Math 5B
Final Examination
March 20, 2006

Name ANSWERS

There are 10 problems, with a total of 100 points.

1. (10pts) Let $\mathbf{u} = 2e^x \sin y\,\mathbf{i} + 4e^x \cos y\,\mathbf{j} + 9z\,\mathbf{k}$

   A. Find div $\mathbf{u}$

$$\nabla \cdot \bar{u} = \frac{\partial}{\partial x} 2e^x \sin y + \frac{\partial}{\partial y} 4e^x \cos y + \frac{\partial}{\partial z} 9z$$

$$= 2e^x \sin y - 4e^x \sin y + 9$$

$$= -2e^x \sin y + 9$$

   B. Find curl $\mathbf{u}$

$$\nabla \times \bar{u} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2e^x \sin y & 4e^x \cos y & 9z \end{vmatrix}$$

$$= 0 \cdot \bar{i} - (0 - 0)\bar{j} + (4e^x \cos y - 2e^x \cos y)k$$

$$= 2e^x \cos y\,\bar{k}$$

2. (10pts) Given that

$$x^2 + y^2 + z^2 - u^2 + v^2 = 1, \qquad x^2 - y^2 + z^2 + u^2 + 2v^2 = 21,$$

find the differentials du, dv in terms of dx, dy, dz at the point $x = 1$, $y = 1$, $z = 2$, $u = 3$, $v = 2$.

$$\begin{cases} 2x\,dx + 2y\,dy + 2z\,dz - 2u\,du + 2v\,dv = 0 \\ 2x\,dx - 2y\,dy + 2z\,dz + 2u\,du + 4v\,dv = 0 \end{cases}$$

At the point,

$$\begin{cases} dx + dy + 2dz - 3du + 2dv = 0 \\ dx - dy + 2dz + 3du + 4dv = 0 \end{cases}$$

$$-3du + 2dv = -dx - dy - 2dz$$
$$3du + 4dv = -dx + dy - 2dz$$

$$du = \frac{\begin{vmatrix} -dx - dy - 2dz & 2 \\ -dx + dy - 2dz & 4 \end{vmatrix}}{\begin{vmatrix} -3 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{\overset{-2}{(-4+2)}dx + \overset{-6}{(-4-2)}dy + \overset{-4}{(-8+4)}dz}{-12-6}$$

$$dv = \frac{1}{-18}\begin{vmatrix} -3 & -dx - dy - 2dz \\ 3 & -dx + dy - 2dz \end{vmatrix} = -\frac{1}{18}\left[\overset{6}{(3+3)}dx + \overset{0}{(-3+3)}dy + \overset{12}{(6+6)}dz\right]$$

(a) The differentials are

$du = \frac{1}{9}dx + \frac{1}{3}dy + \frac{2}{9}dz$

$dv = -\frac{1}{3}dx - \frac{2}{3}dz$

(b) The values of u and v for $x = 1.1$, $y = 1.2$, $z = 1.8$ are approximately

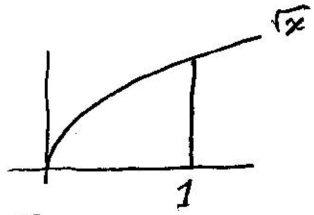
$\Delta u = \frac{1}{9}(1.1 - 1) + \frac{1}{3}(1.2 - 1) + \frac{2}{9}(1.8 - 2) = \frac{.1}{9} + \frac{.2}{3} - \frac{.4}{9} = \frac{.3}{9} = .033\ldots$

$\Delta v = -\frac{1}{3}(.1) - \frac{2}{3}(1.8 - 2) = -\frac{.1}{3} + \frac{2}{3}\cdot .2 = \frac{.3}{3} = .1$

$u \approx 3 + .033\ldots = 3.033\ldots$

$v \approx 2 + .1 = 2.1$

3. (10pts) Evaluate $\iint_R \frac{2y}{x^2+1}\,dA$ where R is the region bounded by the x-axis, $x = 1$, and $y = \sqrt{x}$.



