1. [22 pts] There is a linear transtormation $T : \mathbb{R}^3 \to \mathbb{R}^3$ following the rule $T(x, y, z) = (4x + 6z - 3y,\ 3z + 2x,\ 2y - 4x - 6z)$.

(a) Find the standard matrix for T.

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}4\\2\\-4\end{bmatrix},\quad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-3\\0\\2\end{bmatrix},\quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}6\\3\\-6\end{bmatrix}$$

$$\boxed{\begin{bmatrix}4 & -3 & 6\\ 2 & 0 & 3\\ -4 & 2 & -6\end{bmatrix}}$$

(b) Find the kernel of T.

$$\left[\begin{array}{ccc|c}4 & -3 & 6 & 0\\ 2 & 0 & 3 & 0\\ -4 & 2 & -6 & 0\end{array}\right] \longrightarrow \begin{bmatrix}1 & -3/4 & 3/2 & 0\\ 2 & 0 & 3 & 0\\ -4 & 2 & -6 & 0\end{bmatrix} \longrightarrow$$

$$\begin{bmatrix}1 & -3/4 & 3/2 & 0\\ 0 & 3/2 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & -3/4 & 3/2 & 0\\ 0 & 1 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix} \longrightarrow$$

$$\begin{bmatrix}1 & 0 & 3/2 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \longrightarrow \begin{cases} x + \frac{3}{2}z = 0\\ y = 0\end{cases}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-\frac{3}{2}z\\0\\z\end{bmatrix} = z\begin{bmatrix}-\frac{3}{2}\\0\\1\end{bmatrix}$$

$$\boxed{\mathrm{Ker}(T) = \left\{ z\begin{bmatrix}-3/2\\0\\1\end{bmatrix} \;\middle|\; z \in \mathbb{R} \right\}}$$

(c) What is the dimension of $\mathrm{Ker}(T)$? $\boxed{1}$

2. [16 pts] **Fact**: $\vec{\mathbf{x}}_1(t) = \begin{bmatrix} t \\ -2 \end{bmatrix}$ and $\vec{\mathbf{x}}_2(t) = \begin{bmatrix} t^5 \\ 2t^4 \end{bmatrix}$ are solutions of the homogeneous linear DE system $\vec{\mathbf{x}}'(t) = \begin{bmatrix} 3t^{-1} & 1 \\ 4t^{-2} & 2t^{-1} \end{bmatrix} \vec{\mathbf{x}}(t)$.

(a) Compute the Wronskian $W(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2)$.

$$\begin{vmatrix} t & t^5 \\ -2 & 2t^4 \end{vmatrix} = \boxed{4t^5}$$

(b) Are $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$ linearly independent? $\boxed{\text{YES}}$

(BECAUSE $W(\vec{x}_1, \vec{x}_2) = 4t^5 \neq 0$.)

(c) Find a solution to the DE system satisfying the initial condition $\vec{\mathbf{x}}(1) = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$.

$$\vec{x} = c_1 \begin{bmatrix} t \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} t^5 \\ 2t^4 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 10 \end{bmatrix} = \vec{x}(1) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{cases} c_1 + c_2 = -1 \\ -2c_1 + 2c_2 = 10 \end{cases}$$

$$-c_1 + c_2 = 5$$

$$2c_2 = 4$$

$$c_2 = 2$$

$$c_1 = -3$$

$$\boxed{\vec{x}(t) = -3 \begin{bmatrix} t \\ -2 \end{bmatrix} + 2 \begin{bmatrix} t^5 \\ 2t^4 \end{bmatrix}}$$

3. [22 pts] **(a)** Find all the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$.

$$\begin{vmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 5 = (\lambda-1)(\lambda-5)$$

$$\boxed{\lambda = 1, 5}$$

(b) Find one eigenvector for A corresponding to each eigenvalue. Be sure to label which eigenvector goes with which eigenvalue.

$\underline{\lambda = 1}$ $$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow x + \frac{1}{2}y = 0 \qquad \boxed{\text{E.G.,} \begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

$\underline{\lambda = 5}$ $$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow x - \frac{1}{2}y = 0 \qquad \boxed{\text{E.G.,} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

(c) Write down the general solution for the system $\begin{cases} x' = 3x + y \\ y' = 4x + 3y \end{cases}$.

$$\boxed{\vec{x}(t) = c_1 e^{t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

4. [20 pts] **(a)** **Fact**: The matrix $A = \begin{bmatrix} 1 & -1 \\ 5 & 5 \end{bmatrix}$ has eigenvalues $3 \pm i$. Find one eigenvector for A with eigenvalue $3 + i$.

$$\left[\begin{array}{cc|c} -2-i & -1 & 0 \\ 5 & 2-i & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & \frac{2}{5}-\frac{1}{5}i & 0 \\ -2-i & -1 & 0 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & \frac{2}{5}-\frac{1}{5}i & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow x + \left(\frac{2}{5}-\frac{1}{5}i\right)y = 0$$

$$\boxed{\text{E.G., } \begin{bmatrix} -2+i \\ 5 \end{bmatrix}}$$

(b) Find the general solution for the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ in **real** form (i.e., with no imaginary terms).

$$e^{(3+i)t}\begin{bmatrix} -2+i \\ 5 \end{bmatrix} = e^{3t}(\cos t + i \sin t)\left(\begin{bmatrix} -2 \\ 5 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$= e^{3t}\left(\begin{bmatrix} -2\cos t - \sin t \\ 5\cos t \end{bmatrix} + i\begin{bmatrix} \cos t - 2\sin t \\ 5\sin t \end{bmatrix}\right)$$

$$\boxed{\vec{x}(t) = c_1 e^{3t}\begin{bmatrix} -2\cos t - \sin t \\ 5\cos t \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} \cos t - 2\sin t \\ 5\sin t \end{bmatrix}}$$

(c) What kind of trajectories do these solutions have?

$\boxed{\text{OUTWARD SPIRALS}}$

5. [20 pts] **(a)** **Fact**: The vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector for the matrix $A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$. What is its eigenvalue?

$$\begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \boxed{\lambda = 1}$$

(b) **Fact**: The matrix A has only one eigenvalue (repeated). Find the general solution for the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

$$(A - I)\vec{u} = \vec{v} \longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ -4 & -2 & -2 \end{array}\right] \longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow 2x + y = 1 \qquad \text{E.G.,} \quad \vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{x}(t) = c_1 e^{t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{t}\left(t\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)}$$

$$\text{OR:} \quad c_1 e^{t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} t \\ -2t+1 \end{bmatrix}$$

(c) Find the solution which satisfies the initial condition $\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$c_1 = 2, \quad c_2 = 4$$

$$\boxed{\vec{x}(t) = 2e^{t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 4e^{t}\begin{bmatrix} t \\ -2t+1 \end{bmatrix}} \quad \text{OR:} \quad e^{t}\begin{bmatrix} 4t+2 \\ -8t \end{bmatrix}$$