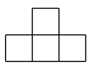
Math 116 - Bonus Problems (11/21/08)

The problems below are meant to be challenging, and correct solutions will be awarded some amount of extra credit. We may eventually discuss the solutions to some in lecture (when everyone is stumped), but solutions will be accepted at any time beforehand. This list will be updated each week, and hints will occasionally be added.

1. For which values of n can an $n \times n$ chess board be tiled by tetronimoes of the shape below? Justify your answer. (Note: You must describe how to obtain a tiling in all the cases where it can be done, AND prove that no tiling exists in the remaining cases.)



2. (see Ex. 39 in Ch. 1) Suppose there are 200 points in the plane, no 3 colinear, and that 100 are colored red and the other 100 are colored blue. Show that it is always possible to connect each red point to a different blue point using 100 straight line segments that do not intersect each other.

3. A computer screen shows an 8 x 8 chessboard, colored in the usual way. One can select with the mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangles switch (black becomes white, white becomes black). What is the minimum number of mouse clicks needed to make the chessboard all one color? Justify your answer. (Once you find the minimum number of clicks necessary, you must also prove that there is no way to do it in fewer clicks.)

4. Let n be a positive integer. How many up-right paths are there from $(0, 0)$ to (n, n) that do not cross the line $y = x$?

5. (Sequel to No. 2). Describe an algorithm that finds a pairing of the red points with the blue points so that none of the connecting line segments intersect. Try to make it as efficient as possible.

6. (Easy!?) Suppose you have 12 pieces of cake, which are all of different sizes. How many ways are there to give all 12 pieces to 3 children so that each child gets at least

one piece? (It might be a lot longer with 5 children! Or maybe not.) Hint: one way of doing this problem (not the easiest) uses exponential generating functions (see the example on p. 257).

7. Give a combinatorial proof (i.e., by counting the same set in two different ways) of the binomial "stocking" identity

$$\sum_{n=k}^{m} \binom{n}{k} = \binom{m+1}{k+1}.$$

Hint: Combinations with repetition.

8. (Different than asked in class) How many binary strings of length n, contain two consecutive 0's and two consecutive 1's? (You may leave your answer in terms of the Fibonacci numbers if necessary.)

9. Let D_n be the number of derangements of the set $\{1, 2, \ldots, n\}$. In class, we proved that

(a) $\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e}$, and

(b) the exponential generating function for D_n is

$$g_e(x) = \sum_{n=0}^{\infty} \frac{D_n}{n!} x^n = \frac{e^{-x}}{1-x}.$$

Can you deduce (a) from (b)? More generally, given a convergent power series $g(x) = \sum_{n=0}^{\infty} c_n x^n$ with $\lim_{n\to\infty} c_n = c$, what is the easiest way of finding c from $g(x)$?