Math 108A - Home Work # 8 Solutions

Spring 2009

1. LADR Problems, p. 94-95:

5. If $(w, z) \in F^2$ is an eigenvector for T with eigenvalue $\lambda \in F$, we have $T(w, z) = (z, w) = \lambda(w, z)$. This implies that $z = \lambda w$ and $w = \lambda z$. Hence $z = \lambda^2 z$ and $w = \lambda^2 w$, and since z and w cannot both be 0, we have $\lambda^2 = 1$. Therefore the only possible eigenvalues of T are $\lambda = 1$ and $\lambda = -1$. We check that both actually occur. For $\lambda = 1$ we need to find (w, z) such that $T(w, z) = (z, w) = (w, z)$. This happens for any $z, w \in F$ such that $z = w$. Thus, (z, z) is an eigenvector with eigenvalue 1 for any nonzero $z \in F$. For $\lambda = -1$ we need to find (w, z) such that $T(w, z) = (z, w) = -(w, z)$. This happens if and only if $z = -w$. Thus $(w, -w)$ is an eigenvector with eigenvalue -1 for any nonzero $w \in F$.

7. If $T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n) = \lambda(x_1, \ldots, x_n)$, then $x_1 + \cdots + x_n = \lambda x_i$ for every i. Adding up these n equations gives $n(x_1 + \cdots + x_n) = \lambda(x_1 + \cdots + x_n)$. Thus, either $\lambda = n$ or $x_1 + \cdots + x_n = 0$. If $\lambda = n$, we have $x_1 + \cdots + x_n = nx_i$ for all i. In particular $nx_i = nx_j$ for all i, j, and hence $x_i = x_j$ for all i, j. Thus the eigenvectors with eigenvalue n are precisely the vectors of the form $(x, x, \ldots, x)$ for some nonzero scalar x. Alternatively, if $x_1 + \cdots + x_n = 0$, then we see that $(x_1, \ldots, x_n) \in null(T)$, and hence it is an eigenvector with eigenvalue 0. The eigenvectors with eigenvalue 0 are precisely the nonzero vectors $(x_1, \ldots, x_n)$ such that $x_1 + \cdots + x_n = 0$.

8. Suppose $T(z_1, z_2, \ldots) = (z_2, z_3, \ldots) = \lambda(z_1, z_2, \ldots)$. This means that $z_{i+1} = \lambda z_i$ for all $i \geq 1$. Hence $z_{i+1} = \lambda z_i = \lambda^2 z_{i-1} = \cdots = \lambda^i z_1$ for all i. Thus we see that $(z_1, \lambda z_1, \lambda^2 z_1, \ldots)$ is an eigenvector with eigenvalue λ for any $z_1 \neq 0$. In particular, every element of F occurs as an eigenvalue of T. (This is only possible in infinite-dimensional vector spaces by Corollary 5.9)

10. Suppose $Tv = \lambda v$ for some $v \neq 0$. Then, by definition of the inverse of T, we have $T^{-1}(\lambda v) = v$, and by linearity we have $T^{-1}(v) = v/\lambda$. This shows that $1/\lambda$ is an eigenvalue of T^{-1}. Conversely, if λ^{-1} is an eigenvalue of T^{-1}, the same argument shows that $(\lambda^{-1})^{-1} = \lambda$ is an eigenvalue of $(T^{-1})^{-1} = T$.

11. Suppose that $ST(v) = \lambda v$ for some $v \neq 0$. First suppose that $\lambda \neq 0$. Then $TST(v) = T(\lambda v) = \lambda T(v)$. Also $T(v) \neq 0$ since $S(T(v)) = \lambda v \neq 0$ (This is ESSENTIAL, and it is why we need to divide the proof into 2 cases). Thus $T(v)$ is an eigenvector of TS with eigenvalue λ.

