Math 116 - Solutions to Take-Home Midterm

1. A certain card game is played with a standard deck of 52 cards. You are dealt 7 cards, and a winning hand consists of 4 cards of the same rank AND 3 consecutive cards in the same suit (including the possibility of King - Ace - 2, etc.). What is the probability of being dealt a winning hand?

   Solution. The total number of possible hands is $\binom{52}{7}$. To create a winning hand, we must choose one rank (from 13) in which we will have 4-of-a-kind. There are 13 ways to make this choice. Next, we must choose three consecutive ranks from the remaining 12 ranks. There are only 10 ways to do this, since we have to avoid the rank that we already chose for the 4-of-a-kind. Finally, we have to choose one of 4 suits for the three cards in a row, which can be done in 4 different ways. By the multiplication principle, we get a total of $13 * 10 * 4 = 520$ winning hands. Hence, the probability of being dealt a winning hand is $520/\binom{52}{7} \approx .000004$.

2. You have 10 (indistinguishable) apples and 10 (indistinguishable) oranges.

   (a) How many ways are there to pass all 20 pieces of fruit out to 10 children?

   (b) How many ways are there to pass all 20 pieces of fruit out to 10 children so that each child gets 2 pieces?

   Solution. a) The number of ways to pass out 10 apples to the 10 children is given by the number of 10-combinations with repetition of the 10-children, which is $\binom{10+10-1}{10-1} = \binom{19}{9}$. This is also the number of ways to pass out 10 oranges. Since any combination of a distribution of the apples with a distribution of the oranges is possible, by the multiplication principle, the total number of ways to distribute both the apples and oranges (with no restrictions) is $\binom{19}{9}^2$.

   b) Each child gets 2 pieces of fruit, and thus either 2 apples, 2 oranges, or 1 of each. We will divide the problem into cases based on the number of children who get 1 of each. Suppose that exactly k children get 1 orange and 1 apple. Obviously, $0 \leq k \leq 10$. Also notice that k must be even, since the remaining $10 - k$ apples must be handed out in pairs. Now for any of these possible values of k, there will be $\binom{10}{k}$ ways to choose which k children get 1 of each type of fruit. Then there will be 10-k children left over, and we must choose exactly half of them who will get 2 apples each (the other half will then get 2 oranges each). There are $\binom{10-k}{5-k/2}$ ways to do this. Letting $k = 2j$, we can now express the total as

$$\sum_{j=0}^{5} \binom{10}{2j}\binom{10-2j}{5-j} = \binom{10}{0}\binom{10}{5} + \binom{10}{2}\binom{8}{4} + \binom{10}{4}\binom{6}{3} +$$

$$
\begin{aligned}
& +\binom{10}{6}\binom{4}{2}+\binom{10}{8}\binom{2}{1}+\binom{10}{10}\binom{0}{0} \\
= \; & 8953.
\end{aligned}
$$

3. A ferris wheel has 8 identical cars, each with 4 seats in a row.

   (a) How many different ways can 32 people be arranged on the ride? (Since the ferris wheel turns, two arrangements that can be rotated into each other are considered to be the same.)

   (b) How many different ways can 30 people be arranged on the ride? (So 2 seats will be empty.)

   Solution. a) There are a total of 32! ways to seat the people on the ride. However, accounting for the rotation of the ferris wheel, we see that each distinct seating arrangement has been counted 8 times. Hence the total number of distinct seating arrangements is 32!/8.

   b) There are $\binom{32}{2}$ ways to pick two empty seats and then 30! ways to arrange everyone else. Again, however, each distinct arrangement gets counted 8 times, corresponding to its various rotations. Hence the total number of distinct seating arrangements is $30!\binom{32}{2}/8 = 32!/16$.

   Alternative solution. Let Mr. X be one of the people. Given any seating arrangement, we can rotate it so that Mr. X's car is at the bottom of the wheel. We have 4 ways to choose his seat, and then 31! ways to arrange everyone else in the remaining 31 seats (which are now in fixed positions). By the multiplication principle, we have a total of $4 * 31!$ possible arrangements. For (b), we have to count the number of ways of placing 29 people in the remaining 31 seats. To do this, we can think of the empty seats as being occupied by two identical ghosts. Then we must count the number of 31-permutations of the 31-element multi-set consisting of 29 distinct people and 2 identical ghosts. The formula yields $31!/(1!)^{29}2! = 31!/2$. Multiplying by 4 (the number of ways to choose X's seat in the bottom car), we get a total of $2 * 31!$ possible arrangements.

