Math 108A - Basis and Dimension Review

Spring 2009

In the following, V always denotes a finite-dimensional vector space over F.

Finding a basis for a subspace. There are several ways that you can find a basis (and prove it) for a subspace U of V. The best method will depend on the information you are given in a particular problem.

1. If you know a spanning set for U, then you can remove any vector that is a linear combination of the rest to obtain a smaller spanning set. Repeat this process until the vectors in your spanning set are linearly independent (you will need to justify why they are linearly independent).

   Example. Find a basis for $U = span\{(1,1,0),(0,-1,2),(2,2,0),(1,0,2)\} \subseteq \mathbb{R}^3$.

2. If you only have a defintion of U – perhaps as the set of vectors satisfying some equations– then you can start by trying to find as many linearly independent vectors in U as possible. Now try to show that these vectors span U, i.e., that any vector in U can be written as a linear combination of them. If this is not possible, any vector in U that is not such a linear combination can be added to your set to get a larger linearly independent set.

   Example. Find a basis for $U = \{p(x) \in \mathcal{P}_4(F) \mid p(1) = p(-1) = 0\}$.

3. Knowing the dimension of U makes things easier. If you know $\dim U = n$, then a basis for U will consist of any n vectors in U that *EITHER* 1) span U *OR* 2) are linearly independent. Of course, you need to justify whichever of 1) or 2) you choose to use.

   Example. Find a basis for $U = \{(x,y,z) \in \mathbb{R}^3 \mid 2x - y + 3z = 0\}$, which is a 2-dimensional subspace. (That $\dim U = 2$ follows easily from the *rank-nullity theorem* which we'll cover later (see below).)

Finding the dimension of a subspace. Usually the easiest way to find the dimension of a subspace U is to find a basis for U and count how many elements it contains. For instance, this is usually easier than trying to find a minimal spanning set for U, since proving that a set is linearly independent is more straightforward than proving that a smaller spanning set does not exist. But here are a couple shortcuts that make use of theorems from class.

1. Dimension of Sum formulas. If $V = U \oplus W$ for subspaces U and W, then we know that

   $$\dim V = \dim U + \dim W, \quad \text{or} \quad \dim U = \dim V - \dim W.$$

   This is useful when

1) U is a subspace of a vector space V whose dimension you know (eg. $V = F^n$); and

2) you can find a subspace W for which

a) $U \cap W = \{0\}$;

b) $U + W = V$; and

c) you know $\dim W$.

Example. Let $U = \{(x, y, z, w) \in \mathbb{R}^4 \mid x = w, y = z\}$. Find $\dim U$ by showing that $\mathbb{R}^4 = U \oplus W$ for $W = span(e_1, e_2)$.

2. Rank-Nullity Theorem. (For future reference.) If U is expressed as a set of vectors satisfying certain linear equations, then you can view U as the kernel (or null-space) of a linear transformation $T : V \to W$. The rank-nullity theorem then says that

$$\dim U = \dim \ker(T) = \dim V - \dim \operatorname{im}(T).$$

Example. Consider U from Example 3 above. By definition $U = \ker(T)$ where $T : \mathbb{R}^3 \to \mathbb{R}$ is the linear map defined by

$$T(x, y, z) = 2x - y + 3z.$$

The image of T is clearly $\mathbb{R}$ (since it is a nonzero subspace of $\mathbb{R}$). Thus

$$\dim U = \dim \mathbb{R}^3 - \dim \mathbb{R} = 2.$$