Math 5B, Midterm 2 Review Problems

Fall 2006

1. (a) Convert the point $(1, -1, 1)$ from rectangular to cylindrical coordinates.

   Solution. (It helps to draw pictures!) To convert to cylindrical coordinates, all we need to do is change the x, y-coordinates to polar coordinates: $r = \sqrt{x^2 + y^2} = \sqrt{2}$, and $\theta = \tan^{-1}(y/x) = \tan^{-1}(-1) = -\pi/4$, which is in the same quadrant as $(1, -1)$. So $(1, -1, 1) = (\sqrt{2}, -\pi/4, 1)$ in cylindrical coordinates.

   (b) Convert $(2, \pi/2, 2\pi/3)$ from spherical to rectangular coordinates.

   Solution. $x = \rho \sin\phi \cos\theta = 2\sin(2\pi/3)\cos(\pi/2) = 0$, $y = \rho \sin\phi \sin\theta = 2\sin(2\pi/3)\sin(\pi/2) = \sqrt{3}$, and $z = \rho\cos\phi = 2\cos(2\pi/3) = -1$. So $(2, \pi/2, 2\pi/3) = (0, \sqrt{3}, -1)$ in rectangular coordinates.

2. Suppose z and w are functions of x and y given by the equations $z = \dfrac{1+y}{y-x} + 2$ and $w = e^{x+2y} - 1$. Find the Jacobian matrix of the inverse mapping when $(z, w) = (2, 0)$, and simplify your answer.

   Solution. Start by finding the Jacobian $\frac{\partial(z,w)}{\partial(x,y)}$ of the given mapping by taking partial derivatives of z and w with respect to x and y. The Jacobian of the inverse mapping will be the inverse of this matrix.

$$J = \frac{\partial(z, w)}{\partial(x, y)} = \begin{pmatrix} \frac{1+x}{(x-y)^2} & \frac{-x-1}{(x-y)^2} \\ e^{x+2y} & 2e^{x+2y} \end{pmatrix}.$$

   Before finding the inverse, we can plug in $z = 2$ and $w = 0$. Since we don't have any z's or w's, we must solve for x and y in the two defining equations for z and w:

$$2 = \frac{1+y}{y-x} + 2 \Rightarrow \frac{1+y}{y-x} = 0 \Rightarrow 1 + y = 0 \Rightarrow y = -1,$$

$$0 = e^{x+2y} - 1 \Rightarrow x + 2y = 0 \Rightarrow x = 2.$$

   Thus $J|_{(2,-1)} = \begin{pmatrix} 0 & -1/3 \\ 1 & 2 \end{pmatrix}$, and $J^{-1}|_{(2,-1)} = \begin{pmatrix} 6 & 1 \\ -3 & 0 \end{pmatrix}$.

3. The two equations $xy + uv = 1$ and $xu + yv = 1$ define u and v implicitly as functions of x and y.

   (a) Find the Jacobian matrix $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution. Let $F(x,y,u,v) = xy + uv - 1$ and $G(x,y,u,v) = xu + yv - 1$. We use the formulas (2.61) to find the partial derivatives.

$$\frac{\partial u}{\partial x} = \frac{-\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = \frac{-\begin{vmatrix} y & u \\ u & y \end{vmatrix}}{\begin{vmatrix} v & x \\ u & y \end{vmatrix}} = \frac{u^2 - y^2}{yv - xu}.$$

$$\frac{\partial u}{\partial y} = \frac{-\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = \frac{-\begin{vmatrix} x & v \\ u & y \end{vmatrix}}{\begin{vmatrix} v & x \\ u & y \end{vmatrix}} = \frac{uv - xy}{yv - xu}.$$

$$\frac{\partial v}{\partial x} = \frac{-\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = \frac{-\begin{vmatrix} v & y \\ x & u \end{vmatrix}}{\begin{vmatrix} v & x \\ u & y \end{vmatrix}} = \frac{xy - uv}{yv - xu}.$$

$$\frac{\partial v}{\partial y} = \frac{-\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = \frac{-\begin{vmatrix} v & x \\ x & v \end{vmatrix}}{\begin{vmatrix} v & x \\ u & y \end{vmatrix}} = \frac{x^2 - v^2}{yv - xu}.$$

So

$$J = \begin{pmatrix} \frac{u^2-y^2}{yv-xu} & \frac{uv-xy}{yv-xu} \\ \frac{xy-uv}{yv-xu} & \frac{x^2-v^2}{yv-xu} \end{pmatrix}.$$

A simpler method is to use formula (2.68), which says that the Jacobian matrix

$$\frac{\partial(u,v)}{\partial(x,y)} = -\left(\frac{\partial(F,G)}{\partial(u,v)}\right)^{-1}\left(\frac{\partial(F,G)}{\partial(x,y)}\right) = -\begin{pmatrix} v & u \\ x & y \end{pmatrix}^{-1}\begin{pmatrix} y & x \\ u & v \end{pmatrix} = \cdots$$

(b) Calculate $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial x \partial y}$.