Now suppose that $\lambda = 0$. As above, we have $TST(v) = \lambda T(v) = 0$, so $T(v) \in null(TS)$ is an eigenvector with eigenvalue 0 if $T(v) \neq 0$. However, if $T(v) = 0$, then we know that T is not injective, and thus not surjective either by 3.21. This means that TS is not surjective, and hence TS is not injective by 3.21 again. This means that $null(TS)$ contains a nonzero vector, which must be an eigenvector with eigenvalue 0.

Altogether, we have shown that any eigenvalue of ST is also an eigenvalue of TS. A symmetric argument, swapping the roles of T and S, shows that any eigenvalue of TS is also an eigenvalue of ST. Thus, we conclude that ST and TS have the same eigenvalues.

12. Suppose $T : V \to V$ has every nonzero $v \in V$ as an eigenvector. We first show that each $v \in V$ has the same eigenvalue. If this is not the case, we can find nonzero vectors u and v such that $T(u) = \lambda_1 u$ and $T(v) = \lambda_2 v$ for $\lambda_1 \neq \lambda_2$. Theorem 5.6 implies that u and v are linearly independent. Then $T(u + v) = \lambda_1 u + \lambda_2 v$, but we must also have $T(u + v) = \lambda_3(u + v)$ for some $\lambda_3 \in F$, since *every* vector in V is an eigenvector of T. Subtracting these two equations gives $0 = (\lambda_1 - \lambda_3)u + (\lambda_2 - \lambda_3)v$ and then the linear independence of u and v implies that $\lambda_1 = \lambda_2 = \lambda_3$, a contradiction. Thus, there is a single eigenvalue $\lambda \in F$ such that $T(v) = \lambda v$ for all $v \in V$. But this is the same as saying $T = \lambda I_V$.

2. As in Ex. 7, consider the matrix ($n = 3$)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(a) Find a change of basis matrix C such that $C^{-1}AC$ is diagonal. What is this diagonal matrix?

Solution. From Ex. 7, we know that the eigenvalues of A are 3 and 0, and $(1, 1, 1)$ is an eigenvector for $\lambda = 3$, while $(1, -1, 0)$ and $(0, 1, -1)$ are linearly independent eigenvectors for $\lambda = 0$. Thus C should have these eigenvectors as its columns:

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

The diagonal matrix $C^{-1}AC$ will have the eigenvalues of A on the diagonal, thus it is

$$D = C^{-1}AC = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) Compute A^{100}.

Solution. $A^{100} = (CDC^{-1})^{100} = CD^{100}C^{-1}$. We compute C^{-1} and multiply:

$$\begin{aligned} A^{100} &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{100} \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3^{100} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -2 \end{pmatrix} /3 \\ &= \begin{pmatrix} 3^{99} & 3^{99} & 3^{99} \\ 3^{99} & 3^{99} & 3^{99} \\ 3^{99} & 3^{99} & 3^{99} \end{pmatrix}. \end{aligned}$$

You could also compute A^{100} by computing A^2 first and noticing that $A^2 = 3A$. Iterating this identity yields $A^n = 3^{n-1}A$ for any $n \geq 1$.

3. **Extra Credit:** The matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has the property that $A^2 = -I$. Find all 2×2 matrices B with this property (i.e., $B^2 = -I$). Hint: think about the eigenvalues of B.

4. **Extra Credit:** Suppose that an $n \times n$ matrix B is diagonalizable, with 0 and 1 as its only eigenvalues. Show that $B^2 = B$. Is the converse true: i.e., if B is diagonalizable and $B^2 = B$, are 0 and 1 the only possible eigenvalues of B?

   Solution. We can write $B = CDC^{-1}$ where D is diagonal with only ones and zeros on the diagonal. Then $D^2 = D$ and $B^2 = CD^2C^{-1} = CDC^{-1} = B$. For the converse, if $B^2 = B$ and λ is an eigenvalue of B with eigenvector v, then $Bv = \lambda v$, and $B^2 v = B\lambda v = \lambda^2 v = Bv = \lambda v$. Thus $\lambda^2 = \lambda$, and this means λ is 0 or 1. (We don't even need to assume B is diagonalizable.)