Name: Solutions

Perm No.: ____________

Math 108A - Final Exam

June 10, 2008

Instructions:

- This exam consists of 5 problems for a total of 65 points.
- You must show all your work and fully justify your answers in order to receive full credit.
- If your justification involves a result proved in class, you should summarize what that result says and, if necessary, explain how you are using it.
- Partial credit will be given for work that is relevant and correct.
- You may assume the results of earlier questions are true, even if you can't prove them, in order to do later questions (eg. to do part (c), you may assume parts (a) and (b)).
- Your proofs will be graded for clarity and organization, in addition to correctness.
- No books, notes or calculators are allowed.
- Write your answers on the test itself, in the space alotted. Scratch paper is available if you need it. You may want to work out your solutions first on scratch paper, so that you can write them on the test as neatly as possible. You may attach additional pages if necessary.

1	
2	
3	
4	
5	
Total	

1. (10 points) Let $V = span\{(1,-1,0,0),\ (2,0,2,1),\ (0,2,2,1),\ (-1,-1,-2,-1)\} \subseteq F^4$. What is $\dim V$? Justify your answer.

We look for linear relations between these vectors:

$$(1,-1,0,0) = -(0,2,2,1) - (-1,-1,-2,-1)$$

$$(-1,-1,-2,-1) = -\tfrac{1}{2}(2,0,2,1) - \tfrac{1}{2}(0,2,2,1)$$

$\Rightarrow V = span\{(2,0,2,1), (0,2,2,1)\}$.

$\Rightarrow$ $\boxed{\dim V = 2}$ since these 2 vectors are linearly independent (they aren't scalar multiples of each other) and thus they form a Basis for V.

2. Let $U = \{(x, y, z) \in F^3 \mid z = 3x + y\}$

(a) (4 points) Find a linear map $T : F^3 \to F$ such that $U = null(T)$.

$U = \{(x,y,z) \in F^3 \mid 3x + y - z = 0\}$.

$= null(T)$

where $\boxed{T(x,y,z) = 3x + y - z}$

(b) (6 points) Find a basis for U.

$T : F^3 \to F$ from above is onto.

(since $Im(T)$ is a nonzero subspace of F)

$\Rightarrow \dim U = \dim null(T) = \dim F^3 - \dim Im(T)$

$= 3 - 1 = 2$.

So we need only find 2 L.I. vectors in U.

take $\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \text{ \& } \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$.

3. (10 points) Let $T \in \mathcal{L}(V)$ for a finite-dimensional vector space V. Show that T is invertible if and only if 0 is not an eigenvalue of T.

$\Rightarrow$ Assume T is invertible.

$\Rightarrow$ T is injective

$\Rightarrow$ $\ker(T) = \{\vec{0}\}$.

If $T(\vec{v}) = 0\vec{v} = \vec{0}$ then $\vec{v} \in \ker(T) = \{\vec{0}\}$.

so $\vec{v} = \vec{0}$.

Thus 0 cannot be an Eigenvalue of T.

$\Leftarrow$: Assume 0 is not an eigenvalue of T.

if $T(\vec{v}) = \vec{0}$ then $T(\vec{v}) = 0 \cdot \vec{v}$

Assumption $\Rightarrow$ $\vec{v} = \vec{0}$.

Thus $\ker(T) = \{\vec{0}\}$ & T is injective

Since $T: V \to V$ and

V is finite-dimensional,

T must be invertible by a Theorem from class. (3.21 in LADR)

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (y + z, x + z, x + y)$.

(a) (3 points) Find the matrix of T with respect to the standard basis.

$$\text{Mat}(T; \mathcal{E}) = \begin{pmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}$$

(b) (6 points) Is T invertible? Justify your answer.

By HW Exercise & Results from class

T is invertible $\Longleftrightarrow$ $\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)\}$ is a basis for F^3

These vectors are the columns of T

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ which we easily see are Linearly Independent.

Since we have 3 L.I. vectors in F^3 they form a basis.

Hence $\boxed{T \text{ is invertible.}}$

(c) (5 points) What are the eigenvalues of T?

$$p(\lambda) = \det(T - \lambda I) = \det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$$

$$= -\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda)$$

$$= -\lambda^3 + 3\lambda + 2$$

$$= -(\lambda+1)^2(\lambda-2)$$

$$\Rightarrow \boxed{\lambda = -1, 2}$$

(d) (6 points) Describe the eigenspaces of T for each eigenvalue.

