2+3x_3=0$ then $ax_1+2ax_2+3ax_3=a(0)=0$. Additive Identity: $0+2*0+3*0=0$, so the set contains the 0 vector.

   (b) Not a subspace. The 0-vector is not in it since $0+2*0+3*0=0\neq 4$.

   (c) Not a subspace. It contains the vectors $(1,1,0)$ and $(0,0,1)$, but not their sum $(1,1,1)$.

   (d) Subspace. Contains the 0-vector since $0=5*0$. Closed under addition: If $x_1=5x_3$ and $y_1=5y_3$ then $x_1+y_1=5x_3+5y+3=5(x_3+y_3)$. Closed under scalar multiplication: If $x_1=5x_3$ then $ax_1=5ax_3$.

   13. & 15: Solution. Both are false, as can be seen by the following counterexample. Let $V=\mathbb{R}^2$ and $U_1=\mathbb{R}(1,0)$, $U_2=\mathbb{R}(1,1)$ and $W=\mathbb{R}(0,1)$. Then $\mathbb{R}^2=U_1\oplus W=U_2\oplus W$, but $U_1\neq U_2$.

2. In class, we saw that the set $\mathcal{C}(\mathbb{R})$ of all continuous functions $f:\mathbb{R}\to\mathbb{R}$ is an $\mathbb{R}$-vector space (with the 0-function $0(x)=0\ \forall x\in\mathbb{R}$ as the 0-vector). Which of the following subsets of $\mathcal{C}(\mathbb{R})$ are subspaces? Justify your answers.

   (a) $\mathcal{C}^2(\mathbb{R})=\{f\in\mathcal{C}(\mathbb{R})\mid f \text{ is twice differentiable }\}$

   Subspace. The sum of any two twice-differentiable functions is twice-differentiable ($(f+g)''=f''+g''$), and a real multiple of a twice-differentiable function is twice-differentiable ($(af)''=af''$). We also know that the 0 function is twice-differentiable.

   (b) $\mathcal{E}=\{f\in\mathcal{C}(\mathbb{R})\mid f(0)=1\}$

   Not a subspace. The 0-function is not in this set, since $0(0)=0\neq 1$.

   (c) $\mathcal{F}=\{f\in\mathcal{C}(\mathbb{R})\mid f(1)=0\}$

   Subspace. If $f(1)=g(1)=0$, then $(f+g)(1)=0+0=0$ and $af(1)=0$, and clearly the 0-function is in this set.

   (d) $\mathcal{G}=\{f\in\mathcal{C}(\mathbb{R})\mid \forall x\in\mathbb{R}\ f(x)\neq 0\}$

   Not a subspace. The 0-function is not in this set.

(e) $\mathcal{B} = \{f \in \mathcal{C}(\mathbb{R}) \mid \exists M \in \mathbb{R}\, \forall x \in \mathbb{R}\, |f(x)| \leq M\ \}$ (The set of all bounded continuous functions.)

Subspace. Clearly the 0-function is in this set, as can be seen by taking $M = 0$. Suppose $|f(x)| \leq M$ and $|g(x)| \leq N$ for all x. Then $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M + N$ for all x. Also, $|af(x)| = |a||f(x)| \leq |a|M$ for all x.

3. Recall the definition of the intersection of a family of sets indexed by a set I: If A_i is a set for each $i \in I$, then

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \forall i \in I\ \}.$$

Suppose that V is a vector space over F, and suppose that V_i is a subspace of V for each $i \in I$. Show that the intersection $\bigcap_{\imath \in I} V_i$ is also a subspace of V.

Solution. Since $0 \in V_i$ for all i, we have $0 \in \bigcap_{\imath \in I} V_i$. Suppose $u, v \in \bigcap_{\imath \in I} V_i$. Then, for each $i \in I$, $u, v \in V_i$. Since V_i is a subspace $u + v \in V_i$ and $av \in V_i$ for any $a \in F$. Thus $av, u + v \in \bigcap_{\imath \in I} V_i$. Hence $\bigcap_{\imath \in I} V_i$ is also a subspace of V.

4. **Extra Credit:** If U is any *subset* of a vector space V, we defined $span(U)$ as the set of linear combinations of elements of U, i.e.,

$$span(U) = \{c_1 u_1 + \cdots + c_n u_n \mid \forall i\ c_i \in F, u_i \in V\},$$

and we showed that $span(U)$ is a subspace of V. Show that $span(U)$ equals the intersection of all subspaces of V that contain the set U. (By the previous exercise, this gives another way of seeing that $span(U)$ is a subspace. We can also interpret this result as saying that $span(U)$ is the smallest subspace of V that contains U.)

Solution. Let W be the intersection of all subspaces of V that contain U. Clearly $span(U)$ is one such subspace, and hence $W \subseteq span(U)$. To prove the reverse inclusion, we show that $span(U) \subseteq W_i$ for any subspace W_i that contains U. In fact, this is trivial since if W_i is a subspace, it is closed under taking linear combinations of its elements; and thus if W_i contains U then it contains all linear combinations of vectors in U. In other words, $span(U) \subseteq W_i$.