Math 108A - Home Work # 3 Solutions

Spring, 2009

1. LADR Problems, p. 35:

1. Solution. Let $v \in V$, and assume $V = span(v_1, \dots, v_n)$. Then we can write

$$\begin{aligned} v &= c_1v_1 + \cdots + c_nv_n \\ &= c_1(v_1 - v_2) + (c_1 + c_2)(v_2 - v_3) + (c_1 + c_2 + c_3)(v_3 - v_4) + \cdots + (c_1 + \cdots + c_n)v_n. \end{aligned}$$

2. Solution. Suppose that $\{v_1, \dots, v_n\}$ is linearly independent, and suppose that

$$c_1(v_1 - v_2) + c_2(v_2 - v_3) + \cdots + c_{n-1}(v_{n-1} - v_n) + c_nv_n = 0.$$

Distibuting and combining each v_i-term gives

$$c_1v_1 + (c_2 - c_1)v_2 + \cdots + (c_n - c_{n-1})v_{n-1} = 0.$$

Linear independence of $\{v_1, \dots, v_n\}$ now implies that

$$c_1 = c_2 - c_1 = \cdots = c_n - c_{n-1} = 0.$$

Hence $c_2 = (c_2 - c_1) + c_1 = 0$ and similarly for each k, we have

$$c_k = (c_k - c_{k-1}) + (c_{k-1} + c_{k-2}) + \cdots + (c_2 - c_1) + c_1 = 0.$$

8. Solution. Since U is defined by 2 equations in $\mathbb{R}^5$, we can guess that U will be $5 - 2 = 3$ dimensional. So we look for 3 linearly independent vectors in U, and then prove that they in fact span U. To find simple vectors in U, notice that we can choose x_2, x_4 and x_5 freely and then $x_1 = 3x_2$ and $x_3 = 7x_4$ will be determined. We thus let one of these 3 numbers equal 1 and the other 2 equal 0, to get

$$u_1 = (3, 1, 0, 0, 0),\ u_2 = (0, 0, 7, 1, 0),\ u_3 = (0, 0, 0, 0, 1) \in U.$$

These vectors are clearly linearly independent since no two of them are nonzero in the same slot. If $x = (x_1, x_2, x_3, x_4, x_5)$ is an arbitrary element of U with $x_1 = 3x_2$ and $x_3 = 7x_4$, then it is easy to see that $x = x_2u_1 + x_4u_2 + x_5u_3 \in span(u_1, u_2, u_3)$. Hence $\{u_1, u_2, u_3\}$ is a basis for U.

Of course, any set of 3 linearly independent vectors in U would also be a valid basis here.

9. Solution. True. Let $p_0 = 1, p_1 = x$ and $p_3 = x^3$. For p_2 we cannot take x^2 since this has degree 2, but we can let $p_2 = x^3 + x^2$. Since $x^2 = p_2 - p_3$, it is obvious that $p_0, \ldots, p_3$ still span $\mathcal{P}_3(F)$, and thus form a basis since $\mathcal{P}_3(F)$ has dimension 4.

2. Let $v_1, \ldots, v_m$ and u be vectors in a vector space V. Show that

$$u \in span(v_1, \ldots, v_m) \iff span(v_1, \ldots, v_m, u) = span(v_1, \ldots, v_m).$$

Solution. $\Rightarrow$: Suppose $u \in span(v_1, \ldots, v_m)$. Thus there exist scalars $c_1, \ldots, c_m \in F$ such that $u = \sum_{i=1}^m c_i v_i$. If $v \in span(v_1, \ldots, v_m, u)$, then $v = \sum_{i=1}^m d_i v_i + d_0 u$ for scalars $d_0, \ldots, d_m \in F$. Substituting the above expression for u, we get $v = \sum_{i=1}^m (d_i + d_0 c_i) v_i \in span(v_1, \ldots, v_m)$. Hence $span(v_1, \ldots, v_m, u) \subseteq span(v_1, \ldots, v_m)$, and the reverse inclusion is trivial since any vector that is a linear combination of $v_1, \ldots, v_m$ can also be written as a linear combination of $v_1, \ldots, v_m$ and u by adding on $0 = 0u$.

$\Leftarrow$: Assume $span(v_1, \ldots, v_m, u) = span(v_1, \ldots, v_m)$, then $u \in span(v_1, \ldots, v_m, u) = span(v_1, \ldots, v_m)$.

3. Suppose that $U_1, \ldots, U_m$ are subspaces of a vector space V such that $V = U_1 + \cdots + U_m$. Show that $V = U_1 \oplus \cdots \oplus U_m$ if and only if *every* set $\{u_1, \ldots, u_m\}$ of nonzero vectors with $u_i \in U_i$ for all i is linearly independent.

Solution. $\Rightarrow$: Assume $V = U_1 \oplus \cdots \oplus U_m$, and let $u_1, \ldots, u_m$ be nonzero vectors with $u_i \in U_i$ for all i. If $c_1 u_1 + \cdots + c_m u_m = 0$, then since $c_i u_i \in U_i$ for all i, by the definition of direct sum, we must have $c_i u_i = 0$ for each i. Since u_i is not the 0-vector, this forces $c_i = 0$ for all i. Hence, $u_1, \ldots, u_m$ are linearly independent.

$\Leftarrow$: Assume that any collection $\{u_1, \ldots, u_m\}$ of nonzero vectors with $u_i \in U_i$ for all i is linearly independent. We are given that $V = U_1 + \cdots + U_m$, so to prove that V is the direct sum of these subspaces, we only need to show that $v_1 + \cdots + v_m = 0$ with $v_i \in U_i$ for all i implies that all $v_i = 0$. However, if some of the v_i were not 0, they would be linearly dependent, which would be a contradiction.