Solution.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{u^2 - y^2}{yv - xu}\right) = \frac{(vy - xu)2uu_x - (u^2 - y^2)(yv_x - u - xu_x)}{(vy - xu)^2}.$$

At this point, we should substitute in u_x and v_x from part (a) and simplify, but you would not be required to write this out on the test.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{u^2 - y^2}{yv - xu}\right) = \frac{(vy - xu)(2uu_y - 2y) - (u^2 - y^2)(v + yv_y - xu_y)}{(vy - xu)^2}.$$

4. Let S be the surface given by the equation $x^3 - xy - yz - xz - x + 2 = 0$.

   (a) Show that the curve C whose equation is $\mathbf{r}(t) = \begin{cases} x &= t+1 \\ y &= t^2 \\ z &= 2 \end{cases}$ is contained in the surface S.

   Solution. Let $F(x, y, z) = x^3 - xy - yz - xz - x + 2$. We must check that $F(t+1, t^2, 2) = 0$. We have $F(t+1, t^2, 2) = (t+1)^3 - (t+1)t^2 - 2t^2 - 2(t+1) - (t+1) + 2 = t^3 + 3t^2 + 3t + 1 - t^3 - t^2 - 2t^2 - 2t - 2 - t - 1 + 2 = 0$.

   (b) Find the equation of the tangent line to C at the point $(2, 1, 2)$.

   Solution. We first find the value of t that corresponds to the point $(2, 1, 2)$ on C: we have $x = 2 = t + 1$, so $t = 1$. The tangent vector to C when $t = 1$ is $\mathbf{v} = (x'(1), y'(1), z'(1)) = (1, 2, 0)$. Thus the equation of the tangent line to C at $(2, 1, 2)$ is $(x, y, z) = (2, 1, 2) + t(1, 2, 0) = (2 + t, 1 + 2t, 2)$.

   (c) Find the equation of the tangent plane to S at the point $(2, 1, 2)$.

   Solution. The normal vector to S at the point $(2, 1, 2)$ is given by the gradient vector of F: $\nabla F = (3x^2 - y - z - 1, -x - z, -y - x)$. So $\nabla F|_{(2,1,2)} = (8, -4, -3)$. The equation of the tangent plane is thus $8(x - 2) - 4(y - 1) - 3(z - 2) = 0$.

5. Find all critical points of the function $f(x, y) = 3x^3 - 6xy + y^2$, and classify each as a relative min, relative max, or saddle point.

   Solution. To find critical points, we look for where ∇f is $\mathbf{0}$ or undefined. We need to solve $\nabla f = (9x^2 - 6y, -6x + 2y) = (0, 0)$, as we can see that ∇f is never undefined. The second equation implies that $y = 3x$, and plugging this into the first equation we have $9x^2 - 18x = 9x(x - 2) = 0$. So $x = 0$ (and $y = 0$) or $x = 2$ (and $y = 6$). So the critical points are $(0, 0)$ and $(2, 6)$. We now calculate the Hessian matrix of second order partial derivatives

   $$H = \begin{pmatrix} 18x & -6 \\ -6 & 2 \end{pmatrix}.$$

   At $(0, 0)$, $\det H = 0(2) - (-6)^2 < 0$, so f has a saddle point at $(0, 0)$. At $(2, 6)$, $\det H = 36(2) - (-6)^2 > 0$ and $trH = 36 + 2 > 0$ so f has a relative minimum at $(2, 6)$.

6. Suppose you want to construct a rectangular wooden box without a top so that the volume is 32 cubic feet. What dimensions (x = length, y = width, z = height) of the box will minimize the amount of wood you need to construct it?

   Solution. The amount of wood needed to construct the box is $f(x, y, z) = xy + 2xz + 2yz$ (xy is the area of the base, there are 2 sides of area xz and 2 sides of area

yz, and no top). So we must minimize the function f subject to the side condition $xyz = 32$, which says that the volume of the box is 32. Let $g(x, y, z) = xyz - 32$, so that $g(x, y, z) = 0$ expresses the side condition. The method of Lagrange multipliers tells us that this minimum must occur where $\nabla f = \lambda \nabla g$ for some real number λ (notice that there are no boundary points to consider here, since we must have $x > 0, y > 0, z > 0$). This gives us the following four equations (the last one is the side condition) which we must solve for x, y, z.