$\lambda = -1$ $\quad V_{-1} = \ker(T+I) = \ker\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\boxed{= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F^3 \,\middle|\, x+y+z=0 \right\}.}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F^3 \,\middle|\, z = -x-y \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ -x-y \end{pmatrix} \in F^3 \,\middle|\, x, y \in F \right\}$$

$$= \left\{ x\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \,\middle|\, x, y \in F \right\}$$

<u>or</u>

$$= \boxed{\operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.}$$

$\lambda = 2$ $\quad V_2 = \ker(T-2I) = \ker\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

$$= \ker\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$ adding last 2 rows to first row.

$$= \ker\begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 3 \\ 1 & 1 & -2 \end{pmatrix}$$ ~~or~~ subtracting 3rd row from 2nd row.

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F^3 \,\middle|\, \begin{matrix} -3y+3z=0 \\ x+y-2z=0 \end{matrix} \right\}.$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F^3 \,\middle|\, \begin{matrix} y=z \\ x=z \end{matrix} \right\}$$

$$\boxed{\begin{aligned} &= \left\{ \begin{pmatrix} z \\ z \\ z \end{pmatrix} \in F^3 \,\middle|\, z \in F \right\} \\ &= \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \end{aligned}}$$

(e) (5 points) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of T, or show that no such basis exists.

From the previous question, choose 2 L.I. eigenvectors from V_{-1} & one eigenvector from V_2.

$$\Rightarrow \boxed{\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}} = \text{Eigenvector basis of } \mathbb{R}^3.$$

5. Let $v = (c_1, \ldots, c_n) \neq 0 \in F^n$ and let $S : F^n \to F^n$ be the unique linear map such that $S(e_i) = v$ for each standard basis vector e_i. (The matrix of S in the standard basis has v as each column.)

(a) (3 points) Describe $null(S)$.

$$null(S) = \{\vec{v} \in F^n \mid S(\vec{v}) = \vec{0}\} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in F^n \;\middle|\; \begin{matrix} c_1x_1 + \cdots + c_1x_n = 0 \\ c_2x_1 + \cdots + c_2x_n = 0 \\ \vdots \\ c_nx_1 + \cdots + c_nx_n = 0 \end{matrix} \right\}.$$

$$\boxed{= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in F^n \;\middle|\; x_1 + \cdots + x_n = 0 \right\}}$$

(b) (2 points) Show that v is an eigenvector of S. What is the corresponding eigenvalue?

$$\begin{aligned} S(\vec{v}) = S(c_1\vec{e}_1 + \cdots + c_n\vec{e}_n) &= c_1S(\vec{e}_1) + \cdots + c_nS(\vec{e}_n) \\ &= c_1\vec{v} + \cdots + c_n\vec{v} \\ &= (c_1 + \cdots + c_n)\vec{v}. \end{aligned}$$

$\Rightarrow \vec{v}$ = eigenvector of S

↳ eigenvalue $\boxed{c_1 + \cdots + c_n}$

(c) (5 points) Prove that S is diagonalizable if and only if $c_1 + \cdots + c_n \neq 0$.

claim $\dim \operatorname{null}(S) = n-1$.

pf By the rank-nullity Theorem

$$\dim \operatorname{null}(S) = \underbrace{\dim F^n}_{=n} - \underbrace{\dim \operatorname{range}(S)}_{=1} = n-1$$

where $\dim \operatorname{range}(S) = 1$ since $\operatorname{range}(S) = \operatorname{span}(\vec{v})$. (= column space of $\operatorname{Mat}(S;E)$)

$$\operatorname{null}(S) = \ker(S) = \ker(S - 0 \cdot I) = \text{Eigenspace of Eigenvalue } 0.$$

if $c_1 + \cdots + c_n \neq 0$, $\lambda = c_1 + \cdots + c_n$ is another eigenvalue of S and $\dim V_0 + \dim V_\lambda \geq n-1+1 = n$.

so $\dim V_0 + \dim V_\lambda = n$ and S is diagonalizable by Theorem 5.21.

If $c_1 + \cdots + c_n = 0$, then we still have $\dim \operatorname{null}(S) = \dim V_0 = n-1$.

we claim that S has no other eigenvalues besides 0.

If $S\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

we get

$$\begin{aligned} c_1(x_1 + \cdots + x_n) &= \lambda x_1 \\ &\vdots \\ c_n(x_1 + \cdots + x_n) &= \lambda x_n \end{aligned}$$

adding these up $\Rightarrow (c_1 + \cdots + c_n)(x_1 + \cdots + x_n) = \lambda(x_1 + \cdots + x_n)$.

So either $x_1 + \cdots + x_n = 0 \Rightarrow \lambda = 0$

or $c_1 + \cdots + c_n = \lambda \Rightarrow \lambda = 0$.

So $\lambda = 0$ is the only Eigenvalue of S.

Since S has only one Eigenspace in this case & its dimension is $n-1 < n$

S is not diagonalizable. ✓