Math 108A - Home Work # 4 Solutions

1. Exercises 13, 14 on p. 36 in LADR.

   13. Suppose U and W are subspaces of $\mathbb{R}^8$ with $\dim U = 3$ and $\dim W = 5$. If $U + W = \mathbb{R}^8$, then $\dim U + W = \dim \mathbb{R}^8 = 8$. Thus $\dim(U \cap W) = \dim U + \dim W - \dim(U + W) = 3 + 5 - 8 = 0$. Since $U \cap W$ is a 0-dimensional subspace of $\mathbb{R}^8$, it must be $\{0\}$.

   14. Suppose U and W are 5-dimensional subspaces of $\mathbb{R}^9$ with $U \cap W = \{0\}$. Then $\dim U \cap W = 0$, and hence $\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 10$. Since $U + W$ must also be a subspace of $\mathbb{R}^9$, it must have dimension ≤ 9. Hence we would have $10 \leq 9$, a contradiction.

2. Consider the subspace $U = \{(x, y, z, w) \in \mathbb{R}^4 \mid x + w = y + z\}$ in $\mathbb{R}^4$.

   (a) Show that $\mathbb{R}^4 = U \oplus \mathbb{R}(0, 0, 0, 1)$.

   (b) What is $\dim U$? (Suggestion: use (a).)

   (c) Find a basis for U, and justify why it is a basis (Part (b) is helpful).

   Solution. a) We first show that $\mathbb{R}^4 = U + \mathbb{R}(0, 0, 0, 1)$. Let $v = (x, y, z, w) \in \mathbb{R}^4$. Then $v = (x, y, z, y + z - x) + (0, 0, 0, w - y - z + x) \in U + \mathbb{R}(0, 0, 0, 1)$.

   Next, we check that $U \cap \mathbb{R}(0, 0, 0, 1) = \{0\}$. Suppose that u belongs to the intersection. Then $u = (0, 0, 0, a) \in U$ for some $a \in \mathbb{R}$. Thus $0 + a = 0 + 0$, whence $a = 0$. So $u = 0$ is the only vector that belongs to the intersection.

   b) By (a), and Theorem 2.18, we have $\dim \mathbb{R}^4 = \dim U + \dim \mathbb{R}(0, 0, 0, 1)$, and thus $\dim U = \dim \mathbb{R}^4 - \dim \mathbb{R}(0, 0, 0, 1) = 4 - 1 = 3$.

   c) By (b), since $\dim U = 3$, to find a basis for U, it suffices to find 3 linearly independent vectors in U. Since the vectors in U are exactly those whose coordinates satisfy the equation $w = y + z - x$, we can get 3 linearly independent elements of U by setting one of x, y, z equal to 1 and the other 2 equal to 0. This produces the vectors $(1, 0, 0, -1), (0, 1, 0, 1)$ and $(0, 0, 1, 1)$ which are clearly linearly independent elements of U. Hence they form a basis for U.

3. Prove or give a counterexample: If $\{v_1, \ldots, v_n\}$ is any linearly dependent set of vectors, then for all i, v_i is a linear combination of the other vectors in the set.

   Solution. The statement is false. For a counterexample, consider the set $\{(0, 0), (1, 0)\}$ of vectors in F^2. It is linearly dependent since it contains the 0-vector, but $(1, 0)$ is not a linear combination of $(0, 0)$.

4. Prove that $\{v_1, \ldots, v_m\}$ is a linearly independent set of vectors if and only if any $u \in span(v_1, \ldots, v_m)$ can be written uniquely as a linear combination $u = c_1 v_1 + \cdots + c_m v_m$ for scalars $c_1, \ldots, c_m \in F$.

   Solution. $\Rightarrow$: Suppose $\{v_1, \ldots, v_m\}$ is linearly independent and let $u \in span(v_1, \ldots, v_m)$. By definition, there exist scalars $c_1, \ldots, c_m \in F$ such that $u = \sum_{i=1}^m c_i v_i$. If there exists another set of scalars $d_1, \ldots, d_m \in F$ such that we also have $u = \sum_{i=1}^n d_i v_i$, then we can subtract the second expression for u from the first to get $0 = \sum_{i=1}^m (c_i - d_i) v_i$. Since $\{v_1, \ldots, v_m\}$ is linearly independent, we must have $c_i - d_i = 0$ for all i. Thus $c_i = d_i$ for all i, and there is only one way to write u as a linear combination of $v_1, \ldots, v_m$.

   $\Leftarrow$: Clearly $0 \in span(v_1, \ldots, v_m)$ and $0 = 0v_1 + \cdots + 0v_m$. If this is the unique way of writing 0 as a linear combination of $v_1, \ldots, v_m$, then these vectors are linearly independent by definition.