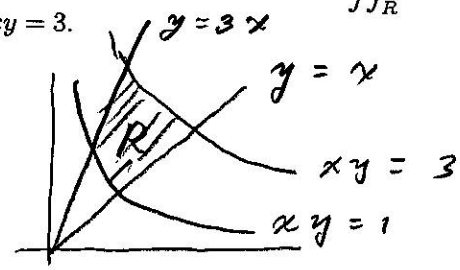
$$\iint_R \frac{2y}{x^2+1}\,dA = \int_0^1 \int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\,dx$$

$$= \int_0^1 \frac{1}{x^2+1}\left[y^2\right]_0^{\sqrt{x}}\,dx = \int_0^1 \frac{x}{x^2+1}\,dx$$

$$= \left[\frac{1}{2}\ln(x^2+1)\right]_0^1 = \frac{1}{2}\left[\ln(1+1) - \ln(0+1)\right]$$

$$= \frac{1}{2}\ln 2 = \ln\sqrt{2}$$

4. (10pts) Use the transformation $x = u/v,\ y = v$ to evaluate $\iint_R xy\, dA$ where R is the area enclosed by $y = x,\ y = 3x,\ xy = 1$, and $xy = 3$.



$$\iint_{R_{xy}} xy\, dA = \iint_{R_{uv}} \frac{u}{v}\cdot v \left| \begin{matrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{matrix} \right| du\, dv = \iint_{R_{uv}} u \left| \begin{matrix} 1/v & 0 \\ -u/v^2 & 1 \end{matrix} \right| du\, dv$$

$$= \iint_{R_{uv}} \frac{u}{v}\, du\, dv$$

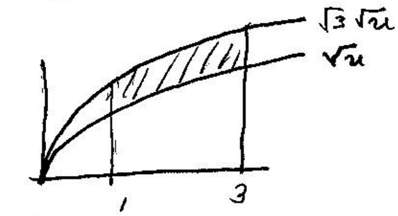
$$y = x \implies v = u/v \implies v = \sqrt{u}$$

$$y = 3x \implies v = 3u/v \implies v = \sqrt{3}\sqrt{u}$$

$$xy = 1 \implies \frac{u}{v}\cdot v = 1 \implies u = 1$$

$$xy = 3 \implies \frac{u}{v}\cdot v = 3 = u = 3$$

so R_{uv} is



$$\iint_{R_{uv}} \frac{u}{v}\, du\, dv = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3}\sqrt{u}} \frac{u}{v}\, dv\, du = \int_1^3 u\left[\ln v\right]_{\sqrt{u}}^{\sqrt{3}\sqrt{u}} du$$

$$= \int_1^3 u\left(\ln(\sqrt{3}\sqrt{u}) - \ln\sqrt{u}\right) du = \int_1^3 u\left(\ln\sqrt{3} + \ln\sqrt{u} - \ln\sqrt{u}\right) du$$

$$= \int_1^3 u \ln\sqrt{3}\, du = \ln\sqrt{3}\ \tfrac{1}{2}\left[u^2\right]_1^3 = \tfrac{1}{2}\ln\sqrt{3}\,(9-1)$$

$$= 4\ln\sqrt{3} = 2\ln 3$$

5. (10pts) Find the length of the curve $x = 2\sin t$, $y = 5t$, $z = 2\cos t$, $-10 \le t \le 10$

$$s = \int_{t=-10}^{t=10} ds = \int_{-10}^{10} \sqrt{4\cos^2 t + 5^2 + (-2\sin t)^2}\, dt$$

$$= \int_{-10}^{10} \sqrt{4\cos^2 t + 25 + 4\sin^2 t}\, dt = \int_{-10}^{10} \sqrt{29}\, dt$$

$$= \sqrt{29}\,[10 - (-10)] = 20\sqrt{29}$$

6. (10pts) Evaluate $\int_C \mathbf{u}\cdot d\mathbf{r}$, where $\mathbf{u} = x^2y^3\,\mathbf{i} - y\sqrt{x}\,\mathbf{j}$, and C is the curve given by $\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\,\mathbf{j}$, $\quad 0 \le t \le 1$.

$$\frac{\partial}{\partial y} x^2y^3 = 3x^2y^2 \,, \quad \frac{\partial}{\partial x}\left(-y\sqrt{x}\right) = -\frac{y}{2\sqrt{x}}$$

These are different, so it's not a gradient.
It's not a closed curve either, so Green's Theorem won't help.