4. A bakery sells 4 different types of cupcakes: Chocolate, Vanilla, Strawberry and Lemon; and currently has 4 of each kind available. How many ways are there to buy a dozen cupcakes if you buy at least one of each kind?

   Solution. If we start by taking one of each kind, we will have 8 cupcakes left to choose and we are allowed to take at most 3 of any single flavor. We will use the inclusion-exclusion principle as follows. Let S be the set of all possible ways of choosing 8 cupcakes of 4 different flavors (with no restrictions). By the formula for counting combinations with repetition $|S| = \binom{8+4-1}{4-1} = \binom{11}{3}$. Now let A_1 (respectively, A_2, A_3, A_4) be the set of ways to choose 8 cupcakes, including at least 4 chocolate (respectively, vanilla, strawberry, lemon) ones. Notice that we are asked to find

$$|\bar{A}_1 \cap \cdots \cap \bar{A}_4| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| + \cdots.$$

We begin by counting A_1. If we first take 4 chocolate cupcakes, then we must choose 4 more from 4 different kinds. There are $\binom{4+4-1}{4-1} = \binom{7}{3}$ ways to do this. Hence $|A_1| = \binom{7}{3}$ and similarly we see that A_i has the same size for each i. Now we count $A_1 \cap A_2$. The number of elements in this set equals the number of ways to choose 8 cupcakes where at least 4 are chocolate and at least 4 are vanilla; but there is only 1 such choice. Similarly $|A_i \cap A_j| = 1$ for all $i < j$. We also see that the intersection of any 3 or more of the sets A_i will be empty, since we are only selecting 8 cupcakes. Hence, by the inclusion-exclusion principle, we get

$$\begin{aligned} |\bar{A}_1 \cap \cdots \cap \bar{A}_4| &= |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| + 0 \\ &= \binom{11}{3} - 4\binom{7}{3} + 6 \\ &= 31. \end{aligned}$$

5. Evaluate the sum

$$\sum_{k=0}^{2008} (-1)^k \binom{2008}{k} \binom{2009}{k}.$$

Hint: Calculate the coefficient of x^{2009} in the expansion of $(1-x)^{2008}(1+x)^{2009}$ in two different ways.

Solution. Following the hint, we first combine the two binomials using $(1-x)(1+x) = 1 - x^2$:

$$\begin{aligned} (1-x)^{2008}(1+x)^{2009} &= (1-x^2)^{2008}(1+x) \\ &= (1+x) * \left[\sum_{k=0}^{2008} \binom{2008}{k} (-1)^k x^{2k}\right] \\ &= \sum_{k=0}^{2008} (-1)^k \binom{2008}{k} (x^{2k} + x^{2k+1}). \end{aligned}$$

In this expression, the coefficient of x^{2009} is $\left(\begin{array}{c} 2008 \\ 1004 \end{array} \right)$, which occurs when $k = 1004$. Next, we use the binomial theorem twice to expand $(1-x)^{2008}$ and $(1+x)^{2009}$ separately, and then multiply these two polynomials together.

$$\begin{aligned} (1-x)^{2008}(1+x)^{2009} &= \left[\sum_{i=0}^{2008} (-1)^i \left(\begin{array}{c} 2008 \\ i \end{array} \right) x^i \right] * \left[\sum_{j=0}^{2009} \left(\begin{array}{c} 2009 \\ j \end{array} \right) x^j \right] \\ &= \sum_{i=0}^{2008} \sum_{j=0}^{2009} (-1)^i \left(\begin{array}{c} 2008 \\ i \end{array} \right) \left(\begin{array}{c} 2009 \\ j \end{array} \right) x^{i+j}. \end{aligned}$$

To find the coefficient of x^{2009} in the last expression, we must set $i + j = 2009$, or equivalently, we can let $j = 2009 - i$. This produces the coefficient of x^{2009} as

$$\sum_{i=0}^{2008} (-1)^i \left(\begin{array}{c} 2008 \\ i \end{array} \right) \left(\begin{array}{c} 2009 \\ 2009-i \end{array} \right) = \sum_{i=0}^{2008} (-1)^i \left(\begin{array}{c} 2008 \\ i \end{array} \right) \left(\begin{array}{c} 2009 \\ i \end{array} \right),$$

which is the summation that we are trying to calculate. By our computation above, we see that it is equal to $\left(\begin{array}{c} 2008 \\ 1004 \end{array} \right)$.