(1) $y + 2z = \lambda yz$

(2) $x + 2z = \lambda xz$

(3) $2x + 2y = \lambda xy$

(4) $xyz = 32$

One way to solve these is to subtract (2) from (1) to get $y - x = \lambda z(y - x)$. From this equation, we see that either $y - x = 0$ or if not, divide both sides by $y - x$, to get $\lambda z = 1$. If $\lambda z = 1$, equation (1) would become $y + 2z = y$, which means $z = 0$, but then $\lambda = 1/z$ is undefined. So we conclude that $y - x = 0$, or $x = y$. Equation (3) now becomes $4x = x^2$, and since x cannot be 0, $x = 4$. Thus $y = x = 4$, and $z = 32/xy = 32/16 = 2$. So the dimensions that minimize the surface area are $(x, y, z) = (4, 4, 2)$.

7. Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a vector field on $\mathbb{R}^3$.

   (a) Find curl($\mathbf{v}$).

   Solution.

$$\text{curl}(\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \cdots = \mathbf{0}.$$

   One can also see this by realizing that $\mathbf{v} = \nabla f$ for $f(x, y, z) = x^2/2 + y^2/2 + z^2/2$ and using the fact that $\text{curl}(\nabla f) = 0$ for any function f with continuous second order partials.

   (b) Show that there is no differentiable vector field $\mathbf{u}$ on $\mathbb{R}^3$ such that $\mathbf{v} = \text{curl}(\mathbf{u})$.

   Solution. Rather than trying to find such a $\mathbf{u}$, recall that $\text{div}(\text{curl}(\mathbf{u})) = 0$ for any vector field $\mathbf{u}$ with continuous second order partials. Since $\text{div}(\mathbf{v}) = 1+1+1 = 3 \neq 0$, $\mathbf{v}$ cannot be equal to curl($\mathbf{u}$) for any $\mathbf{u}$.

8. Let $\mathbf{v} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}$ be a vector field on the surface S defined by the equation $x^2 + y^2 + z^2 = 1$ (i.e., S is the unit sphere in $\mathbb{R}^3$).

(a) Show that at each point of S, the vector field $\mathbf{v}$ is tangent to S.

Solution. Since S is defined by the equation $F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$, the normal vector to S at the point (x, y, z) is $\nabla F = (2x, 2y, 2z)$, and $\mathbf{v} = (y - z, z - x, x - y)$ is tangent to S if $\nabla F \cdot \mathbf{v} = 0$. We have $\nabla F \cdot \mathbf{v} = 2x(y - z) + 2y(z - x) + 2z(x - y) = 0$.

(b) Is $\mathbf{v} = \nabla f$ for some differentiable function $f(x, y, z)$? Justify your answer.

Solution. If $\mathbf{v} = \nabla f$, then $\text{curl}(\mathbf{v}) = \text{curl}(\nabla f) = 0$. So we start by checking whether $\text{curl}(\mathbf{v}) = 0$.

$$\text{curl}(\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - x & x - y \end{vmatrix} = (\frac{\partial}{\partial y}(x-y) - \frac{\partial}{\partial z}(z-x))\mathbf{i} + \cdots = -2\mathbf{i} + \cdots \neq \mathbf{0}.$$

Thus $\mathbf{v}$ cannot equal ∇f for any f.

(c) Is $\mathbf{v} = \text{curl}(\mathbf{u})$ for some differentiable vector field $\mathbf{u}$ on S? Justify your answer.

Solution. As in 7b, we should start by checking that $\text{div}(\mathbf{v}) = 0$. Since this is true here, there is a good chance that $\mathbf{v} = \text{curl}(\mathbf{u})$ for some $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$, and we just need to find $\mathbf{u}$. To simplify things, suppose that $u_z = 0$, so that $\text{curl}(\mathbf{u}) = -\frac{\partial u_y}{\partial z}\mathbf{i} + \frac{\partial u_x}{\partial z}\mathbf{j} + (\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y})\mathbf{k}$. Equating the coordinates of $\text{curl}(\mathbf{u})$ with those of $\mathbf{v}$ we get a system of 3 partial differential equations.

(1) $\frac{\partial u_y}{\partial z} = z - y \Rightarrow u_y = z^2/2 - yz + C(x, y)$

(2) $\frac{\partial u_x}{\partial z} = z - x \Rightarrow u_x = z^2/2 - xz + D(x, y)$

(3) $\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = x - y$.

Integrating (1) and (2) with respect to z gives equations for u_y and u_x as above. We now plug these expressions into (3), and choose the functions $C(x, y)$ and $D(x, y)$ so that (3) is satisfied. Plugging into (3), we have $C_x - D_y = x - y$. So we can choose $C(x, y) = x^2/2$ and $D(x, y) = y^2/2$. Hence, we have shown that $\mathbf{v} = \text{curl}(\mathbf{u})$ for the vector field

$$\mathbf{u} = (z^2/2 + y^2/2 - xz)\mathbf{i} + (z^2/2 + x^2/2 - yz)\mathbf{j} + 0\mathbf{k}.$$