We'll just have ~~to~~ have guts.

$$\int_C \bar{u}\cdot d\bar{r} = \int_C x^2y^3\,dx - y\sqrt{x}\,dy$$

$$= \int_C (t^2)^2(-t^3)^3(2t\,dt) - (-t)^3\sqrt{t^2}\,(-3t^2)\,dt$$

$$= \int_0^1 \left(-t^4t^9\,2t - t^3\,t\,3t^2\right)dt = \int_0^1 -2t^{14} - 3t^6 \quad dt$$

$$= \left[-\frac{2}{15}t^{15} - \frac{3}{7}t^7\right]_0^1 = -\frac{2}{15} - \frac{3}{7} = \frac{-14-45}{105} = -\frac{59}{105}$$

7. (10pts) Evaluate $\int_C xy^3\,ds$ where C is given by $x = 4\sin t,\ y = 4\cos t,\ z = 3t,\ 0 \le t \le \pi/2$.

$$\int_C xy^3\,ds = \int_C (4\sin t)(4\cos t)^3 \sqrt{(4\cos t)^2 + (-4\sin t)^2 + 9}\ dt$$

$$= \int_C 4^4 \cos^3 t \sin t \sqrt{16+9}\ dt$$

$$= 5\cdot 4^4 \int_0^{\pi/2} \cos^3 t \sin t\ dt = 5\cdot 4^4 \left[-\frac{1}{4}\cos^4 t\right]_0^{\pi/2}$$

$$= 5\cdot 4^3 \left[-(0)^4 - (-1^4)\right] = 5\cdot 4^3 = 5\cdot 64 = 320$$

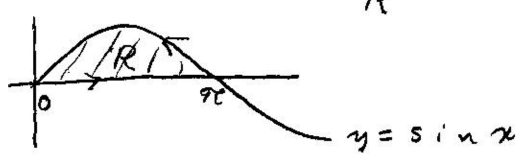
8. (10pts) Use Green's Theorem to find $\oint_C (\sqrt{x} + y^3)\, dx + (x^2 + \sqrt{y})\, dy$ where C is the curve formed by the x-axis from 0 to π and $y = \sin x$.

$$\frac{\partial}{\partial y}(\sqrt{x} + y^3) = 3y^2 \qquad \frac{\partial}{\partial x}(x^2 + \sqrt{y}) = 2x$$

They are different, so the integral may not be 0.

$$\oint_C (\sqrt{x} + y^3)dx + (x^2 + \sqrt{y})dy = \iint_R 2x - 3y^2 \; dA$$

R is



$$\iint_R 2x - 3y^2 dA = \int_0^{\pi} \int_0^{\sin x} 2x - 3y^2 \; dy \; dx$$

$$= \int_0^{\pi} \left[2xy - y^3 \right]_0^{\sin x} dx = \int_0^{\pi} 2x\sin x - \sin^3 x \; dx$$

From the table of integrals,

$\int x \sin x \, dx = \sin x - x\cos x$, $\int \sin^3 x \, dx = -\frac{1}{3}(2 + \sin^2 x)\cos x$

$$\iint_R 2x - 3y^2 dA = 2\left[\sin x - x\cos x\right]_0^{\pi} - \left[-\frac{1}{3}(2 + \sin^2 x)\cos x\right]_0^{\pi}$$

$$= 2\left[0 - \pi(-1) - 0 + 0\right] + \frac{1}{3}\left[(2 + 0^2)(-1) - (2 + 0^2)(1)\right]$$

$$= 2\pi + \frac{1}{3}\left[-4\right] = 2\pi - \frac{4}{3}$$

9. (10pts) Evaluate $\int_C \mathbf{u} \cdot d\mathbf{r}$, where $\mathbf{u} = x^3y^4\,\mathbf{i} + x^4y^3\,\mathbf{j}$, and C is the curve given by $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (1 + t^3)\mathbf{j}, \quad 0 \le t \le 1$.

$$\frac{\partial}{\partial y} x^3 y^4 = 4x^3y^3 = \frac{\partial}{\partial x} x^4 y^3$$

Since these are equal, $\bar{u}$ is a gradient.

In fact, $\bar{u} = \nabla\left(\frac{1}{4} x^4 y^4\right)$. Let $F = \frac{1}{4} x^4 y^4$

The integral is independent of path; so

$$\int_C \bar{u} \cdot d\bar{r} = F\Big|_{t=1} - F\Big|_{t=0}$$

$$F = \frac{1}{4} \sqrt{t}^{\,4} (1+t^3)^4 = \frac{1}{4} t^2 (1+t^3)^4$$

$$F\Big|_{t=0} = \frac{1}{4} \cdot 0^2 (1 + 0^3)^4 = 0$$

$$F\Big|_{t=1} = \frac{1}{4} 1^2 (1 + 1^3)^4 = \frac{1}{4} \cdot 2^4 = 4$$

$$\int_C \bar{u} \cdot d\bar{r} = 4$$

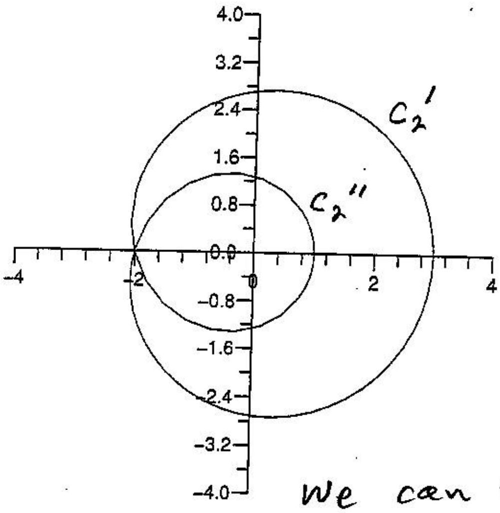
10. (10pts) Assuming that

$$P\,dx + Q\,dy = \frac{x^2y\,dx - x^3\,dy}{(x^2+y^2)^2}$$

(note that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$), what can be said about the relationship between

$$\oint_{C_1} P\,dx + Q\,dy \qquad \text{and} \qquad \oint_{C_2} P\,dx + Q\,dy$$

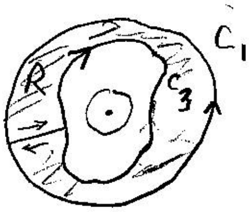
when C_1 is the circle of radius 4 centered at the origin and C_2 is the curve shown? (Namely, the curve given by $x = 2\cos(2t) - \cos t,\ y = 2\sin 2t - \sin t,\ 0 \le t \le 2\pi$, which loops twice around the origin.) Explain your answer, starting from Green's Theorem. (In class, we began by considering the case of one closed curve completely contained in the interior of the other ...)



$$\int_{C_2} P\,dx + Q\,dy = 2 \cdot \int_{C_1} P\,dx + Q\,dy.$$

The domain here is the entire xy-plane, except the origin – there is a hole in it at the origin. Elsewhere, everything is continuous with continuous derivatives.

We can show that if another simple closed curve C_3 is enclosed by C_1, then it has the same integral as C_1:



Connect C_1 + C_2 as shown. From Green's theorem,

$$\oint_{C_1} \sim + \oint_{-C_3} \sim = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = 0$$

(We do C_3 in reverse direction.)

So $\oint_{C_1} \sim = \oint_{C_3} \sim$

Now C_2 can be viewed as two simple closed curves, each enclosed by C_1; call the outer on C_2' and the inner C_2''. $\oint_{C_1} \sim = \oint_{C_2'} \sim$ and $\oint_{C_1} \sim = \oint_{C_2''} \sim$.

But $\oint_{C_2} \sim = \int_{C_2'} \sim + \int_{C_2''} \sim$.